Reprinted from JOURNAL OF COMBINATORIAL THEORY
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Vol. 14, No. 3, May 1973
Printed in Belgium

On a Combinatorial Game

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Received November 8, 1970

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A drawing strategy is explained which applies to a wide class of combinatorial and positional games. In some settings the strategy is best possible. When applied to n-dimensional Tic-Tac-Toe, it improves a result of Hales and Jewett [5].

A family of sets $\{A_k\}$ is said to have property B if there is a set S which meets every A_k and contains none of them. $m(n)$ is the smallest integer so that there is a family $\{A_k\}$, $1 \leqslant k \leqslant m(n)$, $| A_k | = n$ which does not have property B. It is known that [2, 3, 6]

$$2^n \left(1 + \frac{4}{n}\right)^{-1} < m(n) < cn^2\, 2^n.$$

$m(2) = 3$; $m(3) = 7$; $m(4)$ is not known.

Now we define a game connected with property B. Let $\{A_k\}$ be a family of sets. Let

$$\bigcup_k A_k = S = \{a_1, \ldots, b_1, \ldots\}.$$

The players alternately pick elements of S, and that player wins who first has all the elements of one of the A_k. Let $m^*(n)$ be the smallest integer for which there are $m^*(n)$ sets $\{A_k\}$, $| A_k | = n$, $1 \leqslant k \leqslant m^*(n)$, and so that the first player has a winning strategy.

THEOREM. $m^*(n) = 2^{n-1}$.

Proof. First we exhibit 2^{n-1} sets of n elements each so that the first player has a winning strategy. Let G_n be such a collection. Put $G_1 = \{a_1\}$

and $G_1' = \{b_1\}$. By induction define G_{n+1} to be the collection of the sets G_n and G_n' with a_{n+1} adjoined to each and define G'_{n+1} to be the collection of the sets G_n and G_n' with b_{n+1} adjoined to each. The strategy for G_n is clear. First pick a_n. If the second player picks a_i, you pick b_i and vice versa. On each move the second player can block only half the remaining sets. On the n-th move you will complete a set.

Now we must prove that for any smaller collection of sets of n elements the second player has a strategy which keeps the first player from winning. Here is the strategy: Give each element a value which is the sum of the values of each set it belongs to (which has not already been blocked by you). The value of such a set with j elements remaining is 2^{n-j}. Pick an element of largest value. To prove that the first player cannot win, we show that the sum C of the values of all the sets remaining after his i-th move is less than 2^n. So before his next move this sum C is less than $2^n - V$, where V is the sum of the values of the sets just blocked by the second player, i.e., the value of the element picked by him. Now on the first player's next move he doubles the value of each set containing the element picked, i.e., he adds the sum of their previous values V' to C. But clearly $V' \leqslant V$ since V was a maximum.

The same method gives the following slightly more general result: Let $\{A_k\}$ be a family of sets $|A_i| = n_i$ for which

$$\sum_i \frac{1}{2^{n_i}} < 1.$$

Then the next player has a strategy which forces a draw. On the other hand if integers n_i are given for which

$$\sum_i \frac{1}{2^{n_i}} \geqslant \frac{1}{2},$$

we can find sets A_i, $|A_i| = n_i$, so that the first player has winning strategy. Clearly without loss we may assume the sum equals 1/2. After putting a_1 in each set it is again clear that we may put a_2 in "half" the sets and b_2 in the others, i.e., a_2 and b_2 have equal value. Now we may again split the collection in which a_2 occurs into two equal parts, and so on.

Let us now denote by $m_1{}^*(n)$ the smallest integer for which there are $m_1{}^*(n)$ sets $\{A_k\}$, $1 \leqslant k \leqslant m_1{}^*(n)$, $|A_k| = n$, $|A_i \cap A_j| \leqslant 1$, $1 \leqslant i < j \leqslant m_1{}^*(n)$, and so that the first player has a winning strategy. We have no satisfactory estimate for $m_1{}^*(n)$. $m_1{}^*(3) = 6$, the sets being 123, 145, 167, 189, 246, 579. Probably $m_1{}^*(n)$ is considerably smaller than $m^*(n)$.

Hales and Jewett [5] investigated n-dimensional Tick-Tack-Toe in a hypercube of side k. They proved that, if $k \geqslant 3^n - 1$ (k odd) or $k \geqslant 2^{n+1} - 2$ (k even), the second player can force a tie, but for each k there is an n_k so that for $n \geqslant n_k$ the first player can win.

In the n-dimensional hypercube of side k there are $\frac{1}{2}\{(k+2)^n - k^n\}$ sets which form winning lines. Thus our theorem immediately implies that the second player can force a draw if $k > cn \log n$. This result still falls short of their conjecture that the second player can force a draw if

$$k > 2(2^{1/n} - 1)^{-1} \approx \frac{2n}{\log 2} - 1.$$

It is well known [4, 1] that, if we color the edges of a complete graph of n vertices by two colors, there always is a complete subgraph of

$$\left[\frac{\log n}{2 \log 2}\right]$$

vertices all of whose edges have the same color, but there does not have to be such a graph of

$$\left[\frac{2 \log n}{\log 2}\right]$$

vertices.

Now following Simmons we define a game called the Ramsey game connected with this property. The players alternately choose edges. That player wins who first gets all the edges of a complete graph of k vertices. Ramsey's theorem implies that the game is a win for the first player for

$$k \leqslant \left[\frac{\log n}{2 \log 2}\right].$$

Our theorem implies that the game is a draw if

$$2^l > \binom{n}{k}, \qquad \text{where} \quad l = \binom{k}{2} - 1,$$

i.e., it is a draw if

$$k \geqslant (1 + o(1)) \frac{2 \log n}{\log 2}.$$

We did not investigate the Ramsey game for triples since we did not succeed in getting any satisfactory result.

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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.