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EXTREMAL PROBLEMS AMONG SUBSETS OF A SET

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Abstract. This paper is a survey of open problems and results involving extremal size of collections of subsets of a finite set subject to various restrictions, typically on intersections of members.

0. Introduction

The subsets of a (finite) set form a lattice and in fact a Boolean algebra. The following concepts are natural to them.

(A) Intersection.
(B) Union.
(C) Disjointness.
(D) Complement.
(E) Containment.
(F) Rank (size).

In this paper we survey the present status of a number of problems involving maximal or minimal sized families of subsets subject to restrictions involving these concepts.

Problems of this kind arise in a large number of contexts in many areas of mathematics. For example, the divisors of a square free number correspond to the subsets of the prime divisors, so that certain number theoretic problems involving divisors of numbers are of this form. Efficient error correcting codes and block designs can be considered as extremal collections of subsets satisfying restrictions of this kind.

Since the concept of set is as basic in mathematics as the concept of number, one can also investigate the properties considered here for their own sake as one considers similar problems in number theory. Thus we might ask: "What sort of limitations are imposed upon families of subjects of a set by simple restrictions on intersection, union, rank and/or containment among members of the family?"

Questions of this kind have one additional value. Since the concepts involved are all easily understood by non-mathematicians, results and elegant proofs in this area have tutorial value as illustrations of the power of mathematical method that are accessible to the layman.

To facilitate reference, we divide the problems considered here into five areas. These are:

1. Non-intersection.
2. Size limited intersection.
3. Intersection and rank limitations.
4. Containment limitations.
5. Union and intersection restrictions.
6. Miscellany.

Problems and results in these areas are described in the corresponding section below.

1. Non-disjoint families

Let S be a finite set having n elements ($|S| = n$). Among the simplest restrictions that can be placed on families of subsets of S is that no two are disjoint. Thus if $F = \{A_i\}$, $i = 1, ..., \lambda$ with $A_i \subset S$, we may require that $A_i \cap A_j \neq \emptyset$ for all i, j.

With this one restriction there are several questions that can be raised. Among these are:

(a) How large can F be?

(b) If F is "maximal" in that no subset of S can be added to it without violating the restriction, how small can F be?

(c) How many maximal F's are there of any given size?

(d) How many F's are there of any size?

These four kinds of questions can be raised not only about families of subsets restricted as is F above, but also about families satisfying variants of the restriction.

Among possible variant restrictions of the same general kind are:

I.1. Let F be as defined above, and let G consists of the minimal members of F that is the members of F not contained in others.

I.2. Let G_{2k} be the union of k families each restricted as was F above.

I.3. Let G_{3k} be a family containing no k members that are pairwise disjoint.

I.4. Let G_{4k} be a family such that the intersection of every k members is non-empty.

We now describe some results.

No collection of non-disjoint subsets can contain a set and its complement. Thus our family F can have at most half of the subsets $|F| \leq 2^{n-1}$. A maximal family F contains every set containing any member. Since every set disjoint from A is contained in A's complement, if A cannot be added to a maximal family F, $\bar{A}$ is already in it. Thus all maximal families consist of exactly 2^{n-1} subsets, exactly one of A or $\bar{A}$ for each A.

Thus questions (a) and (b) are easily answered for families satisfying the non-disjointness restriction satisfied by F above. The number of maximal families satisfying this restriction on the other hand has not as yet been determined very well.

There exist several levels of inaccuracy in estimates of quantities of this kind. Some of these are listed here. One can have:

(1) An exact formula.

(2) A convergent formula (convergent for large n to the exact result).

(3) An asymptotic formula (ratio to exact result is convergent).

(4) An asymptotic formula for the logarithm.

In addition, one can obtain bounds upon any of these levels, one as well as any others.

We can easily find a level 4 expression for the total number of families F; it is $n(F) = \exp_2 [2^{n-1}(1 + o(1))]$.[1]

The argument will be described below.

The analogous result for the number of maximal families is probably $\exp_2 [\binom{n}{[n/2]} (1 + o(1))/2]$ but this has not been proven. It is, however, a lower bound, and an upper bound of $\exp_2 [\binom{n}{[n/2]} (1 + o(1))]$ is easily obtained.

