UNSOLVED AND SOLVED PROBLEMS IN SET THEORY

P. ERDÖS AND A. HAJNAL

1. Introduction. In 1967 we prepared a collection of unsolved problems for the Set Theory Symposium held at UCLA which finally appeared [**6**] four years later in the Proceedings of the Symposium in 1971. However, we distributed a mimeographed version of the paper in 1967 and since then several people worked on some of these problems and obtained solutions. We tried to keep [**6**] up to date by adding remarks to the page proofs, but we believe it will be more useful if we give a survey of the new results obtained by various people.

We will use this opportunity to mention some entirely new problems, to announce some of our (as yet) unpublished results concerning the old problems, to give some further explanation about those old problems which were either vaguely or incorrectly stated in [**6**], and to revise the so-called "simplest" forms of these unsolved problems which we are still in touch with and for which partial progress has been made.

The notation and terminology will be the same as in [**6**]. A number of the problems here involve Martin's Axiom which is not mentioned in [**6**]. We denote by MA_b the following statement: For every partially ordered set $\langle C, \leqq \rangle$ satisfying the countable chain condition and for every family $\mathscr{F}$, $|\mathscr{F}| \leqq b$, of dense subsets of C, there is $G \subset C$ such that G is C-generic over $\mathscr{F}$. Martin's Axiom—MA—is the assertion that MA_b holds for all $b < 2^{\aleph_0}$. (For references, see [**20**] and [**27**].)

2. About the old problems for the ordinary partition relation. We will assume that the reader is familiar with [**6**].

The status of *Problems* 1–5 concerning the ordinary partition symbol for cardinals did not change. Problem 4 can be stated in the following more comprehensible form:

Let $a \geqq \aleph_0$. Assume $b^{\aleph_0} < a$ for $b < a$, and $c^{\aleph_0} < \mathrm{cf}(a)$ for $c < \mathrm{cf}(a)$. Does then

$$a \to (a, \mathrm{cf}(a))^2$$

hold?

This would be an immediate generalization of the well-known Erdös, Dushnik, Miller theorem ($a \to (a, \aleph_0)^2$ for $a \geqq \aleph_0$) and by the results of **[10]** it is true if either a is regular or a is strong limit. (Note that the conditions of the problem are obviously necessary since $c^{\aleph_0} \not\to (c^+, \aleph_1)^2$ for $c \geqq \aleph_0$ and $\mathrm{cf}(a) \not\to (\mathrm{cf}(a), d)^2$ implies $a \not\to (a, d)^2$ for every d and a.)

There has been considerable progress concerning *Problems* 6–13, stated in [**6**, 3.2, 3.3].

We learned that the theorem of Galvin and Hajnal [**6**, p. 21] was independently discovered by L. Hadded and G. Sabbagh (see **[14]**). A much stronger "canonicity" theorem is proved in Chang's paper **[3]**. For an explicit statement of this theorem see **[21]**.

Problem 6 as it stands is solved. $\omega^5 \to (\omega^3, 6)^2$ holds by a remark of E. C. Milner. However, we know of no general results which would make the computation of $f(k, n)$ of Problem 6 practically possible at least for small values of k and n. For a more detailed exposition of problems arising here see **[21]**.

We already mentioned in **[6]** that *Problem* 7 has been solved by

THEOREM (CHANG). $\omega^\omega \to (\omega^\omega, 3)^2$ **[3]**.

Chang worked out a very deep method for the proof. E. C. Milner somewhat simplified Chang's proof and proved the following theorem:

$$\omega^\omega \to (\omega^\omega, k)^2 \quad \text{for } k < \omega.$$

Finally Jean Larson obtained a relatively simple proof of this general result. She also proved

THEOREM (J. LARSON).

(a) $(\omega^* + \omega)^\omega \to ((\omega^* + \omega)^\omega, k)^2$, *for* $k < \omega$.

(b) *Assume* $a \to (a)_2^{<\aleph_0}$, $\kappa = \Omega(a)$. *Then* $\kappa^\omega \to (\kappa^\omega, n)^2$ *for* $n < \omega$.

It is not known if (b) holds for $\kappa > \omega$ under the weaker hypothesis $a \to (a)_2^2$. The following remains open:

Is $\omega^{\omega^\rho} \to (\omega^{\omega^\rho}, n)^2$ true for $n < \omega$, $0 < \rho < \omega_1$? Galvin remarked that if $\omega^\alpha \to (\omega^\alpha, 3)^2$ then α is of the form ω^ρ if $2 < \alpha < \omega_1$.

It would be interesting to know whether there is an order type Φ for which

$$\Phi \to (\Phi, 3)^2 \quad \text{but} \quad \Phi \not\to (\Phi, 4)^2.$$

There are many results relevant to *Problem* 8, though none of these gives an answer to the problem as it is stated. We know

THEOREM (LAVER). $\mathrm{MA}_{\aleph_1}$ *implies* $\omega_1 \to (\omega_1, [\omega, \omega_1])^2$. [1]

This was generalized in **[2]**.

[1] For typographical reasons we use this notation instead of $\omega_1 \to (\omega_1, \binom{\omega_1}{\omega})^2$.

THEOREM (BAUMGARTNER-HAJNAL). $MA_{\aleph_1}$ *implies* $\omega_1 \to (\omega_1, [\alpha, \omega_1])^2$ *for* $\alpha < \omega_1$.

It is reasonable to ask: Does $MA_{\aleph_1} + 2^{\aleph_0} = \aleph_2$ imply

$$\omega_1 \to (\omega_1, \omega + 2)^2 \quad \text{or} \quad \omega_1 \to (\omega_1, \alpha)^2$$

for $\alpha < \omega_1$?

The following result is relevant to *Problem* 8.

