SOME PROBLEMS IN GRAPH THEORY

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In this short note I discuss two problems which we discussed during the hypergraph meeting.

I. The following conjecture is attributed to I. Zarins by Bondy and Chvatal: Let k be an integer and $n \geq n_0(k)$. Then if $G(n)$ ($G(n)$ is a graph of n vertices) is Hamiltonian and does not have k pairwise independent vertices (i.e. the complementary graph does not contain a complete subgraph of k vertices) then $G(n)$ is pancyclic, in other words it contains a circuit C_r for every $3 \leq r \leq n$. Bondy and Chvatal inform me that Zarins proved this for $k \leq 4$.

Using a method of Bondy, Chvatal and myself [1], I prove this conjecture. In fact I prove the following

Theorem. Let $n > 4k^4$ and $G(n)$ a Hamiltonian graph which does not contain a set of k pairwise independent vertices. Then $G(n)$ is pancyclic.

A theorem of Bondy and myself states that if $n \geq (k-1)(m-1)+1$ and $n \geq k^2 - 2$ then $G(n)$ either contains a C_m or k vertices which are mutually independent. Thus if $m \leq \frac{n}{k}$ our $G(n)$ must contain a C_m, and since by assumption $G(n)$ contains a C_n (i.e. it is Hamiltonian) it suffices to prove that $G(n)$ contains a C_m for $\frac{n}{k} < m < n$.

To complete our proof we only have to show that if

$$(1) \qquad \frac{n}{k} < m \leq n$$

and if

$G(n)$ contains a C_m it also contains a C_{m-1}. Let $x_1, \ldots, x_m$ be the vertices of our C_m; (x_i, x_{i+1}), $(x_{m+1} = x_1)$ $i = 1, \ldots, m$ are its edges. By (1) and $n > 4k^4$ we can assume that $m > 4k^3$. First of all observe that the set of k vertices $x_i, x_{i+2}, \ldots, x_{i+2(k-1)}$ can not all be independent (since otherwise $G(n)$ would contain a set of k pairwise independent vertices). Thus the graph $G(x_1, \ldots, x_m)$ spanned by the vertices $x_1, \ldots, x_m$ contains at least $2k^2$

(2) $\quad (x_{i_r}, x_{j_r})\ ,\ 2k > j_r - i_r \geq 2\ ,\ 1 \leq i_1 < j_1 < \cdots < i_s < j_s \leq n\ ,\ s \geq 2k^2$.

In fact we can assume $j_r - i_r > 2$ for if $j_r - i_r = 2$, we already have our C_{m-1} (our C_m without the vertex $x_{i_r+1} = x_{j_r-1}$) . We can also assume that no two vertices

$$(3) \qquad (x_u, x_v)\ ,\ i_r < u < v < j_r\ ,\ v - u > 1$$

are joined (otherwise we could replace (x_{i_r}, x_{j_r}) by (x_u, x_v)) . An edge (x_{i_r}, x_{j_r}) is called good if the valencies (in $G(x_1, \ldots, x_m)$) of every x_u , $i_r < u < j_r$ is $\geq k+2$. Observe that there must be at least one good edge for otherwise we would have at least $2k^2$ vertices of valency $\leq k+1$ and thus easily get a set of k pairwise independent vertices. Assume now that the edge (x_{i_r}, x_{j_r}) is good. Renumber for convenience of notation the vertices of C_m so that

$$y_1 = x_{i_r+1}\ ,\ y_2 = x_{i_r+2}\ ,\ \ldots,\ y_t = x_{j_r}\ ,\ \ldots,\ y_m = x_{i_r}\ ,\ t = j_r - i_r\ .$$