To illustrate the kind of reasoning that can be employed to obtain estimates of this kind, we sketch the argument here.

A maximal F can be characterized by its minimal members. That is, we can define $G(F)$ to be the family consisting of those members of F which contain no others, and $G(F)$ determines F. The family $G(F)$ is then what is sometimes called a "Sperner family" or an "antichain"; no

[1] For typographical convenience, 2^x will be denoted $\exp_2 x$.

member of G contains another. (We discuss Sperner families in Section 4.)

Some information is available about the number of Sperner families, from this an upper bound to the number of maximal F's can be obtained, the bound being $\exp_2 [\binom{n}{[n/2]} (1 + o(1))]$.

To obtain a lower bound we divide the $\frac{1}{2}n$ element subsets of S into those containing a given element a_0, and the rest (the rest here are the complements of the members of the former collection). There are $\exp_2[\frac{1}{2}\binom{n}{[n/2]}]$ collections Q made up of $\frac{1}{2}n$ element subsets containing a_0. Each of these determines a collection F, with F consisting of all sets with more than $\frac{1}{2}n$ elements, those $\frac{1}{2}n$ element sets containing a_0 in Q and complements of the $\frac{1}{2}n$ element sets containing a_0 not in Q. The argument for n odd is similar.

We expect that the kind of argument used to yield the estimate $\exp_2 [\binom{n}{[n/2]} (1 + o(1))]$ for the number of Sperner families can be applied to show that the number of Sperner families which contain no disjoint members is $\exp_2 [\binom{n}{[n/2]} (1 + o(1))/2]$. Any maximal family having 2^{n-1} members has $2^{2^{n-1}}$ subsets. The total member of subsets of all maximal families, hence the total number of F's, is no more than $\exp_2 [2^{n-1} + \binom{n}{[n/2]} (1 + o(1))]$ which is of the form $\exp_2 [2^{n-1}(1+o(1))]$ as stated above.

The other restrictions (I.1, ..., I.4) have not all been investigated in as much detail. We first present the existent results on all these problems. Open problems are then listed.

I.1. The properties of $G(F)$'s are essentially the properties of maximal F's. They range in size from 1 to $\binom{n-1}{[n-1]/2}$, the number of them can be estimated as discussed above. They are all maximal.

I.2. The number of members in the union of k F's has been shown to be no more than $2^n - 2^{n-k}$ (see [18]). This bound can be achieved by letting the k families be all subsets containing a_j for $1 \leq j \leq k$.

I.3. Bounds on the size of $G_{3k}(n)$, a family containing no k disjoint members, have been obtained (see [21]). For $n = mk - 1$, these bounds are realizable; for other values, they seem to be slightly higher than the best possible results. These results can be obtained by noticing that for any partition of S into k blocks, at least one block must be outside of any $G_{3k}(n)$. This fact, for any given set of block sizes, leads to limitation on the number of members of $G_{3k}(n)$ of these sizes. Manipulation of the limiting identities yields the results mentioned above.

Smallest size of a maximal $G_{3k}(n)$ is no more than $2^n - 2^{n-k}$. This might be conjectured to be the exact result.

I.4. Among the maximal F's are families consisting of all subsets containing some single element. Such families have the property that all intersections are non-empty. Thus the restriction (on $G_{4k}(n)$) that every k members have non-vanishing intersection does not reduce the maximal size of $G_{4k}(n)$ below 2^{n-1}. There are two natural questions which arise here. What is the maximal size of $G_{4k}(n)$'s in which there exist $(k+1)$ members whose intersection vanishes? Also what is the minimal size of a maximal $G_{4k}(n)$? Milner [33, 34] has some results on the first of these questions. The second is open.

We now list some open problems in this area.

(1) What is the number of maximal families no two members of which are disjoint?

(2) How small can a family be that is maximal with respect to the property that is the union of k different maximal families no two members of which are disjoint? It is asymptotic to 2^{n-1} for large nc.

