FUNDAMENTA
MATHEMATICAE
LXXXI (1974)

Some remarks on set theory XI

by

P. Erdös and **A. Hajnal** (Budapest)

Abstract. Let $\varkappa, \lambda$ be infinite cardinals, $F \subset P(\varkappa)$, $A \not\subset B$ for $A \neq B \in F$; $|A| < \varkappa$ for $A \in F$. We give a necessary and sufficient condition (in ZFC) for the existence of an $F' \subset F$ with $|F'| = \varkappa$

$$|\varkappa - \bigcup F'| \geqslant \lambda .$$

§ 1. Let $\varkappa, \lambda$ be infinite cardinals, $F \subset P(\varkappa)$, $|F| = \varkappa$. Problems of the following type were considered in quite a few papers.

(1) Under what conditions for F does there exist $F' \subset F$, $|F'| = \varkappa$ such that $|\varkappa - \bigcup F'| \geqslant \lambda$?

(2) Assume f is a one-to-one mapping with domain $\varkappa$ and range F, $\xi \notin f(\xi)$. Under what conditions for F does the set mapping f have a free subset of cardinality λ, i.e. a subset $R \subset \varkappa$, $|R| = \lambda$ such that $\xi \notin f(\eta)$ for all $\xi, \eta \in R$?

It was proved in [3] that (1) holds with $\varkappa = \lambda$ provided there is a cardinal τ with $|A| < \tau < \varkappa$ for all $A \in F$. In [4] it was proved that the same condition also implies the stronger statement (2) with $\lambda = \varkappa$. It is obvious that if we only assume

$$(3) \qquad |A| < \varkappa \quad \text{for} \quad A \in F$$

we have to impose further conditions on F to obtain results of type (1) and (2).

The aim of this short note is to study the answer to (1) under the following simple condition

$$(4) \qquad A \not\subset B \quad \text{for all } A \neq B \in F .$$

Here we get a complete discussion without using G.C.H. and we give the solution of Problem 73 proposed in our paper [1] as well.

We mention that in a paper with A. Máté [2] we are going to study the answer to (2) under condition (3) and under some additional and more sophisticated conditions.

To have a short notation we say that $P(\varkappa, \lambda)$ is true if (1) holds for all $F \subset P(\varkappa)$, $|F| = \varkappa$, satisfying (3) and (4).

§ 2.

THEOREM 1. *Let $\varkappa$ be regular. Then $P(\varkappa, \lambda)$ holds iff either $\lambda < \varkappa$ and $\nu^\lambda < \varkappa$ for all $\nu < \varkappa$ or $\lambda = \varkappa$ and $\varkappa$ is weakly compact.*

THEOREM 2. *If $\varkappa$ is singular then $P(\varkappa, 1)$ is false.*

Proof of Theorem 1.

First we prove

(5) *If $\nu^\lambda \geqslant \varkappa$ for some $\nu < \varkappa$, $\lambda < \varkappa$ then $P(\varkappa, \lambda)$ is false.*

Proof. Let λ_0 be minimal such that there is $\nu < \varkappa$ with $\nu^{\lambda_0} \geqslant \varkappa$, and let ν_0 be minimal such that $\nu_0^{\lambda_0} \geqslant \varkappa$. Then $\varkappa$ being regular $\nu_0^{\underline{\lambda_0}} < \varkappa$.

It is well known that then there are X, $|X| = \nu_0^{\underline{\lambda_0}}$ and $G \subset P(X)$, $|G| = \nu_0^{\lambda_0}$ such that

(6) $\qquad |A| = \lambda_0$ for $A \in G \quad$ and $\quad |A \cap B| < \lambda_0$ for $A \neq B \in G$.

Let $H = \mathrm{Co}(G) = \{X - A\colon A \in G\}$. We may assume $X \cap \varkappa = \emptyset$. Let $\{B_\xi\colon \xi < \varkappa\} \subset H$ be one-to-one, and put $A_\xi = B_\xi \cup \xi$ for $\xi < \varkappa$; $F = \{A_\xi\colon \xi < \varkappa\}$. Then $|A_\xi| < \varkappa$ for $\xi < \varkappa$, $|X \cup \varkappa| = \varkappa$, $|F| = \varkappa$, $A_\xi \not\subset A_\eta$ for $\xi \neq \eta < \varkappa$ since $|B_\eta - B_\xi| = \lambda_0$. On the other hand if $F' \subset F$, $|F'| = \varkappa$ then, by (6),

$$|X \cup \varkappa - \bigcup F'| < \lambda_0 \leqslant \lambda .$$

This proves (5).