THEOREM (KUNEN [**19.1**]). *Assume a is real valued measurable. Then*

$$a \to (a, \alpha)^2 \quad \textit{for} \quad \alpha < \omega_1.$$

This certainly shows (with reference to a well-known result of Solovay) that

$$2^{\aleph_0} \not\to (2^{\aleph_0}, \alpha)^2$$

cannot be proved for any $\omega + 2 \leqq \alpha < \omega_1$ without some hypothesis on $2^{\aleph_0}$. On the other hand, C.H. $\Rightarrow \omega_1 \not\to (\omega_1, \omega + 2)^2$ can be slightly generalized.

THEOREM (LAVER, UNPUBLISHED). *Assume* $MA_{\aleph_1} + 2^{\aleph_0} = \aleph_2$. *Then*

$$\omega_2 \not\to (\omega_2, \omega + 2)^2.$$

Laver's proof breaks down for $2^{\aleph_0} = \aleph_3$. The following problem is relevant:

Assume $MA + 2^{\aleph_0} = \aleph_3$. Let $|S| = \aleph_2$, $\mathscr{F} \subset [S]^{\aleph_1}$, $|\mathscr{F}| \leqq \aleph_2$ and assume $|A \cap B| < \aleph_0$ for every pair $A \neq B \in \mathscr{F}$. Does $\mathscr{F}$ then possess property $B(\aleph_1)$?

This is a variant of Problem 39, where we asked the same question for $|S| = |\mathscr{F}| = \aleph_{\omega+1}$ assuming the G.C.H.

Galvin proved $\omega_2 \to (\omega_1, \omega + 2)^2$ and mentioned to us that $\omega_2 \to (\omega_1, \omega + 3)^2$, $\omega_2 \to (\omega_1, (\omega + 2)_2)^2$ and many similar problems remain open if we do not assume G.C.H.

The following theorems of Baumgartner show further progress in the direction of two previously mentioned results.

THEOREM (BAUMGARTNER). Con(ZF) *implies*

$$\mathrm{Con}(\mathrm{ZFC} + \omega_1 \to (\textit{stationary subset of } \omega_1, [\alpha, \omega_1])^2) \quad \textit{for } \alpha < \omega,$$

and

$$\mathrm{Con}(\mathrm{ZFC} + \omega_n \not\to (\omega_n, \omega + 2)^2) \quad \textit{for } n < \omega.$$

Problem 9 stands as it is. This seems to be the case with most of the problems where the underlying set has cardinality $\aleph_{\omega+1}$.

As to the *Problems* 10, 10/A, 10/B, 11, 11/A, the following was proved:

THEOREM (BAUMGARTNER-HAJNAL [**2**]). *Assume* $\Phi \to (\omega)^1_{\aleph_0}$. *Then* $\Phi \to (\alpha)^2_k$ *for* $\alpha < \omega_1$, $k < \omega$.

This result yields $\omega_1 \to (\alpha)^2_k$ and $\lambda \to (\alpha)^2_k$ for $\alpha < \omega_1$, $k < \omega$, and so we have a positive answer to Problem 10 with $\rho = 0$, Problem 10/A, Problem 11, 11/A.

The proof of the above mentioned theorem given in [**2**] is of a metamathematical character. It is first proved that if $MA_{|\Phi|}$ holds, then the theorem is true. Then the result is proved by "absoluteness" arguments.

Finally F. Galvin discovered a combinatorial proof of the Baumgartner-Hajnal theorem. He also generalized the theorem for types of partially ordered sets.

THEOREM (F. GALVIN). *Let φ be a partial order type such that $\varphi \to (\eta)^1_{\aleph_0}$. Then*

$$\varphi \to (\alpha)^2_k$$

and

$$\varphi \to (\eta, (\alpha^* \vee \alpha)_k)^2 \quad \textit{for} \quad \alpha < \omega_1, k < \omega.$$

(The meaning of the $\to$ relations extends to partial order types in a self-explanatory way.) Galvin mentions that he does not know if $\varphi \to (\omega)^1_{\aleph_0}$ implies $\varphi \to (\alpha)^2_k$ $(\alpha < \omega_1, k < \omega)$ for partial order types φ.

As to the *earlier history*, some results are already stated in [6]. The topic was extensively studied by F. Galvin, and it is fair to say that even the general conjecture was due to him.

Galvin's old theorems.

(i) $\Phi \not\to (\omega)^1_{\aleph_0} \Rightarrow \Phi \not\to (\omega, \omega + 1)^2$.

(ii) In all the earlier results (stated e.g. in [6]) the conditions $\omega_1, \omega_1^* \nleqq \Phi$, $|\Phi| \geqq \aleph_1$ can be replaced by $\Phi \to (\eta)^1_{\aleph_0}$.

(iii) If $\Phi \to (\eta)^1_{\aleph_0}$ then

(a) $\Phi \to (\alpha)^2_2$ for $\alpha < \omega_1$,

(b) $\Phi \to (\alpha \vee \alpha^*)^2_3$ for $\alpha < \omega_1$,

(c) $\Phi \to ((\omega \cdot 2)_2, \omega^2)^2$.

(iv) If $|\Phi| = \omega_1$ and $\Phi \to (\omega, \alpha_1, \dots, \alpha_k)^2$ then $\omega_1 \to (\alpha_1, \dots, \alpha_k)^2$. Unfortunately, Galvin's nice results remained unpublished for years. The strongest result proved with "conventional methods" before Galvin's new proof is due to Prikry. He proved $\omega_1 \to (\omega^2 + 1, \alpha)^2$ for $\alpha < \omega_1$. (See p. 272 for his general result.)

As to the part $\rho > 0$ of Problem 10, the situation is quite different.

THEOREM (PRIKRY [22]).

$$\text{Con(ZF)} \Rightarrow \text{Con}\left(\text{ZFC} + \text{G.C.H.} + \binom{\omega_2}{\omega_1} \not\to \begin{bmatrix}\omega \\ \omega_1\end{bmatrix}^{1,1}_{\aleph_1}\right).$$

(See the remarks concerning Problem 24 where we will state Prikry's result in greater detail.)