(3) How many such families are there?

(4) Does the smallest maximal G_{3k} have $2^n - 2^{n-k}$ members?

(5) What are the exact upper bounds on $G_{3k}(n)$?

(6) What is the minimal size of a maximal $G_{4k}(n)$?

(7) What are more exact estimates on the number of families of each type indicated?

2. Size limited intersection

In the problems described so far, the basic restriction was that intersections do not vanish. Such restrictions can be replaced by size limitation on intersections. Thus we could instead require that no A_i and A_j in F satisfy

$$|A_i \cap A_j| \geq k, \qquad |A_i \cup A_j| - |A_i \cap A_j| \geq k,$$

$$|A_i \cap A_j| \leq k, \qquad |A_i \cup A_j| - |A_i \cap A_j| \leq k,$$

$$|A_i \cap A_j| \neq k,$$

$$|A_i \cap A_j| = k.$$

The entire range of problems considered above can be raised about families defined by each of these restrictions. The generalization which most retains the flavor of Section 1 is the first. A maximal sized family $F_k(n)$ restricted by it, consists of all subsets having $\frac{1}{2}(n+k+1)$ or more elements, with $\binom{n-1}{(n+k-1)/2}$ other sets if $n+k$ is odd. That this is the largest possible size for $F_k(n)$ was proven by Katona [12]. Few of the other problems have been examined under this restriction.

The opposite restriction that subsets do not intersect "too much" is vaguely related to packing and coding problems. The number of members of size $\geq k$ of a family restricted so that no two members satisfy $|A_i \cap A_j| \geq 1$ is at most $\binom{n}{k}$ and is achieved by choosing all subsets of size k. If we let f_q be the number of members of such a family having q elements. We obtain

$$\sum_{i=k}^{n} f_q \binom{q}{i} \leq \binom{n}{k}$$

as a size restriction.

The coding problem can be described as the study of families limited by the restriction that the "symmetric difference" between any two members be no less than k. The symmetric difference between A_i and A_j is $A_i \cup A_j - A_i \cap A_j$. There are many results on the maximal size of codes under these restrictions and on constructions of optimal codes. Many of these are described in, for example, [4].

Another problem of this general kind is: How large can a family of subsets of S be if the symmetric difference between members is always $\leq q < n$? For even q, it has been show that maximal size families consist of any set α and all other whose symmetric difference with it is $\leq \frac{1}{2}q$. For odd q, $\binom{n-1}{(q-1)/2}$ of the subsets differing from α by $\frac{1}{2}(q+1)$ may also be included (see [19]).

3. Intersection and rank limitations

Another important class of problems involve families of subsets of a given size subject to intersection restrictions of the kinds already discussed.

Erdős, Ko and Rado [7] showed that the maximal size of a family of subsets of S satisfying

(i) all subsets are of size $\leq k \leq \frac{1}{2}n$ (with $|S| = n$); and

(ii) no two are disjoint, no one contains another,
is $\binom{n-1}{k-1}$, the optimum being achieved by choosing all k elements sets which contain a given element.

If $2k = n$, there are a large number $\exp_2\binom{n-1}{[n/2]}$ of such families. If $2k < n$, however, the maximal sized family is unique up to permutation of the elements.

The minimal size of maximal family here may or may not be $\binom{2k-1}{k}$.

Among the questions that have been raised in this area are:

(1) What is the largest family if one excludes families all of whose members contain some element?

(2) Given two families such that the members of one all intersect the members of the other, and subject to the member size limitation described above; what can be said about their sizes?

The following somewhat more general result has been obtained in this direction [23].

Let F and G be two families of subsets of S, with the members of F having k elements and the members of G q elements. Let $k + q$ be no bigger than n; if k is no more than $\frac{1}{2}(n+1)$, then F can have k or fewer members as long as no member contains another; the same possibility for G is allowed. Then, either

$$|F| \leq \binom{n-1}{k-1} \quad \text{or} \quad |G| < \binom{n-1}{q-1}.$$

Milner [33, 34] has certain results on the first problem above.

