Diagonals of Nonnegative Matrices

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Let $(a_1, \ldots, a_n)$, $(r_1, \ldots, r_n)$ and $(c_1, \ldots, c_n)$ be real n-tuples, $n \geqslant 3$, satisfying

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} c_i \quad \text{and} \quad 0 \leqslant a_i \leqslant \min(r_i, c_i), \qquad i = 1, \ldots, n.$$

It is shown that a necessary and sufficient condition for the existence of a nonnegative matrix with main diagonal $(a_1, \ldots, a_n)$, with row sums $r_1, \ldots, r_n$ and column sums $c_1, \ldots, c_n$, is that

$$\sum_{i=1}^{n} (r_i - a_i) \geqslant \max_t (r_t + c_t - 2a_t).$$

Equality can hold if and only if all the off-diagonal positive entries of the matrix are restricted to the kth row and the kth column, for some k, $1 \leqslant k \leqslant n$.

If $A = (a_{ij})$ is an n-square matrix and σ is a permutation on n objects, then the n-tuple $(a_{1\sigma(1)}, a_{2\sigma(2)}, \ldots, a_{n\sigma(n)})$ is called a *diagonal* of A. The sums

$$r_i = \sum_{j=1}^{n} a_{ij}, \qquad i = 1, \ldots, n,$$

are the *row sums*, and

$$c_j = \sum_{i=1}^{n} a_{ij}, \qquad j = 1, \ldots, n,$$

are the *column sums* of A. The matrix $A = (a_{ij})$ is said to be *nonnegative* if $a_{ij} \geqslant 0$ for all i and j.

In this paper we obtain necessary and sufficient conditions for an n-tuple to be a diagonal of a nonnegative matrix with prescribed row sums and column sums. Clearly we may assume without loss of generality that the diagonal in question is the main diagonal.

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THEOREM *Let* $(a_1, \ldots, a_n)$, $(r_1, \ldots, r_n)$ *and* $(c_1, \ldots, c_n)$ *be n-tuples,* $n \geqslant 3$, *satisfying*

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} c_i \quad \text{and} \quad 0 \leqslant a_i \leqslant \min(r_i, c_i), \qquad i = 1, \ldots, n.$$

Then a necessary and sufficient condition for the existence of a nonnegative matrix with main diagonal $(a_1, \ldots, a_n)$, *with row sums* $r_1, \ldots, r_n$ *and column sums* $c_1, \ldots, c_n$, *is that*

$$\sum_{i=1}^{n} (r_i - a_i) \geqslant \max_t (r_t + c_t - 2a_t). \tag{1}$$

Equality can hold in (1) *if and only if the nonzero entries of the matrix are restricted to the main diagonal, the* kth *row and the* kth *column, where* k *is defined by*

$$r_k + c_k - 2a_k = \max_t (r_t + c_t - 2a_t).$$

Proof Let $A = (a_{ij})$ be a nonnegative matrix with row sums $r_1, r_2, \ldots, r_n$ and column sums $c_1, \ldots, c_n$. Clearly the sum of all off-diagonal entries of A cannot be exceeded by the sum of off-diagonal entries in row t and column t; that is,

$$\sum_{i=1}^{n} (r_i - a_{ii}) \geqslant (r_t - a_{tt}) + (c_t - a_{tt}),$$

$t = 1, \ldots, n$. In other words,

$$\sum_{i=1}^{n} (r_i - a_{ii}) \geqslant \max_t (r_t + c_t - 2a_{tt}).$$

We prove the sufficiency by induction on n.

We can assume without loss of generality that

$$r_1 + c_1 - 2a_1 \geqslant r_2 + c_2 - 2a_2 \geqslant \cdots \geqslant r_n + c_n - 2a_n. \tag{2}$$