With a slight generalization Hajnal proved

$$\text{Con(ZF)} \Rightarrow \text{Con(ZFC + G.C.H.} + \omega_2 \not\to [\omega_1 + \omega]^2_{\aleph_1}),$$

and also

$$\text{Con(ZF)} \Rightarrow \text{Con(ZFC + G.C.H.} + \omega_2 \not\to (\omega_1 + 2)^2_{\aleph_0}).$$

Now the following would be interesting to see:

Is $\omega_2 \to (\omega_1 + \omega)^2_2$ or even $\omega_2 \to (\xi)^2_2$ for $\xi < \omega_2$ consistent with G.C.H.?

By the independence results the best possible theorem which still can be true is:

$$\text{G.C.H.} \Rightarrow \omega_2 \to (\xi, (\omega_1 + n)_k)^2 \quad \text{for} \quad \xi < \omega_2, \quad n, k < \omega.$$

We proved $\omega_2 \to (\omega_1 + n)_2^2$ for $n < \omega$ several years ago, but we do not know at present:

Does G.C.H. imply $\omega_2 \to (\omega_1 + 2)_3^2$? (This might be a problem of type D.)

Considering the above mentioned negative consistency results the following problem arises:

Assume G.C.H. Is it true that for every $\alpha < \omega_2$ there is $\xi < \omega_3$ such that $\xi \to (\alpha)_2^2$?

G.C.H. certainly implies e.g. $\omega_2 \cdot 2 \to (\omega_1 + \omega)_2^2$.

The following result of Shelah is a generalization of G.C.H. $\Rightarrow \omega_2 \to (\omega_1 + n)_2^2$.

THEOREM (SHELAH). *Assume* G.C.H. *Let* $\aleph_\alpha$, $\aleph_\beta$ *be regular* $\beta + 2 \leqq \alpha$. *Then*

$$\omega_{\alpha+1} \to (\omega_\alpha + \omega_\beta)_2^2.$$

We wish to comment on *Problem* 12. The problem is vaguely formulated and it really calls for the investigation of the strength of the following assumptions:

Given $\langle R, <\rangle$, typ $R(<) = \Phi$, what is the relation among the assumptions

(i) R contains a countable dense subset, $|R| \geqq \aleph_1$.

(ii) $\omega_1, \omega_1^* \nleqq \Phi$, $|\Phi| \geqq \aleph_1$.

(iii) $\Phi \to (\eta)_{\aleph_0}^1$.

(iv) $\Phi \to (\omega)_{\aleph_0}^1$.

(v) There is $\Psi \leqq \Phi$, with $|\Psi| \geqq \aleph_1$, $\omega_1^* \nleqq \Phi$.

This was realized again by F. Galvin. He proved, e.g., (iv) is equivalent to (iii) $\vee$ $\omega_1 \leqq \Phi$ and also gave a number of necessary and sufficient conditions for (iii) and (iv). These results are unpublished as well.

He stated the problem if (iv) implies (v), the answer is negative by

THEOREM (BAUMGARTNER [**1**]). (a) *There is* Φ, $|\Phi| = \aleph_1$ *which satisfies* (iv) *but not* (v).

(b) *If* $V = L$ *then there is a* Φ, *such that* $|\Phi| = \aleph_2$, $\Phi \to (\omega)_{\aleph_0}^1$ *and* $\Psi \nrightarrow (\omega)_{\aleph_0}^1$ *for every* $\Psi \leqq \Phi$, $|\Psi| < \aleph_2$.

(c) Con(ZFC + $\exists$ *a weakly compact cardinal*) $\Rightarrow$ Con(ZFC + G.C.H. + $\forall\Phi(|\Phi| = \aleph_2 \wedge \Phi \to (\omega)_{\aleph_0}^1 \Rightarrow \exists\Psi(\Psi \leqq \Phi \wedge |\Psi| = \aleph_1 \wedge \Psi \to (\omega)_{\aleph_0}^1)))$.

The first part of *Problem* 13 was solved as

THEOREM (HAJNAL [**15**]). *Assume* G.C.H. *and that* $\aleph_\alpha$ *is regular, then*

$$\omega_{\alpha+1}^2 \nrightarrow (\omega_{\alpha+1}^2, 3)^2.$$

J. Baumgartner improved the previous result and proved that assuming G.C.H.

$$\omega_{\alpha+1}^2 \nrightarrow (\omega_{\alpha+1}^2, 3)^2 \quad \text{holds for every } \alpha.$$

He also proved

THEOREM (BAUMGARTNER). (a) *If* $\aleph_\alpha$ *is a strong limit then* $\omega_\alpha^2 \to (\omega_\alpha^2, 3)^2$ *holds iff* $\omega_{\mathrm{cf}(\alpha)}^2 \to (\omega_{\mathrm{cf}(\alpha)}^2, 3)^2$.

(b) *If a is regular and there is an a-Suslin tree*, $\kappa = \Omega(a)$, *then*

$$\kappa^2 \nrightarrow (\kappa^2, 3)^2.$$

The first part solves a problem of [7] where we discussed several related problems.

Just as in the case of Problem 8, the following problems (of type D) can be stated: Let a be real valued measurable, $\kappa = \Omega(a)$. Does $\kappa^2 \rightarrow (\kappa^2, 3)^2$ hold?

ADDED IN PROOF. Kunen proved that the answer is affirmative.

