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ON THE CONNECTION BETWEEN CHROMATIC NUMBER, MAXIMAL CLIQUE AND MINIMAL DEGREE OF A GRAPH

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Abstract. Let G_n be a graph of n vertices, having chromatic number r which contains no complete graph of r vertices. Then G_n contains a vertex of degree not exceeding $n(3r-7)/(3r-4)$. The result is essentially best possible.

0. Introduction

In this paper we shall use the following notations:

G_n denotes a graph of n vertices, without loops and multiple edges;

$V(G_n)$ respectively $E(G_n)$ the set of vertices respectively the set of edges of G_n;

$(x, y) \in G_n$ means: for $x, y \in V(G_n)$, the edge $(x, y) \in E(G_n)$;

$\sigma(x)$ is the valency of $x \in V(G_n)$;

$V(x)$ is the star of x (i.e., $V(x) = \{y: (x, y) \in E(G_n)\}$), and $S(x)$ the subgraph induced by $V(x)$;

$\chi(G)$ denotes the chromatic number of G;

$A \subset V(G_n)$ is an independent set if no two vertices of A are joined by an edge;

K_r denotes a complete graph of r vertices;

$G\langle v_1, ..., v_r\rangle$ denotes a complete r chromatic graph with independent sets $|V_i| = v_i$ $(i = 1, ..., r)$; if $v_i = v$, we use the notation $G^r\langle v\rangle$.

We remind the reader of the following well-known results:

Theorem 0.1 (Turán's theorem [4]). *For any graph G_n, if $n \equiv l \bmod (r-1)$ and $0 \leqslant l \leqslant r-1$, at most one of the following properties can hold:*

$$(1) \qquad K_r \not\subset G_n,$$

$$(2) \qquad |E(G_n)| > (n^2-l^2)\frac{r-2}{2(r-1)} + \binom{l}{2}.$$

The theorem is best possible in the following sense:
$K_r \not\subset G_n$ and $|E(G_n)| = (n^2-l^2)(r-2)/(2(r-1)) + \binom{l}{2}$ if and only if

$$G_n = G_n \langle v_1, \ldots, v_{r-1} \rangle,$$

where $\Sigma_{i=1}^{r-1} v_i = n$ and $|v_i - v_j| \leqslant 1$ for $1 \leqslant i,j \leqslant r-1$.

A consequence of the above is the following:

Theorem 0.2 (Zarankiewicz's theorem [5]). *For any graph G_n, at most one of the following properties can hold:*

(3) $\quad K_r \not\subset G_n,$

(4) $\quad \min_{x \in V(G_n)} \sigma(x) > \left[n \frac{r-2}{r-1} \right].$

The theorem is best possible too, but for some n and r, there are several extreme graphs. We shall discuss in Section 2 the question of the extreme graphs.

In these theorems we see the connection between the maximal complete subgraph contained in G_n and $|E(G_n)|$ respectively $\min_{x \in V(G_n)} \sigma(x)$. The connection of these quantities with $\chi(G_n)$ is shown already by the following theorem.

Theorem 0.3 (Brook's theorem [2]). *Let $r \geqslant 4$. For any graph G_n, at most two of the following properties can hold*:

(5) $\quad K_r \not\subset G_n,$

(6) $\quad \max_{x \in V(G_n)} \sigma(x) \leqslant r-1,$

(7) $\quad \chi(G_n) \geqslant r.$

P. Erdös, T. Gallai, B. Andrásfai and M. Simonovits [3] determined the largest integer $f_\chi(r,n)$ for which there is a graph G of n vertices and $f_\chi(r,n)$ edges which is χ-chromatic and contains no K_r. It is natural to investigate the analogous question for the problem of Zarankiewicz. It is well known that if we make no assumptions about chromatic numbers, then Zarankiewicz's theorem is an easy consequence of Turán's theorem, and in most of the cases (e.g. for $n > n_0(r)$), the extreme graphs coincide. On the other hand if we make assumptions on chromatic numbers, the situation is completely different.

1.

