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SOME REMARKS ON SIMPLE TOURNAMENTS

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1. Introduction

In this paper we shall prove some results about tournaments which we believe to be interesting both from an algebraic and a set theoretic point of view. The definition of a simple tournament, the subject of our title, was motivated by questions in algebra, but the results and the proofs we give are essentially set theoretical. We assume that the reader is familiar with the current notations of set theory.

A *tournament* $\mathbf{T}=\langle T, \to \rangle$ is a relational structure, where T is a non-empty set and $\to$ is a trichotomous binary relation on T, i.e. for every pair $x, y \in T$ exactly one of the three relations

$$x \to y, x = y, y \to x$$

holds. Here $x \to y$ expresses the fact that $(x, y) \in \to$. We shall also write $b \leftarrow a$ if $a \to b$. It is well known that corresponding to any tournament $\mathbf{T}=\langle T, \to \rangle$, there is an algebraic structure $\mathbf{T}^*=\langle T, \vee, \wedge \rangle$ in which $\vee$ and $\wedge$ are idempotent binary operations which satisfy the condition

$$x, y \in T, x \to y \Rightarrow x = x \vee y, \quad y = x \wedge y.$$

$\mathbf{T}^*$ is a simple algebraic structure iff the tournament $\mathbf{T}$ is simple in accordance with Definition 2 below.

Let $\mathbf{T}=\langle T, \to \rangle$ be a tournament. For $A, B \subset T$ we write $A \to B$ iff $a \to b$ for all $a \in A$ and $b \in B$. If $x \in T$, we put $T(x, \to)=\{y \in T: x \to y\}$, $T(x, \to)=\{y \in T: x \leftarrow y\}$. Thus for any $x \in T$, $T = T(x, \to) \cup \{x\} \cup T(x, \leftarrow)$ and the summands are disjoint.

DEFINITION 1. $K \subset T$ is a *convex subset* of the tournament $\mathbf{T}$ if $K \neq \phi$ and for every $x \in T-K$ either $K \subset T(x, \to)$ or $K \subset T(x, \leftarrow)$.

It is clear that the whole set T is convex and so is any singleton $\{x\}$ and we call these the *trivial convex subsets* of $\mathbf{T}$. We denote by $C(\mathbf{T})$ the set of all non-trivial convex subsets, i.e. $X \in C(\mathbf{T})$ iff X is convex, $X \neq T$ and $|X| > 1$.

DEFINITION 2. The tournament $\mathbf{T}$ is *simple* iff $C(\mathbf{T})=\phi$.

The two-element tournament $\mathbf{T}_0$ is simple and the equational class determined by $\mathbf{T}_0^*$ is the class of distributive lattices.

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The first existence result for simple tournaments was given in [1] where it was shown that there are such tournaments having orders p^k for prime $p \equiv 3 \pmod 4$ and odd k. In [2] it was shown that any tournament $\mathbf{T}=\langle T, \to \rangle$ can be embedded in a simple tournament of order $2|T|+1$ and also, if $|T| \neq 1$, in a simple tournament of order $2|T|+2$. This shows that there are simple tournaments having cardinality $m \neq 4$. It is easy to check (e.g. [2]) that there is no simple tournament of order 4.

In §2 we give an elementary counting argument which shows that, for large n, almost all tournaments of order n are simple. Our main result is the following theorem.

THEOREM 1. *Let* $T \subset T'$, $|T'-T|=2 \neq |T|$. *Then any tournament* $\mathbf{T}=\langle T, \to \rangle$ *on* T *can be extended to a simple tournament* $\mathbf{T}'=\langle T', \to \rangle$ *on* T'.

In general a tournament does not have a one-element simple extension. For example, if $|T|=2n+1$ (n a positive integer) and $\to$ is a linear order on T, then $\langle T, \to \rangle$ cannot be embedded in a simple tournament of order $|T|+1$. However, we do not know necessary and sufficient conditions for a tournament to have such a one-point simple extension.[2]) Our proof of Theorem 1 makes use of Theorems 2 and 3 below. These may not be essential, but the results are interesting in themselves.

