XVIII. ON CHROMATIC NUMBER OF GRAPHS AND SET SYSTEMS

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§ 1. Introduction

The aim of this note is to correct a mistake the first two authors made in [1]. Set theoretical notation will be standard otherwise we will use the graph theoretical notation described in [1].

One of the main results of [1] Corollary 5.6 states that every graph of chromatic number $> \omega$ contains complete bipartite graphs $[i, \omega_1]$ for $i < \omega$.

At the end of our paper in 12.1 we claimed the following generalization of this result for uniform set systems.

If $\mathcal{H} = \langle h, H \rangle$ is a uniform set system with $\chi(\mathcal{H}) = k$, $2 \leq k < \omega$, $\beta \geq \omega$ then either the colouring number of $\mathcal{H}$ is at most β or there is $H' \subset H$, $|H'| = \beta^+$ such that $|\bigcap H'| \geq k-1$. For $k = 2$ this is a trivial consequence of e.g. [1] 5.6. In [1] we omitted the "proof". We verified the result for $\alpha(\mathcal{H}) = \alpha = \beta^+$ and thought that the induction method described in [1] § 4 yields the general result. However, this is not true and we are going to state a correct version of the theorem.

To have a brief notation we introduce a relation $R(\alpha, \beta, \gamma, k, i)$.

<u>Definition</u>. Let α, β, γ be cardinals, $2 \leq k < \omega$ $1 \leq i < k$. $R(\alpha, \beta, \gamma, k, i)$ is said to hold if for every uniform set system $\mathcal{H} = \langle h, H \rangle$, with $\alpha(\mathcal{H}) = \alpha$,

$\kappa(\mathcal{H}) = k$ either there is $H' \subset H$, $|H'| = \gamma$ with $|\bigcap H'| \geq i$ or $Chr(\mathcal{H}) \leq \beta$.

The false theorem claimed that $R(\alpha, \beta, \beta^+, k, k-1)$ holds for $\beta \geq \omega$, $2 \leq k < \omega$, and for every α

<u>Theorem 1</u>. Let $\beta = \omega_\xi$, $3 \leq k < \omega$. Then

$R(\alpha, \beta, \beta^+, k, k-1)$ holds for $\alpha = \beta^+$

$R(\alpha, \beta, \beta^+, k, 2)$ holds for $\alpha \leq \omega_{\xi+k-2}$

Put $exp_0(\beta) = \beta$, $exp_{k+1}(\beta) = 2^{exp_k(\beta)}$ for $k < \omega$

On the other hand we have

<u>Theorem 2</u>. Let $\beta = \omega_\xi$, $3 \leq k < \omega$, $2 \leq i \leq k-1$ and put $\alpha = (exp_{k-i}(\beta))^+$

Then $R(\alpha, \beta, 2, k, i)$ is false.

If we now denote by $f(\beta, k, i)$ the minimal α, for which $R(\alpha, \beta, 2, k, i)$ is false for $\beta \geq \omega$, $3 \leq k < \omega$, $2 \leq i \leq k-1$ then assuming G.C.H. we obtain the following

<u>Corollary</u>:

$$f(\beta, k, 2) = \omega_{\xi+k-1}$$
$$f(\beta, k, k-1) = \omega_{\xi+2}$$

$$f(\beta, k, i) \leq \omega_{\xi+k-i+1} \quad \text{for} \quad 2 \leq i \leq k-1$$

We have an example to show that the upper estimate is not best possible.

<u>Theorem 3</u>. Assume $k = \binom{\ell}{t}$, $\alpha = (exp_{t-1}(\beta))^+$, $k > 1$, $i = \binom{\ell-1}{t} + 1$. Then $R(\alpha, \beta, 2, k, i)$ is false.

As a corollary if G.C.H. is assumed and $\beta = \omega_\xi$ we get

$$f\left(\beta, k, \binom{\ell-1}{t}+1\right) \le \omega_{\xi+t}$$

E.g. in case $k=6$, $\ell=4$ we get

$$f(\beta, 6, 4) \le \omega_{\xi+2} \qquad \text{while } k-i+1=3$$

Note that assuming G.C.H. the simplest unsolved problem is the following:

Does $R(\omega_2, \omega, 2, 5, 3)$ hold?

We can not determine $f(\beta, k, i)$ for other values of i, however with Galvin the first two authors have a number of similar but more complicated results giving sharp upper estimations for the chromatic number of uniform set systems not containing certain types of finite subsystems. These will appear in a forthcoming triple paper of Erdős, Hajnal and Galvin.

Before we turn to the proofs we mention a few other problems which led us to discover our mistake.