In the present paper we prove the following theorem:

Theorem 1.1. *Let $r \geqslant 3$. For any graph G_n, at most two of the following properties can hold*:

(8) $K_r \not\subset G_n$,

(9) $\min_{x \in V(G_n)} \sigma(x) > \frac{3r-7}{3r-4} n$,

(10) $\chi(G_n) \geqslant r$.

The theorem is best possible in the following sense:

Let $3r-4/n$, then there exists a unique extreme graph G_n^* with the following properties:

$$K_r \not\subset G_n^*,$$
$$\min_{x \in V(G_n^*)} \sigma(x) = \frac{3r-7}{3r-4} n,$$
$$\chi(G_n^*) = r.$$

This graph is defined as follows: Let $V = V(G_n^*)$ and

$$\{V_1, ..., V_{r-3}, U_1, ..., U_5\}$$

a disjoint partition of V, where

$$|V_i| = \frac{3n}{3r-4} \quad \text{for} \quad 1 \leqslant i \leqslant r-3,$$

$$|U_j| = \frac{n}{3r-4} \quad \text{for} \quad 1 \leqslant j \leqslant 5.$$

The edges of G_n^* are defined as follows: if $x \in V_i$, then $(x, y) \in G_n^* \Leftrightarrow y \notin V_i$ $(i = 1, ..., r-3)$, if $x \in U_j$, then $(x, y) \in G_n^* \Leftrightarrow y \in \bigcup_{i=1}^{r-3} V_i$ or $y \in U_{j-1} \cup U_{j+1}$, where $U_0 \equiv U_5$, $U_6 \equiv U_1$.

The extremal graph for $r = 3$ respectively $r > 3$ is indicated in Fig. 1 respectively Fig. 2 (where a number 5 in a circle indicates an independent set of 5 vertices, a line between two circles indicates that every vertex in one of these circles is connected with every vertex of the other one by an edge).

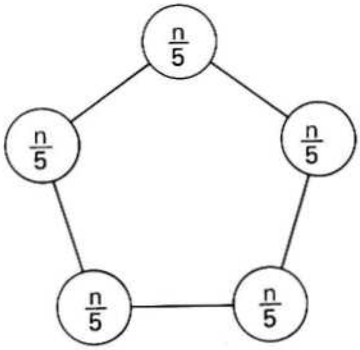

Fig. 1.

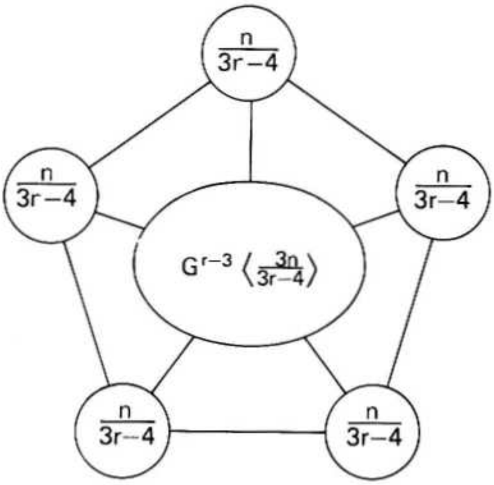

Fig. 2.

Proof of Theorem 1.1. We shall prove the theorem by induction on r. We need the following lemmas.

Lemma 1.2. *The theorem is true for* $r = 3$.

Lemma 1.3. *If the theorem is true for* $r-1$ *and* G_n *satisfies* (8) *and* (9), *then for every* $x \in V$, *we have*

(11) $\chi(S(x)) \leqslant r-2.$

Lemma 1.4. *If G_n would satisfy* (8), (9) *and* (10), *then it has the following property* P:

There is a disjoint partition

$$\{A_1, ..., A_{r-1}, D\}$$

of $V(G_n)$ satisfying the following conditions:

there exist points a_i and subsets B_i $(i = 1, ..., r-1)$ for which

(i) $a_i \in B_i \subset A_i$;

(ii) $(a_i, x) \in G_n$ *if* $x \in B_j$ *and* $i \neq j$, $1 \leqslant j \leqslant r-1$;

(iii) $|B_i| > 2n/(3r-4)$;

(iv) A_i *is independent,* $|\bigcup_{i=1}^{r-1} A_i| \geqslant n - n/(3r-4)(r-2)$;

(v) *for any* $y \in D$ *and any* j $(j = 1, ..., r-1)$, *there exists at least one* $x \in A_j$ *with* $(x, y) \in A_j$;

(vi) *for any* $y \in D$, *there is at least one* j *for which* $(x, y) \notin G_n$ *if* $x \in B_j \setminus \{a_j\}$.