THEOREM 2. (Straightening theorem) *Suppose* $\mathbf{T}=\langle T, \to \rangle$ *is a tournament and* $\to$ *is not a linear order of* T. *Then there is a linear order* $\xrightarrow{\circ}$ *of* T *such that* $C(\mathbf{T}) \subsetneq C(\mathbf{T}^0)$ *where* $\mathbf{T}^0=\langle T, \xrightarrow{\circ} \rangle$.

The linear order $\xrightarrow{\circ}$ in Theorem 2 is not unique and it would be interesting to know to what extent this order is characterized by $\mathbf{T}$. Since the convex subsets of a linearly ordered set are the intervals, an immediate corollary of Theorem 2 is the following.

COROLLARY. *If* $\mathbf{T}_0\langle T=, \to \rangle$ *is a finite tournament, then* $|C(\mathbf{T})| \leqslant \binom{|T|}{2}-1$ *and there is equality only if* $\to$ *is a linear order.*

The rational numbers provide an example of a tournament of order ω having 2^ω non-trivial convex subsets.

An important combinatorial concept for set systems (see [3], [4] and [5]) is the so-called property B.

DEFINITION 3. A family of sets $\mathbf{F}$ is said to have property B (the Bernstein property) iff there is a set B such that $F \cap B \neq \phi$ and $B \not\subseteq F$ for all $F \in \mathbf{F}$.

THEOREM. 3. *For any tournament* $\mathbf{T}$, $C(\mathbf{T})$ *has property* B.

Finally, in §5, we shall prove the following two results.

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2) This problem has now been solved. A tournament $\langle T, \to \rangle$ has a one-point simple extension iff $|T| \neq 3$ and it is not a finite odd chain. This was first proved for finite tournaments by J. W. Moon [8] and extended to arbitrary tournaments by three of us in [6].

THEOREM 4. *If α is any infinite cardinal, there are 2^α pairwise non-isomorphic simple tournaments of order α.*

THEOREM 5. *The class of simple tournaments is not pseudo-elementary.*

2. A proof that almost all finite tournaments are simple

The number of tournaments on n points is $2^{\binom{n}{2}}$. The number of such tournaments in which a fixed k-element set is convex is $2^{\binom{k}{2}} \times 2^{\binom{n-k}{2}} \times 2^{n-k}$. Therefore, the number of non-simple tournaments on n points is

$$N(n) \leqslant \sum_{k=2}^{n-1} \binom{n}{k} 2^{\binom{k}{2}+\binom{n-k}{2}+(n-k)}.$$

Hence

$$\begin{aligned}\frac{N(n)}{2^{\binom{n}{2}}} &\leqslant \sum_{k=2}^{n-1} \binom{n}{k} \frac{1}{2^{(n-k)(k-1)}} \\ &\leqslant \left(\binom{n}{2} + \binom{n}{n-1}\right) 2^{-n+2} + \sum_{k=3}^{n-2} \binom{n}{k} 2^{-2n+6} \\ &\leqslant n(n+1)\, 2^{-n+1} + 2^{-n+6} \to 0\end{aligned}$$

as $n \to \infty$.

3. Proof of Theorem 2

A triple $\{x, y, z\} \subset T$ is called a circuit of the tournament $\mathbf{T} = \langle T, \to \rangle$ if either $x \to y \to z \to x$ or $x \to z \to y \to x$. Let $\Delta(\mathbf{T})$ denote the set of circuits of $\mathbf{T}$. Thus $\to$ is a linear order iff $\Delta(\mathbf{T}) = \phi$. For $z \in T$, put $\Delta(\mathbf{T}, z) = \{\{x, y\} : \{x, y, z\} \in \Delta(\mathbf{T})\}$, and for $\{x, y\} \subset T$, put $\Delta(\mathbf{T}, \{x, y\}) = \{z \in T : \{x, y, z\} \in \Delta(\mathbf{T})\}$. For any set $X \subset [T]^2 = \{Y \subset T : |Y| = 2\}$, we denote by $\mathbf{T}(X)$ the tournament $\mathbf{T}^0 = \langle T, \overset{\circ}{\to} \rangle$ obtained from $\mathbf{T}$ by reversing the relation $\to$ on the pairs belonging to X, i.e. for $u, v \in T$ $u \overset{\circ}{\to} v$ iff $u \to v$ and $\{u, v\} \notin X$ or $u \leftarrow v$ and $\{u, v\} \in X$.

