EXTREMAL PROBLEMS ON GRAPHS AND HYPERGRAPHS

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In this short survey I will state many solved and unsolved problems. I will give almost no proofs and will try to give extensive references, so that the interested reader can find what is omitted here. G_r denotes an r-graph, $G_r(k)$ an r-graph with k vertices and $G_r(k;m)$ an r-graph with k vertices and m r-tuples. $K_r(t)$ denotes the complete r-graph of t vertices, i.e. the r-graph $G_r(t;\binom{t}{r})$. $K_r(t,\ldots,t)$ denotes the r-graph of rt vertices and t^r r-tuples where the vertices are split into r classes of t vertices each and every r-tuple contains one and only one vertex of each class.

If G_r is an r-graph then $f(n;G_r)$ is the smallest integer so that every $G_r(n;f(n;G_r))$ contains our G_r as a subgraph.

In 1940 Turán [1] proved that if $n \equiv s \pmod{t-1}$, then

$$f(n;K_2(t)) = \frac{t-2}{2(t-1)}(n^2 - s^2) + \binom{s}{2}.$$

He also proved that the only $G_2(n;K_2(t)-1)$ which does not contain a $K_2(t)$ is the complete $(t-1)$-partite graph $K_2(m_1,\ldots,m_{t-1})$ where $m_1 + \ldots + m_{t-1} = n$ and the summands are as nearly equal as possible. Turán's paper initiated the systematic study of extremal properties of graphs and hypergraphs. Turán posed the very beautiful and difficult problem of determining $f(n ; K_r(t))$ for $r > 2$ and $t > r$. This problem is unsolved. It is not hard to see (Katona - Nemetz-Simonovits [2]) that

$$\lim_{n=\infty} f(n ; K_r(t)) / \binom{n}{r} = c_{r,t}$$

always exists, but the value of $c_{r,t}$ is unknown for every $r > 2$, $t > r$ though Turán has some plausible conjectures.

In fact very few exact results are known for $r > 2$. Before I state systematically the problems and results in our subject I mention the following recent result of B. Bollobás who proved the following conjecture of Katona: Every $G_3(n ; [\frac{n}{3}][\frac{n+1}{3}][\frac{n+1}{3}] + 1)$ contains three triples so that one of them contains the symmetric difference of the other two. The result is easily seen to be best possible. The paper of Bollobás will be published soon.

$f_r(n ; k, l)$ is the smallest integer so that every $G_r(n ; f_r(n ; k, l)$ contains at least one $G_r(k ; l)$ as a subgraph in other words the structure of our $G_r(k,l)$ is not specified. The study of $f_r(n ; k, l)$ in general is simpler (but perhaps less interesting) than that of $f_r(n ; G_r(k ; l)$. In the first chapter I discuss $r = 2$ and in the second I state some of our meagre knowledge for $r > 2$

$r = 2.$

As far as I know the first paper which tried to study systematically extremal properties of graphs was [3]. First I state the following general theorem of Simonovits - Stone and myself [4]. Let G be a graph of chromatic number k. Then

(1) $$\lim_{n=\infty} f(n; G)/\binom{n}{2} = 1 - \frac{1}{k-1} .$$

In view of (1) we will mostly restrict ourselves to the study of bipartite graphs. A result of Kövari, the Turáns and myself states that [5] (the c's denote absolute constants not necessarily the same if they occur in different formulas)

(2) $$f(n; K_2(t,t)) < c_1 n^{2-\frac{1}{t}}$$

In other words every $G(n; [c_1 n^{2-\frac{1}{t}}])$ contains a complete bipartite graph $K_2(t,t)$ as a subgraph if c_1 is sufficiently large. We conjectured that (2) is best possible but this has been proved only for $t = 2$ and $t = 3$ [6]. Denote by C_k a circuit having k edges. Brown, V. T. Sós, Rényi and I proved that [6]

$$\lim_{n=\infty} f(n; C_4)/n^{3/2} = \frac{1}{2} .$$

Our proof in fact gives

(3) $$f(n; C_4) \le \frac{n^{3/2}}{2} + \frac{n}{4} + \sigma(n)$$

and in fact many of us conjectured that

$$f(n; C_4) = \frac{n^2}{4} + \frac{n}{4} + \sigma(n) .$$

Let $n = p^2 + p + 1$ where p is a power of a prime. Our method gives

(4) $$f(n; C_4) \ge \frac{(p+1)^2 p'}{2} + 1 .$$

It would be nice if we would have equality in (4).