Lemma 1.5. *If G_n has property* P, *and if it also satisfies* (9) *and* (10), *then it will contain a* $K_r \subset G_n$.

Hence to prove our theorem we only have to prove our four lemmas.

Proof of Lemma 1.2. Suppose that (8) and (10) hold with $r = 3$. Let us take in G_n a shortest circuit with odd length, with vertices $a_1, ..., a_k$.

Because of (8), any vertex $x \in V$, $x \neq a_i$ $(i = 1, ..., k)$, is connected with at most two a_i's in G_n. Therefore for the number of edges E^* of type (x, a_i), where $x \neq a_j$ $(i, j = 1, ..., k)$, we have on the one hand

$$|E^*| \leqslant 2(n-k).$$

On the other hand, if $\min_{x \in V} \sigma(x) = \rho$, then

$$|E^*| = \sum \sigma(a_i) - 2k \geqslant k\rho - 2k,$$

which gives

$$\rho \leqslant 2n/k \leqslant 2n/5,$$

i.e., (9) cannot hold.

Remark 1.6. Observe that for the case $r = 3$, we proved a bit more than in Theorem 1.1, namely the following: if $K_3 \not\subset G_n$, $\chi(G_n) \geqslant 3$ and the shortest odd circuit has length k, then

$$\min_{x \in V} \sigma(x) \leqslant 2n/k.$$

Proof of Lemma 1.3. Consider the induced subgraph $S(x)$ and a vertex $y \in V(S(x))$. Let $\sigma^*(y)$ be the degree of y in $S(x)$. If (9) holds for G_n, we have

$$\sigma^*(y) \geqslant \sigma(y) - (n - \sigma(x)) \geqslant \sigma(x) - 3n/(3r-4) > \sigma(x)(1 - 3/(3r-7)),$$

where $|V(S(x))| = \sigma(x)$. This means that if (8) and (9) hold for G_n, then the same holds for $S(x)$ with $r-1$ instead of r. By the induction hypotheses, $\chi(S(x)) \leqslant r-2$.

Proof of Lemma 1.4. Assume that (8) and (9) hold. First of all we construct an induced subgraph with at least $n - n/(3r-4)(r-2)$ vertices which is $r-1$ chromatic and contains a K_{r-1}. Evidently, we may suppose that $K_{r-1} \subset G_n$. Let

$$V(K_{r-1}) = \{a_1, \ldots, a_{r-1}\},$$

$$\chi(S(a_1)) \leqslant r-2,$$

$$|V(S(a_1))| > \frac{3r-7}{3r-4} n.$$

Then there is an a_i, say a_2, which is in an independent set $C_2 \subset V(S(a_1))$ having at most

$$\frac{3r-7}{3r-4} \frac{n}{r-2}$$

vertices. If we consider $S(a_2)$, this contains the vertices $a_1, a_3, \ldots, a_{r-1}$ and so

$$\chi(S(a_2)) = r-2.$$

Let us consider a colouring of $S(a_2)$ by $r-2$ colours, and denote by C_j the independent set of it which contains a_j. For $\mathbf{U}_{i=1}^{r-1} C_i$, we have

$$a_i \in C_i \qquad (i = 1, \ldots, r-1)$$

and

$$\left|\bigcup_{i=1}^{r-1} C_i\right| \geqslant \sigma(a_2) + |C_2| > \left(\frac{3r-7}{3r-4} + \frac{1}{r-2}\,\frac{3r-7}{3r-4}\right)n = n - \frac{n}{(3r-4)(r-2)}.$$

Now we define the sets B_i as the set of all vertices x, for which

(12) $\qquad (x, a_j) \in E(G_n) \quad$ if $i \neq j$.

