

Dissection Graphs of Planar Point Sets

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1. Introduction

Let S be a set of n points in general position (no three collinear) in the plane. For any two points $p, q \in S$, the directed line \overrightarrow{pq} has a certain number, $N(\overrightarrow{pq})$, of points of S on its positive side, that is, the open half plane to the right of \overrightarrow{pq} . We are interested in the directed k -graphs, G_k , of S whose edges are the segments \overrightarrow{pq} with $N(\overrightarrow{pq}) = k$ ($k = 0, 1, \dots, n-2$). Since clearly $G_{n-k-2} = -G_k$, that is, the k -graph with all orientations reversed, it suffices to consider the cases $k \leq (n-2)/2$. If n is even, then the bigraph $B = G_{(n-2)/2}$ is of special interest since each edge occurs in both orientations and it can therefore be considered as an undirected graph.

This case has been studied in several previous papers.

In Section 2, we discuss some general properties of the graphs G_k . In Section 3, we answer the relatively easy question concerning the upper and lower bounds on the number of vertices of G_k and the lower bound on the number of edges of G_k . In Section 4, we tackle the far more difficult problem of the upper bound $e_{n,k}$ on the number of edges in G_k . We obtain upper bounds of the form $cn\sqrt{k}$ and lower bounds of the form $cn \log n$ for $e_{n, rn-1}$, where r is a rational number, $0 < r < 1$, and rn is an integer. Finally in Section 5 we discuss new problems and generalizations.

2. Some structural properties of k -graphs

We can construct the graph G_k as follows. Let l be any oriented line containing no points of S and having $k+1$ points of S on its positive side. Translate l to its left until it meets a point p_1 of S . Call this line $l(0)$. Now rotate $l(0)$ counterclockwise by θ about p_1 into line $l(\theta)$ until it meets a second point p_2 of S at $l(\theta_1) = l_1$. Now rotate counterclockwise about p_2 until $l(\theta)$ meets a point p_3 of S at $l(\theta_2) = l_2$, etc. We thus get a sequence of (not necessarily

† The last author was supported in part by NSF Grant #GP-28696.

distinct) points p_1, p_2, \dots, p_N of S with $p_{N+1} = p_1, p_{N+2} = p_2$ and a sequence of directed lines $l_1, l_2, \dots, l_N, l_{N+1}$ with $l_{N+2} = l_1$.

Theorem 2.1. *The graph G_k consists of those vertices p_i and those edges $\overrightarrow{p_i p_{i+1}}$ for which the orientation $\overrightarrow{p_i p_{i+1}}$ is opposite to that of the line l_i .*

Proof. Clearly the number $N(\theta)$ of points on the positive side of $l(\theta)$ remains constant in any interval which does not contain one of the angles θ_i .

If $\overrightarrow{p_i p_{i+1}}$ is in the direction of l_i then for small $\varepsilon > 0$ we have $N(\theta_i - \varepsilon) = N(\theta_i) = N(\theta_i + \varepsilon)$, since the points p_i, p_{i+1} are either on or to the left of $l(\theta)$

for $\theta_i - \varepsilon \leq \theta \leq \theta_i + \varepsilon$. If $\overrightarrow{p_i p_{i+1}}$ is in the direction opposite to l_i then $N(\theta_i) = N(\theta_i - \varepsilon) - 1 = N(\theta_i + \varepsilon) - 1$ for small $\varepsilon > 0$, since one of p_i, p_{i+1} is to the right of $l(\theta)$ in $\theta_i - \varepsilon \leq \theta \leq \theta_i + \varepsilon$ except for $\theta = \theta_i$ when both are on l_i .

Thus we have $N(\theta) = \text{constant} = k + 1$ for all $\theta \neq \theta_i$ and $N(\theta_i) = k + 1$ or k according as $\overrightarrow{p_i p_{i+1}}$ is in the direction of l_i or not.