Let β be a cardinal. Let $H_0(\beta)$ be the minimal γ, for which there is a partition I of length γ of $\mathcal{P}(\beta)$, i.e. $\mathcal{P}(\beta) = \bigcup_{\nu<\gamma} I_\nu$ such that no I_ν contains three different sets, A, B, C with $A \cup B = C$.

The function H_0 was introduced by Hanson for finite β. Hanson proved

$$(1) \qquad c\sqrt{\beta} < H_0(\beta) \le \frac{\beta}{2}+2 \qquad \text{for } \beta<\omega$$

A theorem of Erdős and Komlós implies that in (1) we have $\frac{\beta}{4} < H_0(\beta)$ as well.

For $\beta \geq \omega$, G. Elekes proved recently that $H_0(\omega) > \omega$ See [2]. For more problems arising here see [3].

Later Erdős considered the following similar problem.

Let $H_1(\beta)$ be the smallest γ for which there is $\mathcal{P}(\beta) = \bigcup_{\nu<\gamma} I_\nu$ such that there are no distinct $A, B, C, D \in \mathcal{P}(\beta)$ in the same I_ν satisfying

$$(2) \qquad A \cup B = C \qquad , \ A \cap B = D$$

For finite β a theorem of Erdős and Kleitman gives

$$c_1 \beta^{1/4} < H_1(\beta) < c_2 \beta^{1/2}$$

Meditation shows that $R(2^\beta, \gamma, 2, 4, 3)$ implies $H_1(\beta) \leq \gamma$ hence, by the false "theorem", $H_1(\beta) \leq \omega$ for every β. Investigation of $H_1(\beta)$ led us finally to the simple proof of Theorem 2.

It is worth to remark that by the above consideration and by Theorem 1 we have

$$(3) \qquad 2^\beta = \beta^+ \qquad \text{implies} \qquad H_1(\beta) = \beta \qquad \text{for} \quad \beta \geq \omega$$

$H_1(\beta)$ will be studied in the forthcoming Erdős-Galvin-Hajnal paper as well.

§ 2. Proofs

Proof of Theorem 1. Let $\mathcal{H} = \langle h, H \rangle$ be a uniform set system with $\alpha(\mathcal{H}) = \alpha$, $\kappa(\mathcal{H}) = k$ and assume $2 \leq \iota \leq k-1$, $\beta < \alpha$ and

$$(4) \quad |\bigcap H'| < \iota \qquad \text{for} \quad H' \subset H, \ |H'| = \beta^+$$

It follows easily from the Lemmas stated in [1] § 4 that then there is a sequence B_ξ, $\xi < \alpha$ of disjoint subsets of α, satisfying the following conditions

(5) $|B_\xi| < \alpha$ for $\xi < \alpha$; $\bigcup_{\xi<\alpha} B_\xi = h$

(6) If $C_\xi = \bigcup_{\eta<\xi} B_\eta$, $X \in H$, $|X \cap C_\xi| \geq i$

then $X \subset C_\xi$ for $\xi < \alpha$.

To prove statement (i) of Theorem 1. let $\alpha = \beta^+$, $i = k-1$ Then, by (5), $|B_\xi| \leq \beta$ for $\xi < \alpha$. Then there are sets D_ν, $\nu < \beta$ such that $h = \bigcup_{\nu<\beta} D_\nu$ and $|D_\nu \cap B_\xi| \leq 1$ for $\nu < \beta$, $\xi < \alpha$. Then, by (6), $X \notin D_\nu$ for $X \in H$ otherwise there is a maximal ξ with $X \cap B_\xi = \{u\}$ for some u, and $X - \{u\} \subset C_\xi$, $|X - \{u\}| = k-1$ implies $u \in C_\xi$ a contradiction. Hence $Chr(\mathcal{H}) \leq \beta$

To prove part (ii) we apply induction on α. We assume that (ii) is true for every $\mathcal{H}'$ with $\alpha(\mathcal{H}') < \alpha$, $\kappa(\mathcal{H}') = \ell$, $3 \leq \ell < \omega$ Since $R(\beta, \beta, \dots)$ is true we may assume $\beta < \alpha \leq \omega_{\xi+k-2}$ and that (4), (5) and (6) hold with $i = 2$ and we have to prove $Chr(\mathcal{H}) \leq \beta$.

By part (i) we may assume $k > 3$.