Does $\mathrm{MA}_{\aleph_1} + 2^{\aleph_0} = \aleph_2$ imply $\omega_1^2 \rightarrow (\omega_1^2, 3)^2$? In [7] we proved $\omega_1^2 \rightarrow (\omega_1 \cdot \alpha, 3)^2$ for $\alpha < \omega_1$, but we do not know if

$$\omega_1^2 \rightarrow (\omega_1 \cdot \omega, 4)^2.$$

Another result of [7] states that assuming C.H. $\rho \nrightarrow (\omega_1^\omega, 3)^2$ holds for every $\rho < \omega_2$; but our proof does not generalize for types corresponding to larger cardinals. The following result is relevant:

THEOREM (JEAN LARSON). *If $\aleph_\alpha$ is regular then $\omega_\alpha^{\omega+1} \nrightarrow (\omega_\alpha^{\omega+1}, 3)^2$.*

3. The old problems for symbol II. As to *Problem* 15, we know

THEOREM (GALVIN, SHELAH [13]). (a) $2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_{\aleph_0}$ *(even more generally* $\mathrm{cf}(2^{\aleph_0}) \nrightarrow [\mathrm{cf}(2^{\aleph_0})]^2_{\aleph_0}$ *and* $\aleph_1 \nrightarrow [\aleph_1]^3_{\aleph_1}$).

(b) $\aleph_1 \nrightarrow [\aleph_1]^2_4$.

(a) is best possible in the sense that if a carries an $\aleph_1$-saturated a-complete ideal, then

$$a \rightarrow [a]^2_{\aleph_1, \aleph_0}$$

holds.

Baumgartner noticed that $2^{\aleph_0} \nrightarrow [\aleph_1]^2_{\aleph_0}$ is consistent with $2^{\aleph_0} =$ anything reasonable (even $2^{\aleph_0} =$ real valued measurable).

S. Shelah proved that $2^{\aleph_\alpha} \leqq \aleph_{\alpha+n}$ implies $\aleph_{\alpha+n} \nrightarrow [\aleph_{\alpha+n}]^{n+1}_{\aleph_{\alpha+n}}$ for $n < \omega$. When $n = 1$ this reduces to the old result of [10].

Problem 16.

THEOREM (R. A. SHORE). *If a is regular and there is an a-Suslin tree then* $a \nrightarrow [a]^2_a$.

Hence $a \nrightarrow (a)^2_2$ implies $a \nrightarrow [a]^2_2$ for regular a if $V = L$.

Problem 17. A positive answer cannot be proved since e.g. assuming G.C.H. this would imply

$$\aleph_2 \nrightarrow [\aleph_1]^3_{\aleph_1} \Rightarrow \aleph_2 \nrightarrow [\aleph_1]^3_{\aleph_1 \cdot \aleph_0} \Rightarrow \aleph_2 \nrightarrow [\aleph_1]^2_{\aleph_1 \cdot \aleph_0},$$

which is independent.

R. A. Shore proved that $\aleph_2 \nrightarrow [\aleph_1]^3_{\aleph_1}$ holds in L. It seems likely that a very general form of the stepping up lemma is true in L but we do not know if anyone has worked out the details yet.

In [10] we claimed

$$\text{G.C.H.} \Rightarrow \omega_2 \nrightarrow [\omega_1]^3_3.$$

We can prove a slightly stronger statement.

THEOREM. *Let b be regular, $k \leqq 4$, $a \nrightarrow [b]^2_k$, then*

$$2^a \nrightarrow [b]^3_k.$$

As a corollary we have

$$2^{\aleph_1} \nrightarrow [\aleph_1]^3_4.$$

Problem 18. Galvin proved: If Ψ is an order type, $\Psi \not\geq \omega_1$, $\Psi \not\geq \eta$ then $\Psi \nrightarrow [\omega, \omega^2, \omega^2, \omega^3, \omega^3, \omega^4, \omega^4, \ldots]^2$. The hypothesis on Ψ is the best possible, in view of the Baumgartner-Hajnal theorem and a result of Galvin that $\eta \rightarrow [\eta]^2_3$. The following generalization was conjectured by Galvin and proved by Laver.

THEOREM (LAVER).

$$\begin{pmatrix} \eta \\ \eta \\ \cdot \\ \cdot \\ \cdot \\ \eta \end{pmatrix} \rightarrow \begin{pmatrix} \eta \\ \eta \\ \cdot \\ \cdot \\ \cdot \\ \eta \end{pmatrix}^{r_1, \ldots, r_n}_{(r_1+\cdots+r_n)!\cdot(r_1+\cdots+r_n-n)!+1}$$

COROLLARY.

$$\eta \rightarrow [\eta]^{<\aleph_0}_{\aleph_0}.$$

4. Symbol III and related problems. Most of the *Problems* 19–21 were asked and investigated quite independently of our asking them. For the convenience of the reader we quote some results we know of.

Problem 19. $\aleph_2 \rightarrow [\aleph_1]^2_{\aleph_1, \aleph_0}$ is equivalent to $\aleph_2 \rightarrow [\aleph_1]^{<\aleph_0}_{\aleph_1 \cdot \aleph_0}$, and this is equivalent to Chang's conjecture which is known to be independent. For results and references see Silver's paper **[26]**. The same applies for *Problems* 19/A, 19/B.

Probem 19/C. J. Baumgartner proved that (it is consistent relative to a large cardinal that)

$$19/\mathrm{A} \not\Rightarrow 19/\mathrm{B},$$

i.e. the almost disjoint transversal property does not imply Kurepa's hypothesis.

Problem 19/D. A positive answer is obviously consistent. We know of no consistency result from the other direction.

M. Magidor remarked recently that $V = L$ implies: There are 2^a almost disjoint stationary subsets of $\Omega(a)$ for every regular a.

Problem 21/A is discussed in Kunen's thesis **[19]**.

Let $\rho_0 = \sup\{\rho$: there is a sequence f^ρ_ν, $\nu < \rho$ satisfying (0 0) of 21/A$\}$. It is proved in **[19]** that assuming G.C.H., $\rho_0 < \omega_3$, but it is consistent to have ρ_0 "arbitrarily large". By Silver **[26]**, $\rho_0 = \omega_2$ is consistent.

Is $\rho_0 > \omega_2$ true in L?