For sufficiently large n and given k, the family consisting of all k element subsets including one particular element is far larger (of the order of $c\, n^k/k!$ as opposed to $c'\, n^{k-1}/(k-1)!$) than any other. Under these circumstances, it is easy to answer many of the related questions that arise here.

Thus, for sufficiently large n for fixed k and q, we can show the following:

(A) The number of members in the union of q sets of k-element non-disjoint subsets of S with $|S| = n$ is no greater than

$$\binom{n-1}{k-1} + \binom{n-2}{k-2} + \ldots + \binom{n-q}{k-q}.$$

(B) The number of members of a set of k element subsets of S under the restriction that no $(q+1)$ are pairwise disjoint is bounded in the same way.

(C) The number of members of a set of k element subsets of S under the restriction that the intersection of each pair has at least q elements in it is at most $\binom{n-q}{k-q}$.

One might conjecture that similar results hold so long as $2k \leq n - q + 1$ for (A) and (B), and that the best result for (C) is the maximum over m of

$$\sum_{p=0}^{k-m} \binom{2m-q}{m+p} \binom{n+q-2m}{k-m-p-1} .$$

Results of this kind have not yet been obtained.

A related problem, also as yet unsolved, is due to Kneser [28]. How many families of k-element subsets of S, each consisting of subsets which are not disjoint from one another, are necessary to cover all k-element subsets? The answer appears to be $n - 2k + 1$ (if this number is at least one).

Restrictions of the kind

subset size $= r$,
size of intersection $\leq q$

represent packing problems, or coding problems involving words of "fixed weight". Problems of the form

subset size $= k$,
intersection size $= q$

describe such structures as projective planes ($q = 2$), Steiner systems and designs. There exists a vast literature on such questions. Neither class of problems will be considered here. Erdös, Ko and Rado [7] conjectured that if $|S| = 4k$ and F consists of subsets of size $2k$ of S which overlap by at least two, then $\max |F| = (\binom{4k}{2k} - \binom{2k}{k}^2)/2$.

4. Containment restriction

In this section we consider families of subsets that are subject to containment restrictions. The prototype of such restrictions is that satisfied by a "Sperner family" or antichain, no member contains another. Sperner [37] in 1927 showed that such a family could have at most $\binom{n}{[n/2]}$ members. Lubell [30] in 1959 and independently Meshalkin [32] in 1963 obtained a somewhat stronger restriction. If f_k is the number of k-element members of a Sperner family of subsets of S with $|S| = n$, then the inequality

$$\sum_{k=0}^{n} f_k/\binom{n}{k} \leq 1$$

holds. Equality can only occur if $f_k = 1$ for some value of k. Sperner's result is a corollary of this inequality since it is trivial that

$$\sum_{k=0}^{n} f_k/\binom{n}{[n/2]} \leq \sum_{k=0}^{n} f_k/\binom{n}{k} .$$

Lubell's argument is so simple that we repeat it here. A maximal chain is a set of $n+1$ subsets of S totally ordered by inclusion. Each k element subset occurs in the same proportion $(1/\binom{n}{k})$ of maximal chains. Since no chain can contain more than one member of a Sperner family, the sum of the proportion of maximal chains containing each member cannot exceed one, which is the Lubell–Meshalkin inequality.

The same argument implies that the maximal number of members in a family which has at most q members in common with any chain is the sum of the largest q binomial coefficients. This result follows from the inequality

$$\sum_{k=0}^{n} f_k/\binom{n}{k} \leq q$$

which must be satisfied by such a family. Lubell's argument can be applied in many other contexts. Thus, by its use, along with certain additional arguments, the following generalization has been obtained [27]. Let f be any function defined in the members of any partial order and let F be a family which has at most k members in common with any chain in the partial order. Let G be a permutation group defined on the partial order which preserves f (for g in G, $f(gA) = f(A)$) and is a symmetry of the partial order ($A \leq B$ if and only if $gA \leq gB$ for every g in G). Then the maximum value of the sum of f over the members of F is achieved for some F which is the union of orbits under G. That is, there is an $\overline{F}$ such that

$$\sum_{A \in F} f(A) \leq \sum_{A \in \overline{F}} f(A)$$

with F the union of complete orbits under G. Lubell [31] has obtained still further generalizations of his result.