For $X \in H$ let $\xi(X) = \max\{\xi < \alpha : B_\xi \cap H \neq \emptyset\}$

$$H_\xi = \{X \in H : \xi(X) = \xi\}; \quad H = \bigcup_{\xi<\alpha} H_\xi$$

Then, by (6), $|X \cap B_\xi| \geq k-1 \geq 3$ for $X \in H_\xi$. Hence there is a uniform set system $\mathcal{H}_\xi = \langle \mathbf{B}_\xi, \hat{H}_\xi \rangle$ such that

(7) $\alpha(\mathcal{H}_\xi) = |B_\xi| \le \omega_{\xi+k-3}$ for each $\xi < \alpha$, $\varkappa(\mathcal{H}_\xi) = k-1$

and there is $Y \in \hat{H}_\xi$, $Y \subset X$ for $X \in H_\xi$

Then obviously $|\bigcap H'| < 2$ for $H' \subset \hat{H}_\xi$, $|H'| = \beta^+$

Applying the induction hypothesis for the set systems $\mathcal{H}_\xi$ we get that there are sets

$$D_{\xi,\nu} \subset B_\xi \quad ; \quad \bigcup_{\nu<\beta} D_{\xi,\nu} = B_\xi$$

such that $Y \not\subset D_{\xi,\nu}$ for $Y \in \hat{H}_\xi$, $\xi < \alpha$, $\nu < \beta$.

Put $D_\nu = \bigcup_{\xi<\alpha} D_{\xi,\nu}$. Then $\bigcup_{\nu<\beta} D_\nu = H$

Let $\nu < \beta$, $X \in H$. Then $X \in H_\xi$ for some $\xi < \alpha$

Then by (7) there is $Y \in \hat{H}_\xi$, $Y \subset X$

Since $Y \not\subset D_{\xi,\nu}$, $X \not\subset D_\nu$. Hence

$$Chr(\mathcal{H}) \le \beta.$$

<u>Proof of Theorem 2</u>. Put

$$h = [\alpha]^{k-i+1} \quad , \quad 2 \le k-i+1 < k$$

Let $X \in [\alpha]^k$, $X = \{x_0, \dots, x_{k-1}\}$, $x_0 < \dots < x_{k-1}\}$

We define $Z(X) \in [h]^k$ by

$$Z(X) = \Big\{ \{u_0^j, \dots, u_{k-i}^j\} \in h : j < k \text{ and } u_\nu^j = x_{j+\nu \bmod k} \text{ for } \nu \le k-i \Big\}$$

i.e. $Z(X)$ consists of the k intervals of length $k-i+1$ of X considered in the cyclical order $x_0 \prec \cdots \prec x_{k-1} \prec x_0$. By $k-i+1 < k$, $|Z(X)| = k$.

Put $H = \{Z(X) : X \in [\alpha]^k\}$

$\mathcal{H} = \langle h, H \rangle$. Then $\alpha(\mathcal{H}) = \alpha = (\exp_{k-i}(\beta))^+$

Let now $X \neq Y \in [\alpha]^k$, then there are $x \in X - Y$, $y \in Y - X$. Hence

$|Z(X) - Z(Y)| \geq k-i+1$, $|Z(Y) - Z(X)| \geq k-i+1$

and thus

$$|Z(X) \cap Z(Y)| \leq \frac{2k - 2(k-i+1)}{2} = i-1$$

Thus $H' \subset H, |H'| = 2$ implies $|\bigcap H'| < i$

We prove $\mathrm{Chr}(\mathcal{H}) > \beta$.

Let $h = \bigcup\limits_{\nu < \beta} D_\nu$ be a partition of h. Then by $h = [\alpha]^{k-i+1}$ and as a corollary of the Erdős-Rado theorem $(\exp_{k-i}(\beta))^+ \to (\beta^+)^{k-i+1}_\beta$ there is $X \subset \alpha, |X| = k$ which is homogeneous, i.e. there is $\nu < \beta$ such that $[X]^{k-i+1} \subset D_\nu$. But then $Z(X) \subset D_\nu$, $Z(X) \in H$ for this X. Hence $\mathcal{H}$ has the properties to show that $R(\alpha, \beta, 2, k, i)$ is false.

As to the proof of Theorem 3, take

$$h = [\alpha]^t, \quad H = \{[X]^t : X \in [\alpha]^\ell\}, \quad \mathcal{H} = \langle h, H \rangle$$

It follows quite similarly as in the proof of Theorem 2 that $\mathcal{H}$ disproves $R(\alpha, \beta, 2, k, i)$

References

[1] P. Erdős and A. Hajnal, On chromatic number of graphs and set systems, Acta Math. Acad. Sci. Hung. 17(1966)61-99.

[2] G. Elekes, On a partition property of infinite subsets of a set. Periodica Math. Hung. to appear.

[3] P. Erdős and A. Hajnal, Unsolved and solved problems in set theory, Proceedings of the Berkeley Symposium 1971, to appear.