Finally we can see that all edges of G_k are included in the lines $l(\theta)$ since any line l' not included in $l(\theta)$ is like-directed to a line $l(\theta')$ so that $N(l') - N(\theta')$

$\neq 0$ since $l(\theta')$ contains a point of S . If $\theta' = \theta_i$ and $\overrightarrow{p_i p_{i+1}}$ is in the direction opposite to l_i , this proves that $N(l') \neq k$. Otherwise $N(\theta') = k + 1$ and l' passes through two points on the right of $l(\theta')$ so that $N(l') \leq k - 1$.

Theorem 2.2. *Given a line L containing no points of the set S so that L divides S into two sets S_1 and S_2 with $|S_1| = m \leq n - m = |S_2|$. Then L intersects $m_0 = \min\{m, k + 1\}$ edges of G_k going from S_1 to S_2 and m_0 edges of G_k going from S_2 to S_1 .*

Proof. Since a small perturbation of L does not affect the hypotheses we may assume that L is not parallel to any line pq joining two points $p, q \in S$. We may therefore pick the point $p_1 \in S_2$ and the directed line $l(0)$ of the family defined in Theorem 2.1 through p_1 ; S_1 lies on the negative side of $l(0)$. As θ increases from 0 to π , the number $N(\theta, S_1)$ of points of S_1 on the positive side of $l(\theta)$ increases from 0 to $m_0 = \min\{k + 1, m\}$. This increase is monotonic if we ignore the values $\theta = \theta_i$.

Now the number $N(\theta, S_1)$ is clearly constant in any interval which does not contain a θ_i . If both points p_i, p_{i+1} of $l(\theta_i)$ are in S_2 then $N(\theta_i - \varepsilon, S_1) = N(\theta_i, S_1) = N(\theta_i + \varepsilon, S_1)$ for small $\varepsilon > 0$. Similarly, if $\overrightarrow{p_i p_{i+1}}$ is in the direction of $l(\theta_i)$ then $N(\theta_i - \varepsilon, S_1) = N(\theta_i, S_1) = N(\theta_i + \varepsilon, S_1)$ for small $\varepsilon > 0$.

If p_i, p_{i+1} are both in S_1 and $\overrightarrow{p_i p_{i+1}}$ is opposite directed to $l(\theta_i)$ then the point p_{i+1} is to the right of $l(\theta_i - \varepsilon)$ and p_i is to the left of $l(\theta_i - \varepsilon)$ for small $\varepsilon > 0$, while p_{i+1} is to the left of $l(\theta_i + \varepsilon)$ and p_i is to the right of $l(\theta_i + \varepsilon)$ for small

$\varepsilon > 0$. Thus we have $N(\theta_i - \varepsilon, S_1) = N(\theta_i + \varepsilon, S_1)$ in this case. Finally if p_i, p_{i+1} are in opposite sides of L and $\overrightarrow{p_i p_{i+1}}$ is opposite directed to $l(\theta_i)$ for $0 < \theta_i < \pi$ then $p_i \in S_1, p_{i+1} \in S_2$ and $N(\theta_i + \varepsilon, S_1) = N(\theta_i, S_1) + 1 = N(\theta_i - \varepsilon, S_1) + 1$ for small $\varepsilon > 0$, since the point p_i is to the right of $l(\theta_i + \varepsilon)$ but on $l(\theta_i - \varepsilon)$. We have thus shown that $N(\theta, S_1)$ increases by one in the interval $0 < \theta < \pi$ whenever L is intersected by a segment $\overrightarrow{p_{i+1} p_i}$ of G_k going from S_2 to S_1 . Since $N(\theta, S_1)$ increases to m_0 there must be m_0 such segments of G_k .

As θ increases from π to 2π , the number $N(\theta, S_1)$ decreases from m_0 to 0 and in a manner entirely analogous to that used above we see that $N(\theta, S_1)$ decreases by one whenever L is intersected by a segment $\overrightarrow{p_{i+1} p_i}$ of G_k going from S_1 to S_2 . Thus there must be m_0 such segments of G_k .