As to *Problem* 20 or rather *Problem* 20/A, one obtains quite similarly that there is $\rho < \omega_3$ such that

$$\rho \to [\omega_2^{\omega_1}]^1_{\aleph_1,\aleph_0}$$

and that it is consistent to have $\rho \not\to [\omega_2^{\omega_1}]^1_{\aleph_1,\aleph_0}$ to arbitrarily large ρ.

We do not know if $\omega_2^{\omega_2} \to [\omega_2^{\omega_1}]^1_{\aleph_1,\aleph_0}$ is consistent. This would follow e.g. from the consistency of the negation of Problem 19/D.

5. Symbol IV and polarized partition relations.

Problem 22. F. Galvin proved that $a \to (b)_2^{<\aleph_0}$ is equivalent to $a \Rightarrow (b)^{<\aleph_0}$.

Problem 24. We know the theorem of Prikry mentioned, i.e. the consistency of

$$\begin{pmatrix}\aleph_2\\ \aleph_1\end{pmatrix} \not\to \begin{bmatrix}\aleph_0\\ \aleph_1\end{bmatrix}^{1,1}_{\aleph_1}.$$

Prikry proved that the following is consistent with ZFC + G.C.H.:

PRIKRY'S PRINCIPLE. *There is $f:\omega_2 \times \omega_1 \to \omega_1$ such that for every $A \in [\omega_2]^{\aleph_0}$, $B \in [\omega_1]^{\aleph_1}$ there is $\alpha \in A$ with*

$$R(f \mid (\{\alpha\} \times B)) = \omega_1.$$

THEOREM (JENSEN). *The Prikry principle is true in L.*

This result probably extends to most generalizations of the Prikry principle we will mention in this paper.

Problem 26. Trivially,

$$\mathrm{MA} \Rightarrow \begin{pmatrix}2^{\aleph_0}\\ \aleph_0\end{pmatrix} \not\to \begin{bmatrix}2^{\aleph_0}\\ \aleph_0\end{bmatrix}^{1,1}_{\aleph_0}.$$

ZFC + $2^{\aleph_0} > \aleph_1$ is very likely consistent with $\binom{2^{\aleph_0}}{\aleph_0} \to \binom{2^{\aleph_0}}{\aleph_0}^{1,1}_2$ but we do not know if it has already been proved. The following results might be relevant.

THEOREM (BAUMGARTNER). *Both*

$$\begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix} \to \begin{pmatrix}2\\ \omega\end{pmatrix}^{1,1}_{\aleph_0} \quad \textit{and} \quad \begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix} \not\to \begin{pmatrix}2\\ \omega\end{pmatrix}^{1,1}_{\aleph_0}$$

are consistent with ZFC + $2^{\aleph_0} > \aleph_1$.

Problem 27. The consistency of

$$\begin{pmatrix}\aleph_2\\ \aleph_1\end{pmatrix} \to \begin{pmatrix}\aleph_0\\ \aleph_1\end{pmatrix}^{1,1}_{\aleph_0}$$

is not known.

The results of **[10]** listed on p. 32 of **[6]** left the following problem open: Does $\binom{\aleph_2}{\aleph_2} \to \binom{\aleph_1}{\aleph_1}^{1,1}_k$ hold for $3 \leqq k < \omega$?

As a corollary of the generalization of the Prikry principle mentioned on p. 272 it is easy to see that the following statements are consistent with G.C.H.:

$$\binom{\aleph_2}{\aleph_2} \nrightarrow \binom{\aleph_1}{\aleph_1}_4^{1,1} \quad \text{and} \quad \binom{\aleph_2}{\aleph_2} \nrightarrow \left(\binom{\aleph_2}{\aleph_1} \vee \binom{\aleph_1}{\aleph_2}, \binom{\aleph_1}{\aleph_1}_2\right)^{1,1}.$$

$$\text{G.C.H.} \Rightarrow \binom{\aleph_2}{\aleph_2} \rightarrow \left(\binom{\aleph_2}{\aleph_1} \vee \binom{\aleph_1}{\aleph_2}, \binom{\aleph_1}{\aleph_1}\right)^{1,1}$$

is proved in [**10**].

On the other hand we have

THEOREM (HAJNAL). G.C.H. $\Rightarrow$

(a)
$$\binom{\aleph_2}{\aleph_2} \rightarrow \binom{\aleph_1}{\aleph_1}_3^{1,1},$$

(b)
$$\binom{\aleph_3}{\aleph_2} \rightarrow \left(\binom{\aleph_0}{\aleph_1}_{\aleph_0}, \binom{\aleph_2}{\aleph_2}\right)^{1,1}.$$

The second statement gives an answer to Problem 12(2) of [**10**].

6. The rest of the old problems.

Problem 29. Shelah pointed out that the problem as stated is obviously false. The problem we had in mind is the following:

Δ is said to be weakly-canonical with respect to $(S_\xi)_{\xi<\Phi}$ if $X, Y \in [S]^r$ and $|X \cap S_\xi| = |Y \cap S_\xi| \leqq 1$ for $\xi < \Phi$ imply that $X \in T_\nu$ and $Y \in T_\nu$ are equivalent for every $\nu < \Omega(c)$. Is Problem 29 true if canonical is replaced by weakly canonical?

Problem 31. The answer to the first part is trivially no since

$$\text{MA}_{\aleph_1} \Rightarrow \binom{\aleph_1}{\aleph_0} \rightarrow \binom{\aleph_1}{\aleph_0}_2^{1,1}$$

and, by the theorem of Laver mentioned on p. 270, $\text{MA}_{\aleph_1} \Rightarrow \aleph_1 \rightarrow (\aleph_1, [\aleph_0, \aleph_1])^2$. As to the second part we only remark that the well-known "Sierpinski partition" trivially shows $\aleph_1 \nrightarrow ([\aleph_1, \aleph_1])_2^2$.