Let C_ℓ, $\ell < m$, be the largest circuit in $G(y_1, \ldots, y_m)$ which contains all the y_u , $t \leq u \leq m$ and perhaps some of the y_u , $1 \leq u < t$. If $\ell = m-1$ our proof is finished (we have our C_{m-1}) . Thus assume $\ell < m-1$, and this assumption will lead to a contradiction. Denote by $v(y_i)$ the valency of y_i . Since the edge $(y_m, y_t) = (x_{i_r}, y_{j_r})$ was good we have

$$(4) \qquad v(y_u) \geq k+2,\ 1 \leq u < t\ .$$

Let y_u be a vertex of C_m which is not a vertex of C_ℓ .
By (3) y_u is joined only to y_{u-1} and y_{u+1} in $G(y_1, \ldots, y_{t-1})$. Thus by (4) y_u is joined to at least k vertices of C_ℓ . Renumber again the vertices of C_ℓ: $z_1, z_2, \ldots, z_\ell, z_{\ell+1} = z_1$, so that (z_i, z_{i+1}) are the edges of C_ℓ . Let y_u be joined to $z_{i_1}, \ldots, z_{i_k}$. The vertices z_{i_r+1} , $1 \leq r \leq k$ must be independent for if any two of them are joined we get a $C_{\ell+1}$ containing $z_1, \ldots, z_\ell$ and y_u . But our graph can not have k pairwise independent vertices and this contradiction proves $\ell = m-1$ and our theorem.

The inequality $n > 4k^4$ is undoubtedly not best possible; perhaps the result holds for $n > c_1 k^2$ if c_1 is sufficiently large. A simple example shows that it certainly fails for $n < k^2/4$.

II. I state a few results on random graphs; proofs and more detailed statements of the results will be published later.

Denote by $G_r(n;k)$ an r-graph of n vertices and k edges (i.e. k r-tuples). It can be shown by probabilistic methods that for every fixed $\alpha > 0$ if $n \to \infty$, there is an r-graph

$$G_r(n;[\alpha n^r])$$

so that for every $m > c(\alpha,\varepsilon)(\log n)^{1/r-1}$ every spanned subgraph $G_r(m)$ of our $G_r(n)$ having m vertices has more than $(\alpha-\varepsilon)m^r$ and **fewer than** $(\alpha+\varepsilon)m^r$ r-tuples. In other words the distribution of the r-tuples is very uniform. It can also be shown that the above result is best possible, in other words it fails if $c(\alpha,\varepsilon)$ is sufficiently small. This and other related results will be discussed in our forthcoming book with Y. Spencer on applications of probability methods to combinatorial analysis.

For simplicity let us restrict ourselves to ordinary graphs, i.e. to $r = 2$ (this is not essential). Consider graphs $G(n;[n^{1+\alpha}])$ where $0 < \alpha < 1$ and in fact let $\alpha = \frac{1}{2}$. Denote by $f(G(m))$ the maximum number of edges of a subgraph of m vertices of our $G(n;[n^{3/2}])$.

<u>Theorem. There is a</u> $G(n;[n^{\frac{3}{2}}])$ <u>so that for every</u> $m < n^{\frac{1}{2}-\varepsilon}$

$$f(G(m)) < \frac{2}{\varepsilon} m$$

<u>but for every</u> $G(n;[n^{3/2}])$ <u>and for some</u> $m < n^{3/2-\varepsilon}$

$$f(G(m)) > \frac{m}{2\varepsilon} .$$

<u>Further there is a</u> $G(n;[n^{3/2}])$ <u>so that</u>

$$f(G[n^{1/2}]) < \frac{c_1 \log n}{\log \log n} n^{\frac{1}{2}}$$

<u>but for every</u> $G(n;[n^{3/2}])$

$$f(G[n^{\frac{1}{2}}]) > \frac{c_2 \log n}{\log \log n} n^{\frac{1}{2}} .$$

The method of proof is similar to those used by Spencer and myself [2].

REFERENCES

1. Bondy, A., and Erdös, P., Ramsey numbers for cycles in graphs, will appear in the Journal of Combinatorial Theory.

2. Chvátal, V., and Erdös, P., A note on Hamiltonian circuits, Discrete Math. 2(1972), 111-113.

3. Erdös, P. and Spencer, I., Imbalances in k - colorations, Networks, 1, (1972), 379-385.