Theorem 2.3. *If n is odd then $G_{(n-3)/2}$ is connected.*

Proof. In this case each of the segments $\overrightarrow{p_{i+1} p_i}$ in Theorem 2.1 is part of the graph. This is certainly the case if $\overrightarrow{p_{i+1} p_i}$ has the orientation of l_i since in that case there are $(n-3)/2$ points to the right of l_i , but it is also the case if $\overrightarrow{p_{i+1} p_i}$ has the opposite orientation of l_i since in that case there are $(n-1)/2$ points to the right of l_i and hence $(n-3)/2$ points to the left of l_i . Thus the construction at the beginning of this section yields a closed oriented Euler path through the graph $G_{(n-3)/2}$.

Except for the trivial case of G_0 , which is the positively oriented boundary of the convex hull of S , this is the only case in which G_k must be connected.

Theorem 2.4. *For any n and any k with $0 < k \leq (n-2)/2, k \neq (n-3)/2$ there exist sets S with n elements so that G_k is not a connected graph.*

Proof. We have either $n = 2k + 2$ or $n \geq 2k + 4$. In the first case we let S consist of the vertices of a convex n -gon and G_k consists of the diagonals joining diametrically opposite vertices.

In the second case let S consist of the vertices of a regular $(2k+4)$ -gon K with the remaining $n - 2k - 4$ points situated closer to the center of K than any of the non-diametric diagonals of K . Then $G_k(S) = G_k(K)$ consists of two closed $(k+2)$ -gons obtained by joining each vertex of K to the one following it by $k+1$ steps in the counter-clockwise direction. All lines which pass through a point of $S \setminus K$ contain at least $k+1$ points of K on each side. Thus no point of $S \setminus K$ is a vertex of $G_k(S)$.

Theorem 2.5. *If we order the oriented lines of the edges of G_k at a vertex, v , in counterclockwise order, then between any two lines containing outgoing*

edges there is a line containing an incoming edge, and between any two lines containing an incoming edge there is a line containing an outgoing edge.

Proof. Let l_1 and l_2 be successive oriented lines through v containing outgoing edges of G_k . Then as l rotates from l_1 to l_2 we have $k+1$ points of S on the positive side of l for l near to l_1 and k points of S on the positive side of l for l near to l_2 . Since the number of points of S on the positive side of l increases by one each time l passes through a point p of S in the oriented angle $\sphericalangle(l_1, l_2)$ and decreases by one each time l passes through a point p of S in the opposite vertical angle $\sphericalangle(-l_2, -l_1)$, it follows that at some stage of the rotation the number of points on the positive side of l decreases from $k+1$ to k so that l contains an incoming edge \overrightarrow{pv} of G_k . The argument for successive lines containing incoming edges is entirely analogous.

Corollary 2.6. *At each vertex of G_k , the number of incoming edges is equal to the number of outgoing edges. Thus each component of G_k has an oriented Euler circuit.*

In the (unoriented) bigraph $(k = (n-2)/2)$ the number of edges at each vertex is odd.

Theorem 2.7. *Let G' be a component of G_k and let S' be the set of vertices of G' . Then there exists a k' , $k' \leq k$, so that $G' = G_{k'}(S')$.*

Proof. The directed lines $l'(\theta)$ which contain the edges of G' form a subset of the lines $l(\theta)$ constructed at the beginning of this section. Let $N'(\theta)$ denote the number of points of S' on the positive side of $l(\theta)$. We first show that $N'(\theta)$ is constant, $k'+1 \geq 1$, for all values of θ which do not correspond to edges of G' where $N'(\theta) = k'$. Clearly $N'(\theta)$ can change only when $l(\theta)$ contains two points of S and if at least one of these points is in S' . Now let

$p_i, p_{i+1} \in l(\theta_i) \cap S$; if p_i, p_{i+1} are both in S' then $\overrightarrow{p_{i+1}p_i}$ is not an edge of G_k

and hence $l(\theta_i)$ has the direction $\overrightarrow{p_i p_{i+1}}$. Thus neither p_i nor p_{i+1} is on the positive side of $l(\theta)$ for $\theta_i - \varepsilon < \theta < \theta_i + \varepsilon$ for sufficiently small $\varepsilon > 0$ and $N'(\theta)$ is constant in this range. If both p_i and p_{i+1} are in S' but $l(\theta_i)$ has direction

$\overrightarrow{p_i p_{i+1}}$ then the same argument applies to keep $N'(\theta)$ constant. Finally, if

$\overrightarrow{p_{i+1} p_i}$ is an edge of G' then p_{i+1} is on the positive side of $l(\theta_i - \varepsilon)$ and p_i is on the positive side of $l(\theta_i + \varepsilon)$ for small $\varepsilon > 0$. Thus $N'(\theta_i - \varepsilon) = N'(\theta_i + \varepsilon) = N'(\theta_i) + 1 = k' + 1$.