We have $a_i \in B_i$. We shall show that

(13) $\qquad a_i \in B_i \subset C_i$.

This is evident for $i \neq 2$, because for any $x \in B_i \neq B_2$,

$$x \in V(S(a_2)) = \bigcup_{i=1}^{r-1} C_i \backslash C_2,$$

but

$$x \notin C_j \qquad \text{for } i \neq j,$$

by (12) and the independence of C_j. For $i = 2$, if $x \in B_2$, then $(x, a_1) \in E(G_n)$, i.e., $x \in S(a_1)$. C_2 was the independent set at a good-colouring of $S(a_1)$ containing a_2; i.e. a colouring of $S(a_1)$ by $r-2$ colours, where the vertices of the same colour form an independent set; all the other classes contain an a_j $(j > 2)$ which is joined to x, consequently, x must be in C_2.

Finally, we show that

(14) $\qquad |B_i| > \dfrac{2n}{3r-4}$.

Let $V(G_n) - (\{a_1, ..., a_r\} \cup B_i) - S$. We count the number of edges E^{**} of type $(a_i, x) \in E(G_n)$, $i \neq 2$, $x \in S \cup B_i$. On the one hand we have from (9) that

(15) $\qquad |E^{**}| \geqslant (r-2)\left(\dfrac{3r-7}{3r-4}n - (r-3)\right)$.

On the other hand, since any $x \in S$ is connected with at most $r-3$ of the a_j's ($j \neq 2$, otherwise it would be in B_i), we have

(16)
$$\begin{aligned}|E^{**}| &\leqslant (r-2)|B_i| + (r-3)|S| \\ &= (r-2)|B_i| + (r-3)(u-|B_i|-r+2) \\ &= |B_i| + n(r-3) - (r-2)(r-3).\end{aligned}$$

(15) and (16) give (14).

Now we define the sets $A_i \subset V(G_n)$ as follows:

(a) $A_i \supset C_i$ $(i = 1, ..., r-1)$;

(b) A_i is independent $(i = 1, ..., r-1)$;

(c) $\bigcup_{i=1}^{r-1} A_i$ is maximal with the properties (a) and (b); for any $y \in \bigcup_{i=1}^{r-1} A_i$ and any $i \in \{1, ..., r-1\}, \{y\} \cup A_i$ is not independent.

To finish the proof of the lemma we only have to prove (vi). Our proof will be indirect.

Because of (10), $V - \bigcup_{i=1}^{r-1} A_i \neq \emptyset$. Assume that we have the vertex $y \in V - \bigcup_{i=1}^{r-1} A_i$ and the vertices $b_i \in B_i - \{a_i\}$, where $(b_i, y) \in E(G_n)$ for $i = 1, ..., r-1$. For these vertices we have from the construction that

$$(a_i, b_i) \notin E(G_n),$$

$$(a_i, b_j) \in E(G_n) \quad \text{for} \quad i \neq j,$$

$$(a_i, a_j) \in E(G_n) \quad \text{for} \quad i \neq j.$$

Let $F_1 = \{y, a_i, b_i \ (i = 1, ..., r-1)\}$ and $F_2 = V(G_n) - F_1$.

We shall count the number of edges E^{***} of type $(x, z) \in E(G_n)$, $x \in F_1$, $z \in F_2$, in two different ways, which will give the desired contradiction.

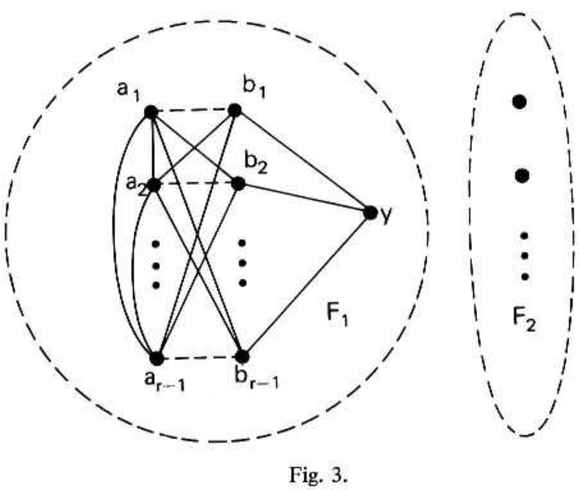

Fig. 3.