LEMMA 1. *Let $\mathbf{T} = \langle T, \to \rangle$ be a tournament, $z \in T$ and suppose that $X = \Delta(\mathbf{T}, z) \neq \phi$. If $\mathbf{T}^0 = \mathbf{T}(X)$, then (i) $\Delta(\mathbf{T}^0) \subsetneqq \Delta(\mathbf{T})$, (ii) $\Delta(\mathbf{T}^0, \{x, y\}) = \phi$ for $\{x, y\} \in X$ and (iii) $C(\mathbf{T}) \subsetneqq C(\mathbf{T}^0)$.*

Proof. Let $U_0 = \mathbf{T}(z, \leftarrow)$, $U_1 = \mathbf{T}(z, \to)$. Then $X = \{\{a, b\} : a \in U_0 \wedge b \in U_1 \wedge a \leftarrow b\}$. Note that $\mathscr{T} = U_0 \cup \{z\} \cup U_1$ and the relations $\overset{\circ}{\to}$ and $\to$ coincide on $U_0 \cup \{z\}$ and $\{z\} \cup U_1$; also $U_0 \overset{\circ}{\to} U_1$. It follows that if D is any circuit of $\mathbf{T}^0$, then $D \subset U_0$ or $D \subset U_1$.

Hence $C(\mathbf{T}^0) \subset C(\mathbf{T})$ and $\Delta(\mathbf{T}^0, \{x, Y\}) = \phi$ for $\{x, y\} \in X$. This proves (i) and (ii) since $z \in \Delta(\mathbf{T}; \{x, y\}) \neq \phi$.

Suppose $K \subset T$ is convex in $\mathbf{T}$. We will show that K is also convex in $\mathbf{T}^0$. Let $u \in T - K$.

Case (a). $K \cap U_0 = \phi$. Since $\rightarrow$ and $\xrightarrow{\circ}$ coincide on $\{z\} \cup U_1$, it follows that if $u \in \{z\} \cup U_1$, then $K \subset \mathbf{T}^0(u, \xrightarrow{\circ})$ or $K \subset \mathbf{T}^0(u, \xleftarrow{\circ})$ according as $K \subset \mathbf{T}(u, \rightarrow)$ or $K \subset \mathbf{T}(u, \leftarrow)$. On the other hand, if $u \in U_0$, then $K \subset \mathbf{T}^0(u, \xrightarrow{\circ})$. Thus K is convex in $\mathbf{T}^0$.

Case (b). $K \cap U_1 = \phi$. A similar argument as in Case (a) shows K is convex in $\mathbf{T}^0$.

Case (c). $K \cap U_0 \neq \phi$, $K \cap U_1 \neq \phi$. Since $U_0 \rightarrow \{z\} \rightarrow U_1$ it follows that, in this case, $z \in K$. Suppose $u \in U_0$. Then $u \rightarrow z$ implies that $K \subset \mathbf{T}(u, \rightarrow) \subset \mathbf{T}^0(u, \xrightarrow{\circ})$. Similarly, if $u \in U_1$, then $K \subset \mathbf{T}(u, \leftarrow) \subset \mathbf{T}^0(u, \xleftarrow{\circ})$. Thus $K \in C(\mathbf{T}^0)$.

Since $U_0 \cup \{z\}$ and $U_1 \cup \{z\}$ are non-trivial convex sets in $\mathbf{T}^0$ (but not in $\mathbf{T}$) it follows that $C(\mathbf{T}) \subsetneqq C(\mathbf{T}^0)$.

Next we prove

LEMMA 2. *Let* $\mathbf{T} = \langle T, \rightarrow \rangle$ *be a tournament and let* $X_\sigma (\sigma < \psi)$ *be an increasing sequence of subsets of* $[T]^2$. *Let* $\mathbf{T}_\sigma = \langle T, \xrightarrow{\sigma} \rangle = \mathbf{T}(X_\sigma)(\sigma < \psi)$ *and suppose that* (i) $\Delta(\mathbf{T}_\sigma) \supset \Delta(\mathbf{T}_\varrho)(\sigma < \varrho < \psi)$, (ii) $\Delta(\mathbf{T}_\sigma, \{x, y\}) = \phi$ *for* $\{x, y\} \in X_\sigma (\sigma < \psi)$ *and* (iii) $C(\mathbf{T}_\sigma) \subset C(\mathbf{T}_\varrho)(\sigma < \varrho < \psi)$. *Then, if* $X = \bigcup_{\sigma < \psi} X_\sigma$ *and* $\mathbf{T}^* = \mathbf{T}(X)$, (i)′ $\Delta(\mathbf{T}_\sigma) \supset \Delta(\mathbf{T}^*)(\sigma < \psi)$, (ii)′ $\Delta(\mathbf{T}^*, \{x, y\}) = \phi$ *for* $\{x, y\} \in X$, (iii)′ $C(\mathbf{T}_\sigma) \subset C(\mathbf{T}^*)$.