With this assumption the condition (1) becomes

$$\sum_{i=1}^{n} (r_i - a_i) \geqslant r_1 + c_1 - 2a_1. \tag{3}$$

For $n = 3$, this condition asserts that

$$r_2 - a_2 + r_3 - a_3 \geqslant c_1 - a_1, \tag{4}$$

or equivalently

$$c_2 - a_2 + c_3 - a_3 \geqslant r_1 - a_1.$$

Suppose first that $c_1 - a_1 \geqslant r_3 - a_3$ and $r_1 - a_1 \geqslant c_2 - a_2$. Then the matrix

$$\begin{bmatrix} a_1 & c_2 - a_2 & r_1 - a_1 - (c_2 - a_2) \\ c_1 - a_1 - (r_3 - a_3) & a_2 & r_2 - a_2 + r_3 - a_3 - (c_1 - a_1) \\ r_3 - a_3 & 0 & a_3 \end{bmatrix}$$

is nonnegative and has the prescribed diagonal, row sums and column sums. If either $c_1 - a_1 \leqslant r_3 - a_3$ or $r_1 - a_1 \leqslant c_2 - a_2$ (and therefore $r_1 - a_1 \geqslant c_3 - a_3$ or $c_1 - a_1 \geqslant r_2 - a_2$, by (2)), then the matrix

$$\begin{bmatrix} a_1 & r_1 - a_1 - (c_3 - a_3) + x & c_3 - a_3 - x \\ r_2 - a_2 - x & a_2 & x \\ c_1 - a_1 - (r_2 - a_2) + x & r_3 - a_3 + r_2 - a_2 - (c_1 - a_1) - x & a_3 \end{bmatrix},$$

where $x = \min(c_3 - a_3, r_2 - a_2)$, is nonnegative and satisfies all the prescribed conditions.

Assume now that the theorem holds for all nonnegative $(n - 1) \times (n - 1)$ matrices. Let

$$\delta = \sum_{i=1}^{n} (r_i - a_i) - r_1 - c_1 + 2a_1 \geqslant 0$$

and set

$$x_1 = \begin{cases} 0, & \text{if } r_n + c_n - 2a_n \leqslant \delta, \\ \min(r_n + c_n - 2a_n - \delta,\ r_1 - a_1), & \text{if } r_n + c_n - 2a_n \geqslant \delta. \end{cases} \tag{5}$$

and

$$y_1 = \max(r_n + c_n - 2a_n - \delta - x_1, 0).$$

By (5), $0 \leqslant x_1 \leqslant r_1 - a_1$. It is also easy to see that $0 \leqslant y_1 \leqslant c_1 - a_1$. For if $y_1 = r_n + c_n - 2a_n - \delta - x_1 > 0$, then $x_1 = r_1 - a_1$, and

$$\begin{aligned} c_1 - a_1 - y_1 &= c_1 - a_1 - \Big\{ r_n + c_n - 2a_n - \sum_{i=1}^{n} (r_i - a_i) \\ &\qquad + r_1 + c_1 - 2a_1 - (r_1 - a_1) \Big\} \\ &= \sum_{i=1}^{n} (r_i - a_i) - (r_n + c_n - 2a_n) \\ &\geqslant 0. \end{aligned}$$

Now, let $x_2, \ldots, x_{n-1}$ and $y_2, \ldots, y_{n-1}$ be any numbers satisfying

$$0 \leqslant x_i \leqslant r_i - a_i, \qquad 0 \leqslant y_i \leqslant c_i - a_i, \qquad i = 2, \ldots, n - 1,$$

and

$$\sum_{i=1}^{n-1} x_i = c_n - a_n, \qquad \sum_{i=1}^{n-1} y_i = r_n - a_n. \tag{7}$$

We have to show that such numbers exist, i.e., that

$$x_1 + \sum_{i=2}^{n-1} (r_i - a_i) \geqslant c_n - a_n \tag{8}$$

and

$$y_1 + \sum_{i=2}^{n-1} (c_i - a_i) \geqslant r_n - a_n. \tag{9}$$

If $x_1 = 0$ and $\delta \geqslant r_n + c_n - 2a_n$, then

$$\begin{aligned} x_1 + \sum_{i=2}^{n-1} (r_i - a_i) - (c_n - a_n) &= \delta - (r_n - a_n) + (c_1 - a_1) - (c_n - a_n) \\ &\geqslant c_1 - a_1 \\ &\geqslant 0. \end{aligned}$$

If $x_1 = r_n + c_n - 2a_n - \delta \geqslant 0$, then

$$\begin{aligned} & x_1 + \sum_{i=2}^{n-1} (r_i - a_i) - (c_n - a_n) \\ &\quad = r_n - a_n - \sum_{i=1}^{n} (r_i - a_i) + (r_1 + c_1 - 2a_1) + \sum_{i=2}^{n-1} (r_i - a_i) \\ &\quad = c_1 - a_1. \end{aligned}$$

Finally, if $x_1 = r_1 - a_1$, then (8) holds by virtue of (1). Inequality (9) is proved similarly. If $r_n + c_n - 2a_n \leqslant \delta$, the proof is a virtual repetition of the first case above.