Now we prove

(7) *Assume $\lambda < \varkappa$, $\nu^\lambda < \varkappa$ for all $\nu < \varkappa$ then $P(\varkappa, \lambda)$ holds.*

Proof. Let F be a system satisfying (3) and (4). Let $\xi < \varkappa$. Put $F_\xi = \{A \in F\colon |\xi - A| \geqslant \lambda\}$. If $|F_\xi| = \varkappa$ for some ξ then by the regularity of $\varkappa$ and by $|\xi|^\lambda < \varkappa$, (1) holds. We assume $|F_\xi| < \varkappa$ for all $\xi < \varkappa$ and we obtain a contradiction. Pick $A_\xi \in F - F_\xi$ for each $\xi < \varkappa$. Put $g(\xi) = \xi - A_\xi$, $h(\xi) = \sup g(\xi)$. We can choose a regular cardinal τ such that $\lambda \leqslant \tau < \varkappa$ otherwise $\lambda^+ = \varkappa$, $\lambda^\lambda \geqslant \varkappa$.

The set $K_\tau = \{\xi < \varkappa\colon \mathrm{cf}(\xi) = \tau\}$ is stationary in $\varkappa$ and $h(\xi) < \xi$ for $\xi \in K_\tau$. By Fodor's theorem there are $\varrho < \varkappa$ and a stationary set $C \subset K_\tau$ such that $h(\xi) = \varrho$ for $\xi \in C$. By $|\varrho|^\lambda < \varkappa$, there is $C' \subset C$, C' cofinal in $\varkappa$ such that $g(\xi) = g(\eta)$ for $\xi, \eta \in C'$. Choose $\xi < \eta \in C'$ such that $A_\xi \subset \eta$. Then $A_\xi \subset A_\eta$ a contradiction.

(5) and (7) prove the first part of our theorem.

We now prove

(8) *Assume $P(\varkappa, \varkappa)$. Then $\varkappa$ is weakly compact.*

Proof. By the assumption $P(\varkappa, \lambda)$ holds for $\lambda < \varkappa$ hence, by (5), $2^\lambda < \varkappa$ for $\lambda < \varkappa$; $\varkappa$ is strongly inaccessible. Assume $\varkappa$ is not weakly compact. Then there is an Aronszajn tree $\langle \varkappa, \prec \rangle$ on $\varkappa$. Let T_ξ denote the set of elements of rank ξ in the tree and put $S_\xi = \bigcup_{\eta<\xi} T_\eta$. P is said to be a *path of length* ξ if P is a chain $\subset S_\xi$ and $P \cap T_\eta \neq \emptyset$ for $\eta < \xi$. It is well-known that there is a set $K \subset \varkappa$, $|K| = \varkappa$ such that there is a maximal path P_ξ of length ξ for each $\xi \in K$.

Put $F = \{S_\xi - P_\xi \colon\ \xi \in K\}$. Assume $\xi < \eta$, $\xi, \eta \in K$. Then by the maximality of P_ξ $S_\xi - P_\xi \not\subset S_\eta - P_\eta$ and obviously $S_\eta - P_\eta \not\subset S_\xi - P_\xi$.

On the other hand let $L \subset K$, $|L| = \varkappa$, $x, y \notin \bigcup \{S_\xi - P_\xi \colon\ \xi \in L\}$. Then there is a $\xi \in L$ such that the ranks of x and y are less than ξ, hence $x, y \in P_\xi$ and $x \leqslant y$ or $y \leqslant x$.

It follows that $\varkappa - \bigcup \{S_\xi - P_\xi \colon\ \xi \in L\}$ is a chain and thus it has cardinality less than $\varkappa$.

Thus F establishes not $P(\varkappa, \varkappa)$. Hence if $\varkappa$ holds $\varkappa$ must be weakly compact. This proves (8) (see Problem 73 of [1]).

Finally we have to prove

(9) *If $\varkappa$ is weakly compact then $P(\varkappa, \varkappa)$ is true.*

Proof. Let F be a system of sets satisfying (3) and (4). It is well known that then there are $A \subset \varkappa$ and $\{A_\xi \colon\ \xi < \varkappa\} \subset F$ such that $A \cap \xi = A_\eta \cap \xi$ for $\xi \leqslant \eta < \varkappa$. First we claim that $\varkappa - A$ is cofinal in $\varkappa$. Otherwise there is ξ such that $\varkappa - \xi \subset A$. Then there is $\xi < \eta$ such that $A_\xi \subset \eta$ and then because of $\eta - \xi \subset A$, $A_\xi \subset A_\eta$.

Then by transfinite induction one can easily choose two increasing sequences σ_η, τ_η; $\eta < \varkappa$ such that $\sigma_\eta \in \varkappa - A$, $A_{\tau_\nu} \subset \sigma_\eta$ for $\nu < \eta$, and $\tau_\nu > \sigma_\eta$ for $\nu \geqslant \eta$. Then

$$\{\sigma_\eta \colon\ \eta < \varkappa\} \subset \varkappa - \bigcup \{A_{\tau_\eta} \colon\ \eta < \varkappa\}\,.$$

This proves (9) and Theorem 1.