Proof. By definition, $u \xrightarrow{\varrho} v$ holds iff there is $\sigma < \psi$ such that $u \xrightarrow{\varrho} v$ for $\sigma \leqslant \varrho < \psi$. Hence $\Delta(\mathbf{T}^*) = \cap \Delta(\mathbf{T}_\sigma)$ and (i)′ holds. If $\{x, y\} \in X$ and $z \in \Delta(\mathbf{T}^*, \{x, y\}) \neq \phi$, then there is $\sigma < \psi$ such that $\{x, y\} \in \mathbf{X}_\sigma$ and there is ϱ such that $\sigma \leqslant \varrho < \psi$ and the relations $\xrightarrow{\varrho}$, $\xrightarrow{*}$ coincide on $\{x, y, z\}$. This implies the contradiction $z \in \Delta(\mathbf{T}_\varrho, \{x, y,\}) \neq \phi$ and so (ii)′ holds. Finally, suppose $K \subset T$ is convex in $\mathbf{T}_\sigma$. Suppose $u, v \in K$, $w \in T$ and $u \xrightarrow{*} w \xrightarrow{*} v$. Then there is ϱ such that $\sigma \leqslant \varrho < \psi$ and $u \xrightarrow{\varrho} w \xrightarrow{\varrho} v$. By (iii) K is convex in $\mathbf{T}_\varrho$ and hence $w \in K$. This proves that K is convex in $\mathbf{T}^*$ and hence (iii)′ holds.

Proof of Theorem 2. It follows from lemma 2 and Zorn's lemma that there is a maximal set $X^0 \subset [T]^2$ such that, if $\mathbf{T}^0 = \mathbf{T}(X^0)$,

(i) $\Delta(\mathbf{T}) \supset \Delta(\mathbf{T}^0)$

(ii) $\Delta(\mathbf{T}^0, \{x, y\}) = \phi$ for $\{x, y\} \in X^0$,

(iii) $C(\mathbf{T}) \subset C(\mathbf{T}^0)$.

We claim that $\mathbf{T}^0 = \langle T, \xrightarrow{\circ} \rangle$ satisfies the requirements of Theorem 2.

Suppose indirectly that $\mathbf{T}^0$ is not a linear order, i.e., $\Delta(\mathbf{T}^0) \neq \phi$. Then there is $z \in T$ such that $X = \Delta(\mathbf{T}^0, z) \neq \phi$. Put $\mathbf{T}^{00} = \mathbf{T}^0(X)$. Then by Lemma 1, (a) $\Delta(\mathbf{T}^{00}) \subset \Delta(\mathbf{T})$, (b) $\Delta(\mathbf{T}^{00}, \{x, y\}) = \phi$ for $\{x, y\} \in X$ and (c) $C(\mathbf{T}^0) \subset C(\mathbf{T}^{00})$. Since $z \in \Delta(\mathbf{T}^0, \{x, y\}) \neq \phi$ for $\{x, y\} \in X$, it follows from (ii) that $X \cap X^0 = \phi$. Therefore, if $X^{00} = X \cup X^0$, we have $\mathbf{T}^{00} = \mathbf{T}^0(X) = \mathbf{T}(X^{00})$. It follows from (a) and (c) that (i) and (iii) both hold with $\mathbf{T}^{00}$ in place of $\mathbf{T}^0$. If $\{x, y\} \in X^0$, then $\Delta(\mathbf{T}^{00}, \{x, y\}) \subset \Delta(\mathbf{T}, \{x, y\}) = \phi$ by (a) and (ii). This, together with (b), shows that $\Delta(\mathbf{T}^{00}, \{x, y\}) = \phi$ for $\{x, y\} \in X^{00}$ contradicting the assumed maximality of X^0. This proves that $\mathbf{T}^0$ is a linear order of T.