For this purpose we prove that every vertex in F_2 is not connected with at least 3 vertices in F_1. If there would be a vertex $z \in F_2$ for which there are at most 2 such vertices, we would have a contradiction either with (8) or with Lemma 1.3.

Namely for the two vertices c and d not joined with x, we have the following 6 possibilities:

(i) $c = a_i$, $d = a_j$. In this case, $S(a_l)$ for $l \neq i, j$ would be $(r-1)$-chromatic.

(ii) $c = b_i$, $d = b_j$, $i \neq j$. In this case, there would be a $K_r \subset G_n$.

(iii) $c = a_i$, $d = b_j$, $i \neq j$. In this case, there would be a $K_r \subset G_n$.

(iv) $c = a_i$, $d = b_i$ for some i. In this case, $S(z)$ would be $(r-1)$-chromatic.

(v) $c = y$, $d = a_i$, and

(vi) $c = y$, $d = b_i$, in these cases, we would have $K_r \subset G_n$.

Therefore, for the number of edges E^{***} of type (x, z), where $x \in F_1$, $z \in F_2$, we have on the one hand

$$(17) \qquad |E^{***}| \leqslant (2r-4)(n-(2r-1)),$$

on the other hand by (9)

$$|E^{***}| \geqslant (2r-1)\left(\frac{3r-7}{3r-4}\,n + 1\right) - 2\,E_0,$$

where E_0 is the number of edges in the induced subgraph $G_n(F_1)$. Since $K_r \not\subset G_n(F_1)$, $|F_1| = 2r-1$ and $\chi(G_n(F_1)) = r$, according to Turán's theorem we have that

$$E_0 < \binom{2r-1}{2} - (r+1);$$

$$(18) \qquad |E^{***}| > (2r-1)\left(\frac{3r-7}{3r-4}\,n + 1\right) - (2r-1)(2r-2) + 2(r+1).$$

From (17) and (18) we have $r < 3$.

Proof of Lemma 1.5. Using Lemma 1.4, we construct a $K_r \subset G_n$ step by step in the following way:

(a) For an arbitrary $x_0 \in D$, let B_1 be the B_i for which

$$(19) \qquad (x_0, b) \notin E(G_n) \quad \text{if } b \in B_1 \setminus \{a_1\}.$$

(From Lemma 1.4 we know that such a B_1 exists.)

(b) Let $x_1 \in A_1$ be a vertex for which

$$(x_0, x_1) \in E(G_n).$$

(c) Let $X_i = \{x\colon x \in A_i, (x, x_0) \notin E(G_n) \text{ for } i \neq 1\}$, and we determine the indices so that

$$|X_2| = \max |X_i|\,. \tag{20}$$

Let $x_2 \in A_2$ be a vertex for which

$$(x_0, x_2), (x_1, x_2) \in E(G_n). \tag{21}$$

Such a vertex exists, because in B_1 we have at least $2n/(3r-4)$ vertices which are connected neither with x_0 nor with x_1, which means (because of (9)) that we have altogether less than $2n/(3r-4)$ vertices which are not joined to at least one of them. Since $|A_2| > 2n/(3r-4)$, we have an $x_2 \in A_2$ for which (21) holds.