If $0 \leqslant r_n + c_n - 2a_n - \delta \leqslant r_1 - a_1$, i.e.,

$$r_n + c_n - 2a_n - \sum_{i=2}^{n} (c_i - a_i) + r_1 - a_1 \leqslant r_1 - a_1,$$

then

$$\sum_{i=2}^{n} (c_i - a_i) \geqslant r_n + c_n - 2a_n. \tag{10}$$

Thus, in this case,

$$\begin{aligned} y_1 + \sum_{i=2}^{n-1} (c_i - a_i) &= 0 + \sum_{i=2}^{n} (c_i - a_i) - (c_n - a_n) \\ &\geqslant r_n - a_n, \end{aligned}$$

by (10). Finally, if $r_n + c_n - 2a_n - \delta \geqslant r_1 - a_1$, then $x_1 = r_1 - a_1$, and

$$\begin{aligned} y_1 &= r_n + c_n - 2a_n - \delta - x_1 \\ &= r_n + c_n - 2a_n - \delta - (r_1 - a_1) \\ &= r_n - a_n - \sum_{i=2}^{n-1} (c_i - a_i) \\ &\geqslant 0. \end{aligned}$$

Thus

$$y_1 + \sum_{i=2}^{n-1} (c_i - a_i) = r_n - a_n.$$

Next we use the induction hypothesis to show that there exists an $(n-1)$-square nonnegative matrix $B = (b_{ij})$ with main diagonal $(a_1, \ldots, a_{n-1})$, row sums $r_i - x_i$, $i = 1, \ldots, n-1$, and column sums $c_i - y_i$, $i = 1, \ldots, n-1$.

We first show that

$$(r_1 - x_1) + (c_1 - y_1) - 2a_1 = \max_i((r_i - x_i) + (c_i - y_i) - 2a_i). \quad (11)$$

In fact we prove that

$$(r_1 - x) + (c_1 - y_1) - 2a_1 \geqslant r_2 + c_2 - 2a_2, \quad (12)$$

and therefore

$$(r_1 - x_1) + (c_1 - y_1) - 2a_1 \geqslant r_i + c_i - 2a_i, \qquad i = 2, \ldots, n - 1.$$

It suffices to prove inequality (12) in case $r_n + c_n - 2a_n - \delta \geqslant 0$. Then $x_1 + y_1 = r_n + c_n - 2a_n - \delta$ and therefore

$$\begin{aligned}
&(r_1 - x_1) + (c_1 - y_1) - 2a_1 - (r_2 + c_2 - 2a_2)\\
&\quad = r_1 + c_1 - 2a_1 - r_2 - c_2 + 2a_2 - (r_n + c_n - 2a_n - \delta)\\
&\quad = r_1 + c_1 - 2a_1 - r_2 - c_2 + 2a_2 - r_n - c_n + 2a_n\\
&\qquad\qquad + \sum_{r=2}^{n} (r_i - a_i) - c_1 + a_1\\
&\quad = \sum_{i=1}^{n} (r_i - a_i) - (r_2 + c_2 - 2a_2 + r_n + c_n - 2a_n)\\
&\quad = \tfrac{1}{2} \sum_{r=1}^{n} (r_i + c_i - 2a_i) - (r_2 + c_2 - 2a_2 + r_n + c_n - 2a_n)\\
&\quad = \frac{1}{2}\Big\{(r_1 + c_1 - 2a_1) - (r_2 + c_2 - 2a_2)\\
&\qquad\qquad + \sum_{r=3}^{n-1} (r_i + c_i - 2a_i) - (r_n + c_n - 2a_n)\Big\}\\
&\quad \geqslant 0,
\end{aligned}$$