The following questions have also been raised about Sperner families. Let F be a Sperner family, let G_+ be the family connecting of all subsets which contain at least one member of F, and let G_I be the family of all subsets ordered by inclusion with respect to at least one of F.

How large can $|F|$ be, given $|G_+|$? Given $|G_I|$? If $|F| > \binom{n}{[n/2]}$, how many pairs A, B with $A \supset B$ must there be in F?

The following results along these lines have been obtained:

(1) If $|F| > \binom{n}{k}$ for $k < \frac{1}{2}n$, then $|G_+| > \Sigma_{g=0}^{k} \binom{n}{g}$ (see [24]).

(2) $|F|/|G_I| < \binom{n}{[n/2]}/2^n$ (see [17]).

(3) The number of "containment pairs" is minimized if F consists of all subsets having $[n/2]$, $[n/2]+1$, $[n/2]-1$, $[n/2]+2$, ... elements and of the remaining members of F all have a number of elements given by the next entry on this list [20].

The minimal number of "containment triples" has not been found as yet, although one could guess the same conclusion.

Sperner's conclusion can be obtained, when the restriction of non-containment is relaxed considerably. Suppose, for example, that S is the union of two disjoint sets T_1 and T_2 (see [13, 17]),

$$S = T_1 \cup T_2, \qquad T_1 \cap T_2 = \emptyset,$$

and suppose that F is restricted such that if $A \supset B$ for $A, B \in F$, then $A - B \not\subset T_1$ and $A - B \not\subset T_2$. Then $|F| \leq \binom{n}{[n/2]}$, that is, Sperner's bound still applies with these weakened requirements on F. An interesting unsolved problem is the analogue of this for $S = T_1 \cup T_2 \cup T_3$ all T's disjoint; under these circumstances the analogous restriction on F is not sufficient to get the same bound on $|F|$. One can ask: What is the best bound? Also: What are the weakest additional restrictions necessary to impose upon F to get back to the Sperner bound in this case? One can also ask: What analogue of Lubell's inequality can be obtained for the $S = T_1 \cup T_2$ problem?

Katona [15], Schönheim [36] and Erdös [11] have obtained further generalizations of Sperner's theorem.

The number of Sperner families of subsets of S has been investigated by many authors beginning with Dedekind. The best recent result [25] is that this number is greater than $\exp_2 [\binom{n}{[n/2]} (1 + cn^{-1/2} \log n)]$.

Katona [14] and Kruskal [29] have considered a related question. Given an f member family F of k-element subsets of S. Let G consist of the $(k+1)$ element subsets which contain one or more members of F.

How small can $|G|$ be, given f? His result is an exact one: f can be uniquely expressed as

$$\binom{r_1}{k-1} + \binom{r_2}{k-2} + \binom{r_3}{k-3} + \ldots + \binom{r_m}{k-m}$$

with $r_1 > r_2 > \ldots > r_m$. Then $|G| \geq \binom{r_2}{k} + \binom{r_3}{k-2} + \ldots + \binom{r_m}{k-m}$.

Meshalkin [32] has obtained a result on families of partitions of n-element sets into k labelled blocks restricted so that no blocks properly contains a block with the same label. The result, the largest k-nomial coefficient, is really a corollary of the Lubell–Meshalkin identity.

5. Union and intersection restrictions

There are a number of problems that have been studied which involve intersection restriction involving three or more subsets. The following set of limitations have been considered.

(a) F_1 is limited in that no three members A, B, C satisfy $A \cup B = C$ ($A \cap B = C$ would be equivalent).

(b) F_2 obeys the restriction that no four members A, B, C, D satisfy $A \cup B = C$, $A \cap B = D$.

(c) No three members of A, B, C of F_3 satisfy $A \cup B = C$ or $A \cap B = C$.

(d) No three members of F_4 satisfy $A \cup B \supset C$ (equivalently, $A \cap B \subset C$).