(d) If $x_0, ..., x_j$ ($x_v \in A_v$ for $1 \leqslant v \leqslant j$) are determined already, we define x_{j+1} ($j < r-1$) in the following way:

$$(x_v, x_{j+1}) \in E(G_n) \quad \text{for } 0 \leqslant v \leqslant j.$$

The following reasoning shows that such a vertex exists:

For $x_v \in A_v$ $(1 \leqslant v \leqslant j)$, let

$$d_v = |\{a\colon (a, x_v) \notin E(G_n), a \in \textstyle\bigcup_{i=1}^{r-1} A_j \setminus A_v\}|\,.$$

Because of $x_v \in A_v$ and A_v is independent, we have

$$d_v \leqslant 3n/(3r-4) - |A_i|,$$

and consequently, using $\bigcup_{i=1}^{r-1} A_i \geqslant n - n/(3r-4)(r-2)$ and Lemma 1.4(iv),

$$\begin{aligned} \sum_{v=1}^{j} d_v &< (j-1)\,\frac{3n}{3r-4} - \sum_{i=1}^{j-1} |A_i| \\ &\leqslant (r-1)\,\frac{3n}{3r-4} - \sum_{i=1}^{r-1} |A_i| \\ &\leqslant \frac{n}{3r-4} + \frac{n}{(3r-4)(r-2)}\,. \end{aligned} \tag{22}$$

Let

$$d_0 = |\{a\colon (a, x_0) \notin E(G_n), a \in \bigcup_{i=3}^{r-1} A_i\}| .$$

From (19), (20) and (9), we have that

$$(23) \qquad d_0 \leqslant (n - \sigma(x_0) - |B_1| + 1)\left(1 - \frac{1}{r-2}\right) \leqslant \frac{n}{3r-4}\frac{r-3}{r-2} .$$

(22) and (23) give that

$$\sum_{v=0}^{j} d_v \leqslant \frac{2n}{3r-4} .$$

Since $|A_{j+1}| > 2n/(3r-4)$ for $j \geqslant 2$, we have an $x_{j+1} \in A_{j+1}$ for which $(x_v, x_{j+1}) \in E(G_n)$ for $0 \leqslant v \leqslant j$. This completes the construction of our $K_r \subset G_n$, hence Lemma 1.5 and Theorem 1.1 is proved.

With a little more detailed reasoning, our proof gives the uniqueness of the extreme graph in case $(3r-4)/n$.

The following questions seem interesting: Assume $K_r \not\subset G_n$ and $\chi(G_n) \geqslant l \geqslant r$. What can be said about $\min_{x \in V(G_n)} \sigma(x)$? Erdös and Simonovits proved that if $r = 3$, then $\min_{x \in V(G_n)} \sigma(x) \geqslant (\frac{1}{3} + o(1))n$.

2.

We denote by T respectively Z graphs, the extreme graphs belonging to Turán's respectively Zarankiewicz's theorem.

For $r \geqslant 2$, a graph G_n is a Z graph if

$$K_r \not\subset G_n ,$$
$$\min_{x \in V(G_n)} \sigma(x) = \left[n\frac{r-2}{r-1}\right].$$

Theorem 0.1 implies Theorem 0.2 but the Z graphs are not known generally. One can see easily that for $n = q(r-1)$, the Z graphs and the T graphs are the same. In general, any **T** graph is a Z graph too even if we omit "a few" of its edges. It is interesting that for fixed r, all the **T** graphs are $(r-1)$-chromatic but there exists a Z graph with chromatic numbers $\geqslant r$. In the simple case $r = 3$, all Z graphs are given in [1, Theorem 2.4]. In general, there exist many types of the Z graphs although giving all of them seems to be hopeless. However, in the case $r = 4$ we can give all the Z graphs.

Proposition 2.1. *In the case $r = 4$, there are exactly seven* Z *graphs with chromatic number* $\geqslant 4$ *(see Fig.* 4).

The proof of Proposition 2.1 is a somewhat lengthy discussion of several cases and we leave it to the reader.