by (2). We now show that

$$\sum_{i=1}^{n-1} ((r_i - x_i) - a_i) \geqslant (r_1 - x_1) + (c_1 - y_1) - 2a_1$$

and thus, by the induction hypothesis, that the matrix B exists. If

$$x_1 + y_1 = r_n + c_n - 2a_n - \delta \geqslant 0,$$

then

$$\begin{aligned}
\sum_{i=1}^{n-1} ((r_i - x_i) - a_i) &= \sum_{i=1}^{n} (r_i - a_i) - (r_n - a_n) - \sum_{i=1}^{n-1} x_i\\
&= r_1 + c_1 - 2a_1 + \delta - (r_n - a_n) - (c_n - a_n)\\
&= r_1 + c_1 - 2a_1 - x_1 - y_1.
\end{aligned}$$

If $x_1 + y_1 = 0$, that is $r_n + c_n - 2a_n - \delta \leqslant 0$, then

$$\begin{aligned}
\sum_{i=1}^{n-1} ((r_i - x_i) - a_i) &= r_1 + c_1 - 2a_1 + \delta - (r_n - a_n) - (c_n - a_n)\\
&\geqslant r_1 + c_1 - 2a_1\\
&= (r_1 - x_1) + (c_1 - y_1) - 2a_1.
\end{aligned}$$

Thus there exists an $(n-1)$-square matrix $B = (b_{ij})$ with

$$b_{ii} = a_i, \qquad i = 1, \ldots, n-1,$$

and with rows sums $r_i - x_i$, $i = 1, \ldots, n-1$, and column sums $c_i - y_i$, $i = 1, \ldots, n-1$. It follows that the $n \times n$ matrix

$$A = (a_{ij}) = \left[\begin{array}{c|c} B & \begin{matrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \end{matrix} \\ \hline y_1\ y_2 \cdots y_{n-1} & a_n \end{array}\right]$$

where

$$\begin{aligned} a_{ij} &= b_{ij}, & i,j &= 1, \ldots, n-1, \\ a_{in} &= x_i, & i &= 1, \ldots, n-1, \\ a_{nj} &= y_j, & j &= 1, \ldots, n-1, \end{aligned}$$

and

$$a_{nn} = a_n,$$

has the required properties, i.e., main diagonal $(a_1, a_2, \ldots, a_n)$, row sums $r_1, r_2, \ldots, r_n$ and column sums $c_1, c_2, \ldots, c_n$.

It remains to discuss the case of equality. With assumption (2), we have to show that if

$$A = \begin{bmatrix} a_1 & c_2 - a_2 & c_3 - a_3 & \cdots & c_n - a_n \\ r_2 - a_2 & a_2 & 0 & 0 \cdots & 0 \\ r_3 - a_3 & 0 & a_3 & 0 \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & 0 \\ r_n - a_n & 0 & 0 & 0 \cdots 0 & a_n \end{bmatrix} \tag{13}$$

and A is nonnegative, then (3) is an equality, and conversely if equality holds in (3) then the only nonnegative matrix with main diagonal $(a_1, \ldots, a_n)$, row sums $r_1, \ldots, r_n$ and column sums $c_1, \ldots, c_n$ is the matrix in (13). These conclusions are quite obvious, since

$$\sum_{i=1}^{n} (r_i - a_i) = (r_1 - a_1) + (c_1 - a_1)$$

asserts that the sum of all off-diagonal entries is equal to the sum of the off-diagonal entries in the first row and those in the first column.

By setting $r_1 = \cdots = r_n = c_1 = \cdots = c_n = 1$ we obtain

COROLLARY *An n-tuple* $(a_1, \ldots, a_n)$, *where* $0 \leqslant a_i \leqslant 1$, $i = 1, \ldots, n$, *is a diagonal of a doubly stochastic matrix if and only if*

$$\sum_{i=1}^{n} a_i \leqslant n - 2 + 2 \min_i a_i.$$

The result in the corollary is due to A. Horn [1].

Reference

[1] Alfred Horn, Doubly stochastic matrices, *Amer. J. Math.* **76** (1954), 621–630.

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