Since, by hypothesis, $\mathbf{T}$ is not a linear order, it follows from Lemma 1 that there is $\mathbf{T}'$ such that $C(\mathbf{T}) \subsetneqq C(\mathbf{T}')$. Applying the above to $\mathbf{T}'$ instead of $\mathbf{T}$, we obtain a linear order $\mathbf{T}^0$ such that $C(\mathbf{T}) \subsetneqq C(\mathbf{T}') \subset C(\mathbf{T}^0)$.

4. Proof of Theorem 3

If $\mathbf{F}' \subset \mathbf{F}$ and $\mathbf{F}$ has property B, then so does $\mathbf{F}'$. Therefore, in view of the straightening theorem, it is sufficient to prove Theorem 3 under the assumption that $\mathbf{T} = \langle T, \rightarrow \rangle = \langle T, < \rangle$ is a linearly ordered set. For $a, b \in T$, we shall write $[a, b]$ to denote the closed interval $\{x \in T : a \leqslant x \leqslant b \vee b \leqslant x \leqslant a\}$. Since every non-trivial convex set of $\mathbf{T}$ contains a proper interval $[a, b]$ with $a \neq b$, it will be enough to prove that $C_0 = \{[a, b] : a, b \in T \wedge a \neq b\}$ has property B.[3])

We first show that the set $C_1 = \{[a, b] : |[a, b]| = 2\}$ of two-element convex sets has property B. Let $\sim$ be the equivalence relation defined on T by putting $x \sim y$ iff $[a, b]$ is finite. Then T is the disjoint union of equivalence classes (intervals) which are either finite, or have order types ω or ω^* or $\omega^* + \omega$. If $[a, b] \in C_1$, then a, b are neighbouring elements of one of these equivalence classes D and $|D| > 1$. Consequently, to prove our assertion, it will be enough to prove that if D is such an equivalence class, $|D| > 1$, then the intervals of D have property B. On D define another equivalence relation $\vdash\!\dashv$ so that $x \vdash\!\dashv y$ iff $[x, y]$ is odd. This equivalence partitions D into two disjoint sets $B_D \cup (D - B_D)$ and B_D establishes property B for the intervals of D. The set $B' = \cup B_D$, where the union extends over all $\sim$ equivalence classes D with $|D| > 1$, shows that C_1 has property B.

Now let $\{I_\xi : \xi < \varphi\}$ be a well ordering of C_0 so that C_1 is an initial segment, i.e. $C_1 = \{I_\xi : \xi < \varphi_0\}$ for some $\varphi_0 < \varphi$. We are going to define elements x_ξ, y_ξ for $\xi < \varphi$ by transfinite induction as follows. Let $\xi < \varphi$ and suppose x_ζ, y_ζ have been defined for $\zeta < \xi$ so that $B_\zeta = \{x_\eta : \eta < \zeta\}$ and $C_\zeta = \{x_\eta : \eta < \zeta\}$ are disjoint for $\zeta \leqslant \xi$, and for $\zeta < \xi$

(o) if $B_\zeta \cap I_\zeta \neq \phi$, then $x_\zeta = x_\tau$ where $\tau = \min\{\tau' < \zeta : \mathbf{X}_{\tau'} \in I_\zeta\}$,

(oo) if $C_\zeta \cap I_\zeta \neq \phi$, then $y_\zeta = y_\sigma$ where $\sigma = \min\{\sigma' < \zeta : y_{\sigma'} \in I_\zeta\}$.