We have thus shown that $G' \subset G_{k'}(S')$. The argument that $G' \supset G_{k'}(S')$ is exactly as in the proof of Theorem 2.1.

Corollary 2.8. *Theorem 2.2 applies to each component G' of G_k . That is, if a directed line L which contains no vertices of G' divides the vertices of G'*

into sets of m' and $n' - m'$ vertices with $m' \leq n' - m'$, then L intersects $\min\{m', k' + 1\}$ edges of G' crossing L from right to left and the same number crossing L from left to right.

Corollary 2.9. Let $G', G'', \dots, G^{(r)}$ be the components of G_k and set $G^{(i)} = G_{k_i}(S^{(i)})$. Then $k = k_1 + \dots + k_r + r - 1$ and each directed line containing an edge of $G^{(i)}$ is crossed by k_j edges of $G^{(j)}$ from left to right and k_j edges of $G^{(j)}$ from right to left for each $j \neq i$.

Each union

$$G^{(i_1)} \cup G^{(i_2)} \cup \dots \cup G^{(i_s)} = G_t(S^{(i_1)} \cup \dots \cup S^{(i_s)}),$$

where $t = k_{i_1} + \dots + k_{i_s} + s - 1$.

Proof. As shown in the proof of Theorem 2.7, an edge of $G^{(i)}$ is contained in a line whose positive side contains k_i points of $S^{(i)}$ and $k_j + 1$ points of $S^{(j)}$ for each $j \neq i$. Thus $k = k_1 + \dots + k_r + r - 1$.

3. On the number of vertices of G_k .

Lemma 3.1. A point p of S is a vertex of $G_k(S)$, $k \leq (n-2)/2$, if and only if there exists a directed line through p whose positive side contains no more than k points of S .

Proof. The necessity is obvious since any line of an edge of G_k with vertex p has that property. The sufficiency follows from the fact that, as a directed line l is rotated around p , the number of points on its positive side ranges through all values from the minimum m to the maximum M . We have $m \leq k$ and $M = n - m - 1 \geq n - k - 1 \geq k + 1$. Thus there must be an instant at which the number of points on the positive side of l changes from k to $k + 1$. This can only happen when l contains an edge \overrightarrow{pq} of G_k .

Corollary 3.2. All points on the convex hull of S are vertices of every $G_k(S)$. In particular, if all points of S are on its convex hull then $G_k(S)$ has n vertices.

Theorem 3.3. Every point of S is a vertex of $G_{\lfloor (n-2)/2 \rfloor}$. If $k < \lfloor (n-2)/2 \rfloor$ then G_k has at least $2k + 3$ vertices, and for every v with $2k + 3 \leq v \leq n$ there exists an S so that $G_k(S)$ has exactly v vertices.

Proof. The first part of the theorem is an immediate consequence of Lemma 3.1. If $k < \lfloor (n-2)/2 \rfloor$ then choose a point p of S which lies on its convex hull and a line l through p which contains at least $\lfloor (n-1)/2 \rfloor$ points of S on each side and is not parallel to any line joining two points of S . Then there exist two oppositely directed lines l_1, l_2 on either side of l parallel to l through points of S so that their positive sides, which exclude l , contain exactly k points of S each. According to Lemma 3.1, each of the $2k + 2$ points to the right of or on one l_i ($i = 1, 2$) and the point p is a vertex of G_k so that G_k has at least $2k + 3$ vertices.

Finally, if $2k+3 \leq v \leq n$, let S consist of the vertices of a regular v -gon inscribed in the unit circle K and of $n-v$ points located so near to the center of K that a line through one of them cuts K in arcs exceeding $2\pi(k+1)/v$. Thus by Lemma 3.1 and Corollary 3.2, the vertices of G_k are exactly those of the regular v -gon.