Szemerédi remarked that

$$\text{MA} \Rightarrow \binom{2^{\aleph_0}}{\aleph_0} \rightarrow \left(\binom{2^{\aleph_0}}{\aleph_0}, \binom{b}{\aleph_0}\right)^{1,1} \quad \text{for} \quad b < 2^{\aleph_0}.$$

Problem 32. Shelah proved that the answer is affirmative.

Problem 33. Baumgartner proved that $V = L$ implies $(m, <\aleph_0, 2) \nrightarrow \aleph_0$ provided $m \nrightarrow (\aleph_0)_2^{<\aleph_0}$.

This was proved independently but later by K. J. Devlin as well in [**4**]. In [**4**] there are many interesting and relevant results we do not quote here.

The most up to date results are now in the preprint of Devlin and Paris [**5**].

THEOREM (DEVLIN, PARIS). (a) *If there is an a such that* $(a, <\aleph_0, 2) \rightarrow \aleph_1$ *then* $O^{\#}$ *exists.*

(b) *Assume* $V = L[D]$ *where D is a normal ultrafilter on a measurable cardinal. Then, for all cardinals a and all regular cardinals b,*

$$(\kappa, <\aleph_0, 2) \rightarrow \lambda \quad \textit{iff} \quad \kappa \rightarrow (\lambda)_2^{<\aleph_0}$$

where $\kappa = \Omega(a)$, $\lambda = \Omega(b)$.

Problem 34. (A) is false as stated since if m is real valued measurable $n < m$, then

$$(m, <\aleph_0, n) \rightarrow m.$$

This was realized by many people.

(B) Hajnal remarked that

$$\mathrm{Con(ZF)} \Rightarrow \mathrm{Con(ZFC + G.C.H.} + (\aleph_{k+2}, 3, 2) \not\rightarrow \aleph_{k+1}) \quad \text{for } k < \omega.$$

Is it true that $(\aleph_2, 3, 2) \rightarrow \aleph_1$ is consistent with ZFC + G.C.H.?

Is $(\aleph_3, 4, 2) \not\rightarrow \aleph_1$ consistent with ZFC + G.C.H.?

Problem 35. The problem was badly stated. In [**10**, Theorem 49] we already proved that under the conditions of the problem there is a free set of power $\aleph_\omega$. The real problem is if there is a free set of power $\aleph_{\omega+1}$.

Problem 36. A positive answer follows from $\omega_1 \rightarrow (\alpha)_2^2$. However, the following stronger results are true:

THEOREM (SHELAH [**25**]). *Assume* G.C.H., typ $S(\prec) = \omega_{\alpha+1}$, $\aleph_\alpha$ *is regular. Let f be a set mapping on S such that* $|f(x) \cap f(y)| < \aleph_\alpha$ *for* $x \neq y \in S$. *Then for every* $\xi < \omega_{\alpha+1}$ *there is a free subset of type* ξ.

THEOREM (PRIKRY). *Let* $[\omega_1]^2 = I_0 \cup I_1$, $I_0 \cap I_1 = \varnothing$ *be an arbitrary partition.*

(i) *If there are no* $A, B \subset \omega_1$ *such that* $|A| = \aleph_0$, $|B| = \aleph_1$, $[A]^2 \subset I_0$ *and* $[A, B] \subset I_0$ *then for every* $\alpha < \omega_1$ *there is a C such that* typ $C = \alpha$ *and* $[C]^2 \subset I_1$.

(ii) *If there are no* $A, B \subset \omega_1$ *such that* $|A| = \aleph_0$, $|B| = \aleph_1$, $[A]^2 \subset I_1$ *and* $[A, B] \subset I_0$, *then either for every* $\alpha < \omega_1$ *there is a C such that* typ $C = \alpha$ *and* $[C]^2 \subset I_1$ *or there is* $D \subset \omega_1$, $|D| = \aleph_1$, $[D]^2 \subset I_0$.

Problem 37.

ADDED IN PROOF. Let $P(\alpha)$ be the following statement. If typ $S(<) = \alpha$, then there is $\mathscr{F} \subset [S]^\omega$ such that $|A \cap B| < \aleph_0$ for $A \neq B \in \mathscr{F}$ and for all $X \in [S]^{\omega^2}$ there is $A \in \mathscr{F}$ with $A \subset X$. First Devlin and Shelah proved independently that

$$2^{\aleph_0} = \aleph_1 \Rightarrow P(\omega_2)$$

and later Galvin and Hajnal proved independently that

$$2^{\aleph_0} = \aleph_1 \Rightarrow \forall n < \omega(P(\omega_n)).$$

Let $P^*(\alpha)$ be the following statement. If typ $S(<) = \alpha$, then there is $\mathscr{F} \subset [S]^\omega$ such that $|A \cap B| < \aleph_0$ for $A \neq B \in \mathscr{F}$ and for all $X \in [S]^\omega$ if X is not contained in the union of finitely many members of $\mathscr{F}$, then there is $A \in \mathscr{F}$ with $A \subset X$.

Baumgartner proved that if $2^{\aleph_0} = \aleph_1$, then even $P^*(\omega_n)$ holds for all $n < \omega$.

Problem 38. S. Hechler [**18**] proved

THEOREM. *If* C.H. *is true then* 38/A *is false even if* $f(x)$ *is a sequence with the only limit point* x *for every* $x \in R$.

38/A is independent from ZFC + $2^{\aleph_0} > \aleph_1$.

If MA is true then 38/C is false even if $f(x)$ is of measure 0 for $x \in R$.

Problem 40. Of course MA implies a positive answer. We do not know if the negation is consistent.

ADDED IN PROOF. Baumgartner and Laver proved using iterated Sacks forcing that the negation of Problem 40 is consistent with $2^{\aleph_0} = \aleph_2$.

Problem 42. (C) A well-known result of Jensen states that in L there is a stationary subset B of ω-limits less than ω_2 such that every section of B is non-stationary. This implies that (C) holds in L.

Problem 46. The answer is yes. This problem is relevant to the following problem of W. Taylor [**28**].