We have to define x_ξ, y_ξ so that $B_{\xi+1} = B_\xi \cup \{x_\xi\}$ and $C_{\xi+1} = C_\xi \cup \{y_\xi\}$ are disjoint and so that (o), (oo) hold for $\zeta = \xi$. Let B' be the set defined in the preceding paragraph which establishes property B for the set C_1. We distinguish two cases (i) $\xi < \varphi_0$, (ii) $\varphi_0 \leqslant \xi$. In case (i) we simply put x_ξ and y_ξ as the unique elements of $B' \cap I_\xi$ and $I_\xi - B'$ respectively. It is easy to see that (o) and (oo) hold for $\zeta = \xi$ in this case. Now suppose (ii) $\xi \geqslant \varphi_0$. We first show that $I_\xi - C_\xi \neq \phi$. Suppose this is false, i.e. that $I_\xi \subset C_\xi$. Let y_ϱ, y_σ be the two elements of I_ξ having minimal suffices. If $[y_\varrho, y_\sigma] \in C_1$, then $[y_\varrho, y_\sigma] = I_\lambda$ for some $\lambda < \varphi_0 \leqslant \xi$ and so $x_\lambda = y_\varrho$ or y_σ contrary to the assumption that $B_\xi \cap C_\xi = \phi$. Hence there is τ such that $\max(\varrho, \sigma) < \tau < \xi$, $y_\tau \in [y_\varrho, y_\sigma]$ and, by the

[3]) F. Hausdorff [7, Satz, X.] proved this in the case when $\langle T, < \rangle$ is dense.

assumed minimality of ϱ and σ, $y_\tau \neq y_\alpha$ for $\alpha \leqslant \max(\varrho, \sigma)$. Therefore, by assumption (oo) $y^\varrho \notin I_\tau$ and $y_\sigma \notin I_\tau$. Therefore, $I_\tau \subset [y_\varrho, y_\sigma]$ and $x_\tau \in I_\tau - C_\xi \subset I_\xi \neq C_\xi$. This proves that $I_\xi - C_\xi \neq \phi$. If $I_\xi \cap B_\xi \neq \phi$ then we put $x_\xi = x_\lambda$ where $\lambda = \min\{\lambda' < \xi : x_{\lambda'} \in I_\xi\}$. On the other hand, if $I_\xi \cap B_\xi = \phi$, then we choose an arbitrary $x_\xi \in I_\xi - C_\xi$. Clearly (o) holds with this definition for x_ξ. A similar argument shows that $I_\xi - B_{\xi+1} \neq \phi$ and then y_ξ can be chosen so that $B_{\xi+1} \cap C_{\xi+1} = \phi$ and (oo) holds. This defines the x_ξ, y_ξ for $\xi < \varphi$ and the set $B = \{x_\xi : \xi < \varphi\}$ establishes the property B for $C(\mathbf{T})$.

Remark. We mention that the proof of Theorem 3 can be carried out without appealing to the straightening theorem. If, instead of intervals, we consider the convex subsets of an arbitrary tournament which are generated by two elements, essentially the same proof works but the argument is slightly more involved.

5. Proof of Theorem 1

We may assume $|T| > 2$. Let $T' = T \cup \{x, y\}$, where $x \neq y$ and $x, y \notin T$. We extend the relation $\to$ to T' as follows.

By Theorem 3 there is a set $B \subset T$ which establishes property B for $C(\mathbf{T})$. In the special case that $\cap C(\mathbf{T}) \neq \phi$, then we assume that $B = \{u\}$ where $u \in \cap C(\mathbf{T})$. We define $\to$ on $[T']^2 - [T]^2$ by putting (i) $\{x\} \to B \to \{y\}$, $\{x\} \leftarrow T - B \leftarrow \{y\}$ and (ii) $x \to y$.

Let $K \in C(\mathbf{T}')$. If $\{x, y\} \subset K$, then (i) implies that $B \subset K$, $T - B \subset K$ and hence $K = T'$, a contradiction. Therefore, $|\{x, y\} \cap K| \leqslant 1$. Suppose $|K \cap T| \geqslant 2$. Then $K \cap T \in C(\mathbf{T})$ and hence $K \cap T \cap B \neq \phi$ and $K \cap T \cap (T - B) \neq \phi$. By (i) $T - B \to \{x\} \to B$, $B \to \{y\} \to T - B$ and so $\{x, y\} \subset K$, a contradiction. Therefore, we can assume that $K = \{x, v\}$ or $\{y, v\}$ for some $v \in T$. If $v \in T - B$, then by (i) and (ii) $v \to x \to y \to v$ so that both $x, y \in K$, a contradiction. Therefore, we can suppose that $v \in B$. Assume that there is $D \in C(\mathbf{T})$ such that $v \notin D$. There are $a \in D \cap B$ and $b \in D - B$. If $K = \{x, v\}$, then the convexity of K implies $a \to v \leftarrow b$, and if $K = \{y, v\}$, then $a \to v \to b$. In either case this contradicts the assumption that $v \notin D$. It follows that there is no such D and hence $\cap C(\mathbf{T}) \neq \phi$, i.e. $B = \{u\}$ and $u = v$. The convexity of K implies that $\{u\} \leftarrow T - \{u\}$ if $K = \{x, u\}$ and $\{u\} \to T - \{u\}$ if $K = \{y, u\}$. Since $|T| > 2$, this implies that $T - \{u\} \in C(\mathbf{T})$ contrary to fact that $B (= \{u\})$ meets every member of $C(\mathbf{T})$.