4. On the number of edges of G_k

The lower bound on the number of edges is easily settled by the result of the preceding section. According to Corollary 2.6, each vertex of G_k is incident to at least two directed edges, so that the number of its edges can be no less than the number of its vertices.

Theorem 4.1. *The graph $G_{\lfloor (n-2)/2 \rfloor}$ has at least n directed edges. In particular, if n is even, the bigraph $B = G_{(n-2)/2}$ has at least $n/2$ undirected edges. If $k < \lfloor (n-2)/2 \rfloor$ then G_k has at least $2k+3$ edges. For every number e with $2k+3 \leq e \leq n$ there is a set S so that $G_k(S)$ has exactly e edges.*

Proof. By Theorem 2.2, every vertex of the convex hull is incident to exactly two edges of G_k . Thus the theorem follows from the constructions made in the proof of Theorem 3.3.

Using Lemma 3.1, we can get an upper bound on the valence of a vertex of G_k .

Theorem 4.2. *The valence of the vertices of G_k does not exceed $2k+2$.*

Proof. Let p be a vertex of G_k , and assume $k \leq (n-3)/2$. The edges of G_k through p clearly lie on the directed lines joining p to other points of S whose positive side contains no more than $(n-3)/2$ points of S . There are no more than $n-1$ such lines and if n is even there are no more than $n-2$ such lines.

By Lemma 3.1, the point p is also a vertex of $G_{k+1}, G_{k+2}, \dots, G_{\lfloor (n-3)/2 \rfloor}$ all of which are edge-disjoint from each other and from G_k . Since the valence v_p of p is not less than 2 in any of these graphs, we get for odd n

$$v_p \leq n-1-2\left(\frac{n-3}{2}-k\right) = 2k+2$$

and for even n

$$v_p \leq n-2-2\left(\frac{n-4}{2}-k\right) = 2k+2.$$

Since the vertices of G_k include those of G_0, G_1, \dots, G_{k-1} , it follows that all these vertices have valences less than $2k+2$, including vertices of valence 2 which are points on the boundary of the convex hull of S . We could use this to get a poor upper bound for the number of edges of G_k . A better upper bound is obtained through the use of Theorem 2.2.

Theorem 4.3. *The number of edges of G_k , $k \neq (n-2)/2$, is less than*

$$E_{n,k} = 4\sqrt{(k+1)}\sqrt{(n-k-1)}\sqrt{n}.$$

Proof. Pick a direction which is not parallel to any line joining two points of S and draw $n-1$ parallel lines in this direction separating the points of S . According to Theorem 2.2, the total number of intersections of these lines with the edges of G_k is

$$\begin{aligned} N &= 2(2+4+\dots+2k)+2(n-1-2k)(k+1) \\ &= 2n(k+1)-2(k+1)^2. \end{aligned}$$

Now we choose an integer α , to be determined later, and divide the edges of G_k into two classes according to whether they intersect at least α of the parallel lines or not. The first class clearly contains no more than N/α elements. The second class contains no more elements than there are pairs of points of S separated by fewer than α of the parallel lines, that is

$$2[(n-2)+(n-3)+\dots+(n-\alpha)] = 2(\alpha-1)n - \alpha(\alpha+1) + 2.$$

Thus the number of edges of G_k is certainly less than $2\alpha n + N/\alpha$ which is minimal when we choose $\alpha = \sqrt{(N/2n)}$. Let α be the integer just above $\sqrt{(N/2n)}$; then the number of edges is less than $2\sqrt{(2nN)} = E_{n,k}$.

Definition 4.4. Let $e_{n,k}$ denote the maximal number of edges of a graph $G_k(S)$ where S contains n points. For (undirected) bigraphs we write $e_n = \frac{1}{2}e_{2n,n-1}$.

Theorem 4.5. *If $k \neq (n-2)/2$ then*

$$e_{2n,2k+1} \geq 2e_{n,k} + n.$$

Proof. We first show that a $G_k(S)$ with a maximal number of edges must have n vertices. For, assume that there is a point $p \in S$ which is not a vertex of G_k . According to Lemma 3.1, this means that p is on the negative side of all the edges of G_k . If we move p across the line of an edge of G_k then that edge is removed from the graph but there will be at least two new edges incident to p , contrary to the maximality assumption for G_k .