Taylor's problem. Let b be a cardinal. Let $m(b)$ be the minimal cardinal having the following property. For every graph G of chromatic number $\geqq m(b)$ and for every n, there is a graph G_n of chromatic number $\geqq n$, such that $G' \subset G_n$, $|G'| < b$ implies that G' is isomorphic to a subgraph of G. Is it true that $m(\aleph_0) = \aleph_1$? (The existence of $m(b)$ is obvious for every b.)

A negative answer to Problem 46 would have shown $m(\aleph_0) > \aleph_1$. We still have a number of special classes of finite graphs for which we can prove the result for $\geqq \aleph_2$-chromatic graphs, but not for $\geqq \aleph_1$-chromatic graphs.

Problem 47. The following is a finite version of the problem.

Let $0 < k < \omega$, and let $\mathscr{G}$ be a graph of chromatic number $\geqq k$; $N = \{i < \omega$: there is a circuit of length i contained in $\mathscr{G}\}$. Is it true that

$$\sum_{i \in N} \frac{1}{i} \geqq \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k}\ ?$$

Problem 54. Folkman's paper appeared in [**11**]. A positive answer to Problem 54 would be implied by the following. There is an order type Φ for which

$$\Phi \rightarrow (\Phi, d)^2 \quad \text{but} \quad \Phi \not\rightarrow (\Phi, d+1)^2.$$

Graham and Spencer proved $(23, 5) \xrightarrow{\text{VIII}} (2, 3)$.

Problem 55. (B) and (C) are not difficult. By standard methods one can prove that G.C.H. implies

$$\aleph_{\omega+1} \xrightarrow{\text{IX}} [\aleph_{\omega+1}, \aleph_n]^{\aleph_\omega} \quad \text{for } n < \omega$$

and

$$\aleph_{\omega_1} \xrightarrow{\text{IX}} [\aleph_{\omega_1}, \aleph_\alpha]^{\aleph_\beta}_{\aleph_\gamma} \quad \text{for all } \alpha, \beta, \gamma < \omega_1.$$

Here only (A) seems to be a genuine problem. The rest of the problems not covered by the above remark are of the same type.

Problem 58. A solution is published in [**12**].

Problems 59, 60, 61. Yes. See the remarks in [**6**]. We mention that R. Solovay pointed out to us, in the theorem of Hajnal, $\binom{a^+}{a} \to \binom{a}{a}_c^{1, <\omega}$ [**6**, p. 42], if a normal measure is given on a, the homogeneous set can be shown to be of measure 1.

Problem 62. Note that we do not even know the answer if $r = 3$ and we change the requirement (2) to

(2″) $S' \subset S, |S'| = \aleph_1$ implies that there is $S_i \subset S, |S_i| = 4, [S_i]^3 \subset T_i$ for $i < 2$.

Problem 63. In the remark before the problem there is a misprint. We mean $b < \aleph_0$ in the equivalence (1). For $b = \aleph_0$, this is false since $m \not\to (\aleph_0)_2^{\aleph_0}$ holds for every m while $(\aleph_0, \aleph_1) \xrightarrow{\text{XII}} (\aleph_0, \aleph_1)$ is trivial. We remark that we do not know if there is an m for which $(m, \aleph_1) \xrightarrow{\text{XII}} (\aleph_1, \aleph_1)$ holds.

Problem 65. The answer is yes. It follows from $\binom{\omega_1}{\omega^2} \to \binom{\omega}{\omega^2}_2^{1,1}$. See [**2**].

Problem 66. For the proof for $\rho \leqq \omega + 1$ see [**8**].

Problem 67. Yes (Laver). He also proved

THEOREM (LAVER). *Assume $\varphi \not\to (\eta)^1_{\aleph_0}$. Then there is $n < \omega$ such that*

$$\varphi \to [\varphi]^1_n.$$

Problem 69. Erdös proved that the complement of a Suslin tree yields a negative answer even if we assume that in every $g' \subset g, |g'| = \aleph_1$ there is $g'' \subset g', |g''| = 2$ such that all but $\aleph_0$ vertices are adjacent to at least one of them. With this stronger assumption it follows that $\mathscr{G}$ at least contains a complete $[\aleph_1, \aleph_1]$.

Problem 71. For Shelah's results see [**24**]. We state some instances of his theorems.

THEOREMS (SHELAH). (A) *Assume* G.C.H. *The answer to Problem* 71/A *is negative even if* $\aleph_1$ *is replaced by any cardinal* a.

(B) *Assuming* C.H. *the answer to Problem* 71/B *is affirmative for* $\xi < \omega \cdot 2$.

(C) *Assume* $|A| = \aleph_{\alpha+1}$, $|\mathscr{F}| > \aleph_{\alpha+1}$. *Then there exists a sequence of type* $\omega_\alpha + 1$ *which is strongly cut either by* $\mathscr{F}$ *or by the family of complements.*

Shelah asks the following problem. Assume G.C.H. and $\aleph_\alpha$ is regular, $|A| = \aleph_\alpha$, $\alpha > 0$, $|\mathscr{F}| = \aleph_{\alpha+1}$. Does there exist a sequence of power $\aleph_\alpha$ strongly cut either by $\mathscr{F}$ or by the family of the complements? By the result of Prikry mentioned after Problem 8 it is consistent that this is false.

Problem 72. It is easy to see that a Suslin tree yields a counterexample.

Problem 73. The answer is yes. The easy proof will be given in a forthcoming joint paper of ours.

Problem 74. Shelah [**25**] proved that the answer is affirmative for every singular strong limit cardinal in place of $\aleph_\omega$.

Problem 75. Shelah remarked that a positive answer to Problem 75 would imply that every uniform ultrafilter on $\aleph_\omega$ is $\aleph_1$-descendingly incomplete.

Problem 77/A. Jensen pointed out to us that the answer is negative in L.
Problem 78.