6. Proofs of Theorems 4 and 5

Proof of Theorem 4. It is well known that there are order types $\Xi_\xi = tp(X_\xi, <_\xi)$ for $\xi < 2^\alpha$ such that $|X_\xi| = \alpha$ and

(i) $\Xi_\xi \neq \Xi_\eta$ for $\xi \neq \eta$

(ii) $tp(X_\xi - X) = \Xi_\xi$ for $X \subset X_\xi$, $|X| < \omega$.

For example, one such construction is to put $\varphi(0) = \omega^*$, $\varphi(1) = \omega^* + \omega$ and $\Xi(f) = \sum_{v < \alpha} \varphi(f(v))$ for $f \in {}^\alpha 2$.

By Theorem 1, for each $\xi < 2^{\alpha}$, there is a simple tournament $\mathbf{T}_{\xi} = < T_{\xi}, \overset{\xi}{\rightarrow} >$ such that $|T_{\xi} - X_{\xi}| = 2$, $X_{\xi} \subset T_{\xi}$ and $\overset{\xi}{\rightarrow}$ is an extension of $<_{\xi}$. Suppose ψ is an isomorphic mapping of T_{ξ} onto T_{η}. Then ψ is an order preserving mapping from $X_{\xi} - P$ onto $X_{\eta} - Q$ where $P = \psi^{-1}(T_{\eta} - X_{\eta})$, $Q = \psi(T_{\xi} - X_{\xi})$. It follows from (i) and (ii) that $\xi = \eta$.

Proof of Theorem 5. We first give some definitions and prove a lemma. Let $\mathbf{T} = \langle T, \rightarrow \rangle$ be a tournament and let $X \subset T$. We put $K'(\mathbf{T}, X) = \{u \in T : u \in X \vee v \rightarrow u \rightarrow w$ for some $v, \omega \in X\}$. Now put $K_0(\mathbf{T}, X) = X$, $K_{n+1}(\mathbf{T}, X) = K'(\mathbf{T}, K_n(\mathbf{T}, X))$ for $n < \omega$. Then $K(\mathbf{T}, X) = \bigcup_{n<\omega} K_n(\mathbf{T}, X)$ is the convex hull of X in $\mathbf{T}$.

LEMMA 3. *There is a sequence of tournaments* $\mathbf{T}_n = \langle T_n, \overset{n}{\rightarrow} \rangle$ *and pairs* $\{a_n, b_n\} \in [T_n]^2$ for $n < \omega$ so that:

(i) $\mathbf{T}_n$ *is simple,*

(ii) $\mathbf{T}_m$ *is a subtournament of* $\mathbf{T}_n$ *for* $m < n$,

(iii) $K_{n+1}(\mathbf{T}_n, \{a_0, b_0\}) = T_n$,

(iv) *there is* $c_n \in T_n - K_n(\mathbf{T}_n, \{a_0, b_0\})$.

Proof. Let $\mathbf{T}_0 = \langle T_0, \overset{\circ}{\rightarrow} \rangle$, where $T_0 = \{a_0, b_0, c_0\}$ and $a_0 \overset{\circ}{\rightarrow} b_0 \overset{\circ}{\rightarrow} c_0 \overset{\circ}{\rightarrow} a_0$. Clearly (i) (i)–(iv) hold for $n = 0$. Assume that $\mathbf{T}_n$ has been defined so that (i)–(iv) hold. Put $T_{n+1} = T_n \cup \{a_n, b_n\}$, where a_n, b_n are distinct elements not in T_n. Define an extension $\overset{n+1}{\rightarrow}$ of $\overset{n}{\rightarrow}$ to T_{n+1} by putting

$$a_{n+1} \overset{n+1}{\longrightarrow} b_{n+1},\ a_{n+1} \overset{n+1}{\longrightarrow} c_n \overset{n+1}{\longrightarrow} b_{n+1}$$