I proved that

$$f(n, C_{2k}) < c_1 n^{1+\frac{1}{k}} \tag{5}$$

I never published a proof of (5) since my proof was messy and perhaps even not quite accurate and I lacked the incentive to fix everything up since I never could settle various related sharper conjectures--all these have now been proved by Bondy and Simonovits--their paper will soon appear. Probably (5) is best possible but this has been proved only for $k = 2$ and $k = 3$ (Singleton). For further results on cycles see the papers of Bondy and Woodall [7].

Gallai and I proved that every $G(n\,;[\frac{1}{2}(k-1)n]+1)$ contains a path of lenght k and V. T. Sós and I conjectured that every such graph contains every tree of k edges [1]. No progress has been made with this conjecture [8].

Let G be a bipartite graph. I conjectured that $f(n\,;G)/n^{1+\alpha}$ tends to a finite non zero limit for some α of the form $\frac{1}{k}$ or $1-\frac{1}{k}(k = 2,3,\ldots)$. Simonovits and I disproved this conjecture [9]. We still think that for every bipartite G there is an $\alpha, 1 < \alpha < 2$, for which

$$\lim_{n=\infty} f(n\,;G)/n^{\alpha} = c(G) \quad , \quad 0 < c(G) < \infty \tag{6}$$

but the set of these α's is everywhere dense in (1,2). Probably the α in (6) is always rational.

Let G be the skeleton of a cube. Simonovits and I proved [9]

$$f(n\,;G) < cn^{8/5} \tag{7}$$

We could not decide whether (7) is best possible.

Simonovits and I determined $f(n;G)$ if G is the skeleton of an octahedron [10] and Simonovits determined $f(n;G)$ if G is the skeleton of the icosahedron

Before I close this chapter I state two simple unsolved questions considered by Simonovits and myself. Let G_k be the graph having the $1+k+\binom{k}{2}$ vertices $x_1; y_1,\dots,y_k$, and $z_{i,j}$, $1 \le i < j \le k$. x_1 is joined to $y_1,\dots,y_k$ and $z_{i,j}$ is joined to x_i and x_j. Is it true that

$$f(n;G_k) < c_k n^{3/2} \tag{8}$$

I proved (8) for $k = 3$ [11]. G_k contains rectangles so that (8) if true is best possible. Denote by $G - x$ the graph obtained from G by removing the vertex x and all edges incident to it. Is it true that for every k

$$\lim_{n=\infty} f(n;G_k - x)/n^{3/2} = 0 .$$

II

Now we discuss some problems and results for $r > 2$. A few years ago I proved that for every r and t there is an $\varepsilon_{r,t}$ so that every $G_r(n,[n^{r-\varepsilon_{r,t}}])$ contains a $K_r^{(r)}(t,\dots,t)$ [12]. For $r = 2$ this is the theorem of Kövári and the Turáns stated in (2). For $r > 2$, $t \ge 2$ the exact value of $\varepsilon_{r,t}$ is not known. This result implies that every $G_r(n;[\varepsilon n^r])$ contains a subgraph of $m = m(n) \to \infty$ as vertices which has at least m^r/r^r edges. I conjecture that the following result is true: There is an absolute constant $c > \frac{1}{r^r}$ so that every

$G_r(n;[\frac{n^r}{r^r}(1+\varepsilon)])$ contains a subgraph $G_r(m;[cm^r])$ where $m = m(n) \to \infty$

as $n \to \infty$. The case $r = 2$ is completely cleared up by the result of Stone and myself [13][4]. For $r > 2$ and for $r = 2$ and directed graphs or multigraphs many unsolved problems remain (see a forthcoming paper of Brown, Simonovits and myself).