Now associate an outgoing edge e_p to each vertex p of G_k and construct a set S' with $2n$ points by splitting each point p into two points p' and p'' at a small distance ε from p , with $\overrightarrow{p'p}$ and $\overrightarrow{pp''}$ in the direction of e_p . Consider $G_{2k+1}(S')$. First for each $p \in S$ we get $p'p''$ as an edge of G_{2k+1} since each point on the positive side of e_p has become two points; and, if $e_p = \overrightarrow{pq}$, then exactly one of the two points q', q'' is on the positive side of e_p . In addition, both $\overrightarrow{p'q'}$ and $\overrightarrow{p''q''}$ are edges of G_{2k+1} whose positive sides contain the points arising from those on the positive side of e_p and, respectively, the point p' or that point q', q'' which lies on the positive side of e_p .

Finally, if \overrightarrow{pq} is an edge of G_k other than e_p , then the edge which joins the point p' or p'' on the negative side of \overrightarrow{pq} to the point q' or q'' on the positive side of \overrightarrow{pq} as well as the edge which joins the point p' or p'' on the positive side of \overrightarrow{pq} to the point q' or q'' on the negative side of \overrightarrow{pq} are edges of G_{2k+1} .

Thus in this splitting process each edge of G_k yields two edges of G_{2k+1} and the n edges e_p of G_k yield an additional edge $\overrightarrow{p'p''}$. Thus

$$e_{2n,2k+1} \geq e(G_{2k+1}) = 2e_{n,k} + n.$$

In order to get an analog to Lemma 4.5 for bigraphs B , we must avoid the possibility of associating the same edge in its two orientations with both of its endpoints, that is, $e_p = \overrightarrow{pq}$ and $e_q = \overrightarrow{qp}$. Since $e_1 = 1$, $e_2 = 3$, such associations cannot be avoided for bigraphs on 2 or 4 vertices, but they can be avoided for $2n \geq 6$.

Lemma 4.6. $e_{n+1} \geq e_n + 2$.

Proof. Let B be a bigraph of S with $2n$ vertices and e_n unoriented edges. By an affine transformation we can assume that all points of S are close to the x -axis and that all edges of B make small angles with the x -axis. Now we add two points p, q to S , where p has sufficiently large positive y -coordinate and q has sufficiently large negative y -coordinate and both have, say, x -coordinates smaller than the x -coordinates of the points of S ; then all the edges of $B(S)$ are also edges of $B(S')$, where $S' = S \cup \{p, q\}$, and since pq is not an edge of $B(S')$, there are two new edges incident to p and q , respectively. Thus $e_{n+1} \geq e(B(S')) = e_n + 2$.

Corollary 4.7. For $n \geq 3$, we have $e_n \geq 2n$.

Lemma 4.8. If B is a bigraph of S with $2n$ vertices and e_n unoriented edges, $n \geq 3$, then each component of B has at least 6 vertices.

Proof. By Corollary 2.9, each union of components of B is itself a bigraph. Thus if B has a component B' with no more than 4 vertices, we can write $B = B' \cup B''$ where B' has $2i \leq 4$ vertices and therefore $\leq 2i - 1$ edges and B'' has $2n - 2i$ vertices and therefore $\leq e_{n-i}$ edges. This implies $e_n \leq e_{n-i} + 2i - 1$, in contradiction to Lemma 4.6.

Lemma 4.9. For a bigraph B with $2n$ vertices and a maximal number e_n of unoriented edges, $n \geq 3$, it is possible to associate to each vertex p an edge e_p of B so that $e_p \neq e_q$ whenever $p \neq q$.

Proof. By Lemma 4.8 and Corollary 4.7, each component B' of B has at least as many edges as it has vertices. It therefore contains a circuit C . If we arrange the vertices of C in cyclic order p_1, p_2, \dots, p_s with $p_{s+1} = p_1$ then we can associate with each p_i the edge $p_i p_{i+1}$. If there are vertices B' not included in C then there exist immediate neighbors, q , of vertices p_i in C . To

each such neighbor we associate the edge qp_i . If this still does not exhaust the vertices of B' we get additional vertices joined to the immediate neighbors of C etc.