(o)

$$\{a_{n+1}\} \overset{n+1}{\longleftarrow} T_n - \{c_n\} \overset{n+1}{\longleftarrow} \{b_{n+1}\}.$$

We will show that $\mathbf{T}_{n+1} = \langle T_{n+1}, \overset{n+1}{\rightarrow} \rangle$ satisfies (i)–(iv) with $n+1$ in place of n.

The conditions (o) relate $\mathbf{T}_{n+1}$ to $\mathbf{T}_n$ in exactly the same way that the conditions (i) and (ii) in the proof of Theorem 1 relates $\mathbf{T}'$ to $\mathbf{T}$ if we put $B = \{c_n\}$, Considering that $C(\mathbf{T}_n) = \phi$ by (i), the set $B = \{c_n\}$ does establish (vacuously) the property B for $C(\mathbf{T}_n)$. Consequently, as in the proof of Theorem 1, $\mathbf{T}_{n+1}$ is simple. (ii) holds trivially with $n+1$ in place of n since $\mathbf{T}_{n+1}$ is an extension of $\mathbf{T}_n$. Now assume that $X \subset T_n - \{c_n\}$ and $u \in K'(\mathbf{T}_{n+1}, X) - X$. Then there are $v, w \in X$ such that $v \overset{n+1}{\rightarrow} u \overset{n+1}{\rightarrow} w$ and hence $u \neq a_{n+1}$, $u \neq b_{n+1}$ by (o). Using this remark and (iv) it follows by induction on i that

$$K_i(\mathbf{T}_{n+1}, \{a_0, b_0\}) = K_i(\mathbf{T}_n, \{a_0, b_0\}) \quad \text{for} \quad i \leqslant n+1.$$

Since $a_{n+1}, b_{n+1} \in K'(\mathbf{T}_{n+1}, T_n)$ by (o), and since $K_{n+1}(\mathbf{T}_{n+1}, \{a_0, b_0\}) = K_{n+1}(\mathbf{T}_n, \{a_0, b_0\}) = T_n$, it follows that (iii) holds with $n+1$ in place of n. Now put $c_{n+1} = a_{n+1}$. Then $c_{n+1} \notin T_n = K_{n+1}(\mathbf{T}_n, \{a_0, b_0\}) = K_{n+1}(\mathbf{T}_{n+1}, \{a_0, b_0\})$ and so (iv) holds for $\mathbf{T}_{n+1}$ as well. This proves that the $\mathbf{T}_n$ defined for $n < \omega$ satisfy the requirements of the lemma.

In order to prove Theorem 5 it is sufficient to show that a reduced product of simple tournaments is not neccessarily simple. Let $\mathbf{T}_n (n<\omega)$ be tournaments satisfying Lemma 3 and let U be a nontrivial ultrafilter on ω. Consider the tournament $\mathbf{T}=\prod_{n<\omega} \mathbf{T}_n/U=\langle T, \rightarrow \rangle$. Let $a=(a_0,\ldots, a_0,\ldots)$, $b=(b_0,\ldots, b_0,\ldots)$ and $c=(c_0,\ldots, c_n,\ldots)\in T$. Then $a\neq b$. By Loś' theorem [q], for $t<\omega$ and any $x=(x_0,\ldots, x_n,\ldots)\in T$, $x\in K_t(\mathbf{T}, \{a, b\})$ iff $\{i\in\omega : x_i\in K_t(\mathbf{T}_i, \{a_0, b_0\})\}\in U$. By (iv) of Lemma 3, $c_n\notin K_n(\mathbf{T}_n, \{a_0, b_0\}) \supset K_t(\mathbf{T}_n, \{a_0, b_0\})$ for $n\geqslant t$ and hence $c\notin K_t(\mathbf{T}, \{a, b\})$ for any $t<\omega$. This shows that $K(\mathbf{T}, \{a, b\})$ is a non-trivial convex set in $\mathbf{T}$ and so $\mathbf{T}$ is not simple.

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