(e) No three members of F_5 satisfy $A \cup B \subset C$.

(f) No 2^k members of F_{6k} form a Boolean algebra under union and intersection.

(g) Given any k members $A_1 \ldots A_k$ of F_{7k}, the intersection $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_k$ is nonempty and the same restriction holds if any or all A_j's are replaced by their complements.

(h) Given two disjoint members of F_8, their union is a nonmember $A \cup B = C$, $A \cap B = \emptyset$ is excluded.

Results on these areas have been as follows:

(a) The restriction $A \cup B \neq C$ would seem to limit F_1 to $\binom{n}{[n/2]}(1+cn^{-1})$ members. The best limitation [26] obtained has been $\binom{n}{[n/2]}(1+c/n^{-1/2})$.

(b) Under the restriction $A \cup B \neq C$ or $A \cap B \neq D$, F_2 can have $c\, 2^n n^{-1/4}$ members. Upper and lower bounds of this form have been obtained; they may or may not be equal [8].

(c) The restriction stated above probably requires that F_3 can have at most $\binom{n}{[n/2]} + 1$ members for n even. Clements (private communication) has found examples having this many members.

(d) The number of members of F_4 is exponentially small compared to 2^n. Little is known about this limitation.

(e) Under $A \cup B \not\subset C$, the size of F_5 cannot exceed $\binom{n}{[n/2]}(1+c/n)$ which bound can be achieved.

(f) Little is known beyond case (b) above for this restriction.

(g) This problem has been considered by Joel Spencer (private communication). For $k = 2$, it is resolved that the bound is $\binom{n-1}{[n-1/2]}$. For $k = 3$, upper and lower bounds of the form C^n with $1 \leq c \leq 2$ have been obtained. They are not close to one another. This restriction includes that of (d), namely $A_1 \cap A_2 \not\supset A_3$ for $k \geq 3$.

(h) Roughly speaking, under these restrictions, the family G can contain all sets having $\frac{1}{3}n$ to $\frac{2}{3}n$ elements. Best results have been obtained for $n = 3k + 1$. For $n = 3k$, $3k + 2$, there is a slight gap between the best bound and the best existing results.

Another set of related problems are due to Erdös and Moser [9]. Rewards for their solution are available from the former author.

"Find bounds for $f(n)$ = the least number of subsets of a set A of n elements such that every subset of A is the union of two of the $f(n)$ subsets. It is easy to prove that

$$\sqrt{2} \cdot 2^n < f(2n) \leq 2 \cdot 2^n .$$

We offer \$25.00 deciding (with proof) whether $f(2n)$ is $>$ or $<$ $(1.75)^2$ for sufficiently large n."

"Find bounds on $f(n)$ = the largest number of subsets $A_1, A_2, \ldots, A_{f(n)}$ of a set of n elements such that the $\binom{f(n)}{2}$ sets $A_i \cup A_j$, $1 < i \leq f(n)$, are distinct. We can prove that for large n,

$$(1+\epsilon_1)^n < f(n) < (1+\epsilon_2)^n ,$$

where $0 < \epsilon_1 < \epsilon_2 < 1$, and offer \$25.00 for finding ϵ_1, ϵ_2 with $\epsilon_2/\epsilon_1 \leq 1.01$."

6. Miscellany

Another kind of problem involves families of sets of a specified size out of a not necessarily specified set.

Two problems of this kind are:

(1) Suppose that no three subsets have pairwise the same intersection, and they are of size k. How many can there be?

(2) Suppose that any subset which interests all members of the family contains at least one member. How few members can the family have?

The property mentioned in (2), called "Property B", has been extensively studied. For $n = 3$, one can find a 7 member family with this property. For $n = 4$, the smallest family size is unknown but probably around 20. Erdös [5, 6] has an upper bound of $cn^2 2^n$ and Schmidt [35] has a lower bound of $2^n(1 + 4/n)^{-1}$. These results have recently been improved slightly by Herzog and Schönheim (private communication).

The best bound for problem (1) here is probably of the form c^k. The best result obtained so far for an upper bound has been of the form $k!\, c^k$[3, 10].

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