Theorem 4.10. For $n \geq 2$, we have $e_{2n} \geq 2e_n + 2n$.

Proof. For $n = 2$, we do this by inspection since $e_2 = 3$ is obtained whenever the vertices form a non-convex quadruple and $e_4 = 9$ by a modified splitting procedure. For $n \geq 3$, we use Lemma 4.9 to associate to each vertex a different incident edge of B and then employ the splitting process used in the proof of Theorem 4.5 to complete the proof.

By iterated application of Theorems 4.5 and 4.10, we get

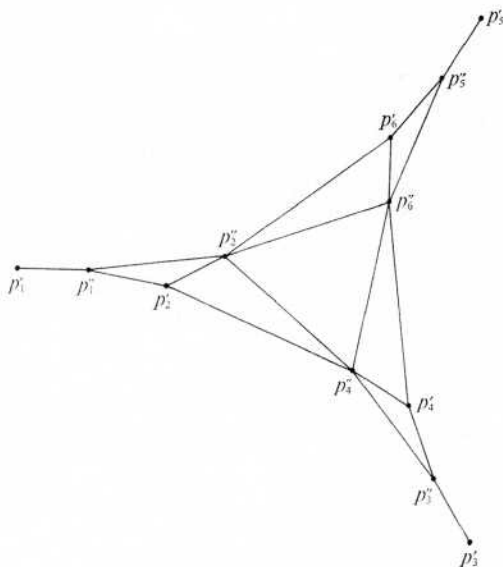


Fig. 1.

Corollary 4.11. (1) $e_{2^m n, 2^m(k+1)-1} \geq 2^m e_{n,k} + m2^{m-1}n$.

For $n \geq 2$, (2) $e_{2^m n} \geq 2^m e_n + m2^m n$.

Theorem 4.12. For all n , we have $e_n > \frac{1}{2}n \log_2(2n/3)$.

Proof. Apply Corollary 4.11 to get

$$e_{3 \cdot 2^m} \geq 2^m e_3 + 3m \cdot 2^m = 3 \cdot 2^m(2+m). \quad (4.13)$$

Now choose m so that $3 \cdot 2^m \leq n < 3 \cdot 2^{m+1}$, then $m+1 > \log_2(n/3)$ and $3 \cdot 2^m > \frac{1}{2}n$, so that (4.13) yields

$$e_n \geq e_{3 \cdot 2^m} > \frac{1}{2}n \log_2(2n/3).$$

If we modify Theorem 4.3 to include the unoriented bigraph, we replace n by $2n$, k by $n-1$ and N by $\frac{1}{2}N$ so that the proof of Theorem 4.3 yields

Theorem 4.14. For all n , $e_n < (2n)^{\frac{3}{2}}$.

We can generalize the lower bound of Theorem 4.12 to arbitrary $e_{n,k}$. For convenience, we restrict our attention to the cases $n \geq 3(k+1)$.

Lemma 4.15. For all $m \geq 2$, we have

$$e_{m,0} = m;$$

and hence for all $m \geq 3$ and all $l \geq 0$,

$$e_{2^l m, 2^l - 1} \geq (l+2)2^{l-1}m. \quad (4.16)$$

Proof. The first part follows from the fact that $G_0(S)$ consists of the boundary of the convex hull of S traversed in the counter-clockwise direction. The second part then follows through l -fold application of Theorem 4.5 to $e_{m,0}$.

Lemma 4.17. If $n \geq 3(2^l - 1)$ then

$$e_{n, 2^l - 1} \geq (l+2)2^{l-1} \lceil n/2^l \rceil.$$

Proof. Choose $m \geq n/2^l$. Then according to Lemma 4.15 and its proof there exists a set S with $2^l m$ points whose vertices consist of clusters of 2^l points in arbitrarily small neighborhoods of vertices of a convex (say, a regular) m -gon C so that $e(G_{2^l-1}(S)) \geq (l+2)2^{l-1}m$. Now we can place $n - 2^l m$ points so near to the centroid of C that every line through one of these points has at least 2^l points of S on either side. Thus if the new set is called S' then $e(G_{2^l-1}(S')) = e(G_{2^l-1}(S)) \geq (l+2)2^{l-1}m = (l+2)2^{l-1} \lceil n/2^l \rceil$.