In two forthcoming papers W. Brown, V. T. Sós and I began a systematic study of extremal problems for r-graphs. Before stating some of our results I state the most attractive unsolved problem:

Is it true that

$$f(n;G_3(6,3)/n^2 \to 0 \quad . \tag{9}$$

We proved $f(n;G_3(6;3) > cn^{3/2}$ and it seems likely that in fact $f(n;G_3(6;3)) < n^{2-\varepsilon}$ for some $\varepsilon > 0$, but we could not even prove (9). Very recently Szemerédi states that he proved (9).

We prove that every $G_3(n;[c_1 n^{5/2}])$ contains a triangulation of the sphere for sufficiently large c_1 (the result fails if c_1 is small). Simonovits independently proved that every $G_3(n;[cn^{3-\frac{1}{k}}])$ contains a k- tuple pyramid and that for $k = 2$ and $k = 3$ the exponent is best possible.

To conclude I state some of the problems, results, for 3-graphs. In our paper for simplicity we take $r = 3$ (some of the results hold with appropriate change for $r > 3$). We have

$$\lim_{n=\infty} \frac{1}{n^2} f(n;G_3(4,2) = \frac{1}{6} \quad .$$

but the determination of

$$\lim_{n=\infty} \frac{1}{n^3} f(n;G_3(4;3))$$

seems to be very difficult, perhaps as difficult as Turán's problem on $f(n;K_3(4))$.

(10) $$c_1 n^{5/2} < f(n;G_3(5;4) < c_2 n^{5/2}$$

We have not been able to get an asymptotic formula for $f(n;G_3(5;4)$.

By the probabilistic method we proved

(11) $$f(n;G_3(k,k-1) > n^{2+\varepsilon_k}$$

but except for $k = 5$ we do not know the exact value of ε_k . It is easy to see that for every $k > 3$ 0 ,

(12) $$c_1^{(k)} n^2 < f(n;G_3(k,k-2) < c_2^{(k)} n^2 .$$

As stated previously $c_1^{(k)} = c_2^{(k)} = \frac{1}{6}$, but for $k > 4$ we have no asymptotic formula for $f(n;G_3(k,k-2))$. I would not be surprised if it would turn out that for every k

(13) $$\lim_{n=\infty} \frac{1}{n^2} f(n;G_3(k,k-2)) = \frac{1}{6} .$$

The only argument in favor of this conjecture (the conjecture may easily turn out to be nonsense) is the following

Theorem. Every $G_3(n;[\frac{1}{3}\binom{n}{2}]+1)$ contains either a $G_3(5;3)$ or a $G_3(6;4)$.

Let $x_1,\dots,x_n$ be the vertices of our graph and $T_1,\dots,T_\ell$, $\ell = [\frac{1}{3}\binom{n}{2}]+1$ its triples. Since $3\ell > \binom{n}{2}$ at least one pair say

(x_1,x_2) is contained in two triples say T_1 and T_2. We can clearly assume that no pair is contained in three triples for otherwise our graph would contain a $G_3(5;3)$. Also if say $T_1 = (x_1,x_2,x_3)$ and $T_2 = (x_1,x_2,x_4)$, no T_i can contain (x_3,x_4) since if $T_i = (x_3,x_4,x_5)$, then $G_3(x_1,x_2,x_3,x_4,x_5)$ contains three triples. Thus to every pair which is contained in two triples there corresponds a pair not contained in any triple. This correspondence can not be one to one since otherwise ℓ would be x at most $[\frac{1}{3}\binom{n}{2}]$. Thus there must be two triples (x_1,x_5,x_6) and (x_2,x_5,x_6) which have a common pair and also exclude (x_1,x_2). But then $G_3(x_1,\dots,x_6)$ contains four triples and thus our theorem is proved. I hope this argument can be improved.

It is clear that many more problems could be formulated.

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