We can get an upper bound for the number of vertices of the Z graphs with chromatic number $\geqslant r$. For general r, Gallai conjectured that every Z graph with chromatic number $\geqslant r$ has fewer than cr^2 vertices. This conjecture follows easily from Theorem 1.1. In fact, let $\chi(G_n) \geqslant r$ and G_n a Z graph. Put $n = q(r-1) + d$ $(d = 1, ..., r-2)$, then we have

$$(24) \qquad \sigma_{min} = \min_{x \in V(G_n)} \sigma(x) = \left[n \frac{r-2}{r-1} \right] = q(r-2) + d - 1,$$

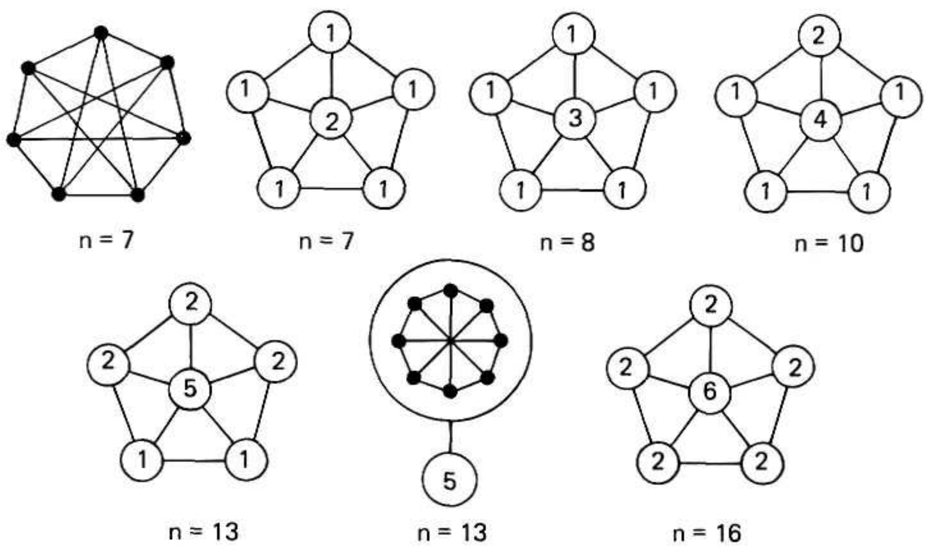

Fig. 4.

and by Theorem 1.1,

$$q(r-2)+d-1 \leqslant \frac{3r-7}{3r-4}\,(q(r-1)+d).$$

Thus $q \leqslant 3r-4-3d$. Hence

(25) $\qquad n \leqslant 3r^2-r(7+3d)+4(d+1);$

equality if and only if $G_n = G_n^*$.

Let us consider the special case of the greatest remainder $d = r-2$, and let G_n be a Z graph with chromatic number $\geqslant r$ ($r \geqslant 4$). By (24) and (25), we have $\sigma_{\min} = q(r-2)+r-3$, $q \leqslant 2$ and $n \leqslant 3r-4$. Now if $q = 1$, then $\sigma_{\min} = 2r-5$ and $n = 2r-3$, and if $q = 2$, then $\sigma_{\min} = 3r-7$ and $n = 3r-4$. In the case $q = 2$ by Theorem 1.1, G_n is identical to G_{3r-4}^*. We are going to show that there is no Z graph G_n for which $n = 2r-3$, $\sigma_{\min} = 2r-5$ and $\chi(G_n) \geqslant r$. To see this observe that since $\sigma_{\min} = 2r-5$ and $K_r \not\subset G_n$, we obtain our graph by omitting $r-2$ independent edges from a K_{2r-3}. However, we have $\chi(G_n) = r-1$, and it is a contradiction. Hence from Theorem 1.1 we obtain the following corollary:

Corollary 2.2. *For fixed $r > 4$, let G_n be a Z graph with chromatic number $\geqslant r$ and $n = q(r-1)+r-2$. Then $G_n = G_{3r-4}^*$ given in Theorem* 1.1 *(see Fig.* 5*).*

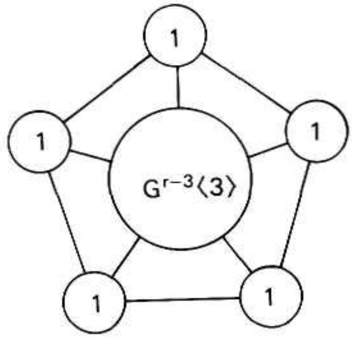

Fig. 5.

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