Lemma 4.18. $e_{n+2, k+1} \geq e_{n, k} + 4$.

Proof. The proof follows from a construction which is entirely analogous to that used in the proof of Lemma 4.6.

Theorem 4.19. If $n \geq 3(k+1)$ then

$$e_{n, k} \geq \frac{1}{2}(n-3k) \log_2(2k+2).$$

Proof. Write $k = 2^l - 1 + a$, where $0 \leq a < 2^l$, and $n = 2^l m + 2a + b$, where $0 \leq b < 2^l$. Then $2^{l+1} > k+1$, so that $l+2 \geq \log_2(2k+2)$ and $2^l m = n - 2a - b \geq n - 3(2^l - 1) \geq n - 3k$. Now, combining Lemmas 4.17, 4.18 and 4.19, we get

$$e_{n, k} \geq (l+2)2^{l-1}m \geq \frac{1}{2}(n-3k) \log_2(2k+2).$$

5. Generalizations and problems

There are several obvious generalizations of the graphs G_k considered above. For example, if we have a set S of n points in general position in E^d (no $d+1$ on a hyperplane) then we can consider the hypergraph G_k^d consisting of oriented hyperplanes containing d points of S and having k points of S on their positive sides. By symmetry we may again assume $k \leq (n-d)/2$. The elements of the graph are now vertices and hyperplanes (d -tuples of vertices). However, the construction in Section 2 is no longer applicable for

the construction of G_k^d since the spatial rotations do not form a one-dimensional group.

It is now clear that every $(d-1)$ -tuple of points of S , through which there passes a hyperplane containing no more than k points of S on one side, is contained in at least 2 faces of G_k^d . We must get a trivial lower estimate for $e_{n,k}^d$, the maximal number of faces of G_k^d , obtained whenever the points of S are vertices of its convex hull,

$$e_{n,k}^d \geq 2 \binom{n}{d-1}. \quad (5.1)$$

The other lower bound estimates, such as Theorems 3.3 and 4.1 can also be generalized without difficulty. However, the more difficult estimates based on Theorems 2.2 and 4.5 do not generalize as easily. In particular, if we try to extend the splitting process of Theorem 4.5, we would turn each point into d points and the distributions on opposite sides of the dividing planes would not be easy to establish.

Another generalization would be to use other classes of dividing curves and surfaces instead of straight lines and planes. For example, in the plane we could use circles through 3 points or conic sections through 5 points, etc.

It appears likely that the lower bound obtained for e_n is closer to the truth than the upper bound.

Conjecture 5.2. *The lower bound $e_n > cn \log n$ obtained in Theorem 4.12 cannot be substantially improved. In particular, we conjecture that $e_n = o(n^{1+\varepsilon})$ for all $\varepsilon > 0$.*

Finally, the upper bound construction in Theorem 4.3 leads to the following

Problem 5.3. *Given any graph with n vertices in general position in the plane (the graph need not be planar, so its edges are permitted to intersect), what is the minimal number $e = f(n, k)$ of edges that guarantees that there exists a straight line intersecting at least k of the edges?*

Obviously, $f(n, 1) = 1$, $f(n, 2) = 2$ while $f(n, 3) = n+1$ since a convex n -gon has n edges no three of which are intersected by a straight line. In the same manner we get

$$f(2ml, 2l^2 + 1) > ml(2l - 1)$$

since we can place complete graphs on $2l$ vertices in small neighborhoods of the vertices of a regular m -gon. Then no straight line can intersect more than two of these complete graphs and therefore no straight line intersects more than $2l^2$ edges of the graph. This leads us to a final conjecture:

Conjecture 5.4. *If a graph with n vertices in general position in the plane has more than nk edges, then there exists a straight line which intersects k^2 edges.*