Proof of Theorem 2. Assume $\mathrm{cf}(\varkappa) < \varkappa$. Let $\{\varkappa_\nu \colon\ \nu < \mathrm{cf}(\varkappa)\}$ be a normal sequence of type $\varkappa$ of cardinals less than $\varkappa$, tending to $\varkappa$ such that $\varkappa_0 = \mathrm{cf}(\varkappa)$. Then

$$\varkappa = \varkappa_0 \cup \bigcup_{\nu < \mathrm{cf}(\varkappa)]} \varkappa_{\nu+1} - \varkappa_\nu\,.$$

For $\varkappa_0 \leqslant \xi < \varkappa$ let $\nu(\xi)$ be the unique ν for which $\xi \in \varkappa_{\nu+1} - \varkappa_\nu$. Put $A_\xi = \varkappa_{\nu+1} - \{\nu(\xi), \xi\}$ for $\varkappa_0 \leqslant \xi < \varkappa$ and $F = \{A_\xi \colon\ \varkappa_0 \leqslant \xi < \varkappa\}$.

Assume $\xi \neq \eta < \varkappa$. If $\nu(\xi) \neq \nu(\eta)$ then $\nu(\xi) \in A_\eta - A_\xi$. If $\nu(\xi) = \nu(\eta)$ then $\xi \in A_\eta - A_\xi$. Hence $A_\eta \not\subset A_\xi$. On the other hand if $L \subset \varkappa - \varkappa_0$ is cofinal in $\varkappa$ then obviously

$$\bigcup \{A_\xi \colon\ \xi \in L\} = \varkappa\,.$$

§ 3. Remarks.

1) First we mention that the weak assumption (4) is insufficient to obtain set mapping theorems of type (2) as is shown by the following example

For $n \in \omega$ define

$$f(n) = \{m < n:\ m \text{ is even}\} \cup \{m+1\} \quad \text{if } n \text{ is even}$$

and

$$f(n) = \{m < n:\ m \text{ is odd}\} \cup \{n+1\} \quad \text{if } n \text{ is odd}.$$

Then $f(n) \not\subset f(m)$ if $n \neq m$ and there is no free set of three elements. (Two independent points obviously exist.)

2) The following would be a Ramsey-type generalization of the positive part of Theorem 1.

(10) *Let $2 \leqslant k < \omega$ and let F: $[\omega]^k \to [\omega]^{<\omega}$ be such that $F(X) \not\subset F(Y)$ for $X \neq Y \in [\omega]^k$. Then there is $A \subset \omega$, $|A| = \omega$ such that*

$$|\omega - \bigcup \{F(X):\ X \in [A]^k\}| \geqslant \omega .$$

We have examples to show that (10) is false for $k = 2$ even if we assume that $F = \{F(X):\ X \in [\omega]^k\}$ satisfies the following stronger condition.

(11) *No member of F is contained in the union of l others for some $2 \leqslant l < \omega$.*

We suppress the proof.

3) We also mention that some of the counterexamples can be obtained with set-systems F satisfying the stronger condition (11).

Using the fact that for each $1 \leqslant l < \omega$ there is $G \subset P(\omega)$ such that the intersection of l members of G is infinite and the intersection of $l+1$ members of G is finite one can strengthen the counterexample of Theorem 1 to

(12) *For $\omega_1 \leqslant \varkappa \leqslant 2^\omega$ there is $F \subset P(\varkappa)$, $|F| = \varkappa$ satisfying* (11) *and such that*

$$|\varkappa - \bigcup F'| < \omega \quad \textit{for} \quad F' \subset F,\ |F'| = \varkappa .$$

The existence of the required G was pointed out to us by L. Pósa.

Assuming C. H., we know that there is an F satisfying (12) and the following condition stronger than (11). No member of F is contained in the union of finitely many others. We did not investigate how far these results can be generalized.

4) Finally we mention a rather technical problem. Let F: $[\omega]^2 \to [\omega]^{<\omega}$ be such that $F(X) \not\subset F(Y)$ for $X \neq Y \in [\omega]^2$. Does there exist an infinite path $I \subset [\omega]^2$ such that $|\omega - \bigcup \{F(X):\ X \in I\}| \geqslant \omega$

References

[1] P. Erdös and A. Hajnal, *Unsolved problems in set theory*, Proceedings of Symposia in Pure Mathematics, 13, Part 1. A.M.S. Providence, R. I. (1971), pp. 17–48.
[2] — — and A. Máté, *Chain conditions on set mappings and free sets*, Acta Sci. Math. 34 (1973), pp. 69–79.
[3] G. Fodor, *On a problem in set theory*, Acta Sci. Math. 15 (1953–54), pp. 240–242.
[4] A. Hajnal, *Proof of a conjecture of S. Ruziewicz*, Fund. Math. 50 (1961), pp. 123–128.

Reçu par la Rédaction le 24. 4. 1973