THEOREM (HAJNAL-JUHÁSZ). $\text{Con(ZF)} \Rightarrow \text{Con}(\text{ZFC} + 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} =$ *anything reasonable* + *there is a hereditarily separable subspace* R *of power* $2^{\aleph_1}$ *of* 2^{ω_1}).

Here 2^{ω_1} denotes the topological product of $\aleph_1$ discrete spaces of 2 elements and hereditarily separable means that every subspace R' of R contains a denumerable dense subset.

Let $\sigma(R) = |\{U \subset R: U \text{ is open}\}|$. It is an easy corollary of the above result that there is a (very good) Hausdorff space R with $\sigma(R) > 2^{\aleph_0}$ such that $\sigma(R)$ is *not* of the form $m^{\aleph_1}$.

One can conjecture that the following is true (in ZFC): $\sigma(R)^{\aleph_0} = \sigma(R)$ for every infinite Hausdorff space.

Problem 79.

THEOREM (HAJNAL-JUHÁSZ). $\text{Con(ZF)} \Rightarrow \text{Con}(\text{ZFC} + \text{G.C.H.} +$ *there is a completely regular, hereditarily Lindelöf topological space* R, $|R| = \aleph_1$, *such that every countable subspace is closed discrete in* R, *and every uncountable subspace has weight* $\aleph_2$).

Both theorems are generalizations of the Prikry result mentioned on p. 272.

Problem 80. The problem should have been asked in the form whether a system $A_{\xi,\mu}$ exists for $\xi < \eta \in A$, for a stationary subset A of $\Omega(a)$.

THEOREM (HAJNAL [**17**]). *The answer is affirmative if a is weakly inaccessible and not* ω*-weakly Mahlo, or a is strongly inaccessible and not* ω*-strongly Mahlo. It also follows from the result of* [**17**] *and from a well-known result of Jensen that if* $V = L$, *the answer is affirmative iff a is not weakly compact.*

Problem 81. Prikry pointed out that $\binom{\aleph_2}{\aleph_1} \not\to \binom{\aleph_0}{\aleph_1}^{1,1}_2$ implies a negative answer to this question. Thus a negative answer to Problem 81 is consistent with ZFC + G.C.H.

Problem 82. The problem was independently asked by H. J. Keisler.

THEOREM (PRIKRY [**23**]). *Assume* $V = L$. *Then the answer to Problem* 82 *is negative.*

7. Some new problems.

Problem I. *Assume* G.C.H. *Let* $|S| = \aleph_2$, *and let* $[S]^2 = I_0 \cup I_1$ *be a partition of* $[S]^2$ *establishing* $\aleph_2 \not\to (\aleph_2)^2_2$, *i.e.* $[X]^2 \subset I_i$ *implies* $|X| < \aleph_2$ *for* $X \subset S$ *and* $i < 2$. *Does, then, there exist a subset* $Z \subset S$, $|Z| = \aleph_1$ *such that* $I \restriction Z$ *establishes*

$$\aleph_1 \not\to (\aleph_1)^2_2$$

i.e. $[Y]^2 \subset I_i$ *implies* $|Y| < \aleph_1$ *for* $Y \subset Z$ *and* $i < 2$?

A variant of the problem: Let $\langle S, \prec \rangle$ be an ordered set, $|S| = \aleph_2$. Assume S

does not contain increasingly or decreasingly well-ordered subsets of type ω_2.

Does then there exist $Z \subset S$, $|Z| = \aleph_1$ such that Z does not contain increasingly or decreasingly well-ordered subsets of type ω_1?

This obviously is a "Kurepa-Chang conjecture" type problem.

ADDED IN PROOF. Devlin proved that the answer is no, provided $V = L$.

Problem II. *Let* $|S| = \aleph_1$. *Does there exist a 2-partition I of length 2 of S (i.e.* $[S]^2 = I_0 \cup I_1$*) such that I establishes* $\aleph_1 \nrightarrow (\aleph_1)^2_2$, *but for every* $Z \subset S$, $|Z| = \aleph_1$ *and for* $i < 2$ *there are* $A, B \subset Z$, $|A| = |B| = \aleph_1$ *with* $[A, B] \subset I_i$?

ADDED IN PROOF. Both Galvin and Shelah pointed out to us that the answer is obviously yes. See **[13]**.

The following should be read knowing the remarks given on pp. 271–274 concerning Problems 10–13.

THEOREM (HAJNAL). *Let* $\kappa = \Omega(a^+)$, $a \geqq \aleph_0$. *Assume* $\Phi \to (\kappa)^1_{2^a}$. *Then*

$$\Phi \to (\kappa + 1)^2_a.$$

This is an easy Galvin type generalization of the Erdös-Rado Theorem.

Let us now assume G.C.H., $a = \aleph_0$. We do not know if one can prove the possible generalizations of $\omega_2 \to (\omega_1 + 1)^2_{\aleph_0}$ mentioned on p. 272.

E.g. does $\Phi \to (\omega_1)^1_{2\aleph_0}$ imply $\Phi \to (\omega_1 + n)^2_k$ for $k, n < \omega$?

We make the following

DEFINITION 1. Let $a, b_\nu, \nu < \Omega(c)$, be cardinals or order types, and c, r cardinals. Assume that each b_ν is a cardinal if a is a cardinal.

We write $a \xrightarrow{R} (b_\nu)^r_c$ if the following statement is true.

Let S be a set if a is a cardinal and let $\langle S, \prec\rangle$ be an ordered set if a is an order type such that $|S| = a$ or typ $S(\prec) = a$ respectively.

Let $\langle I_\nu : \nu < \Omega(c)\rangle$ be an r-partition of type c of S. Then there exist $\nu < \Omega(c)$, e, d, and a subset $S' \subset S$ such that

(i) $|S| = e$ or typ $S(\prec) = e$ respectively.

(ii) $e \to (d, b_\nu)^r$ but $b_\nu \nrightarrow (d, b_\nu)^r$.

(iii) For all $X \subset S'$, $|X| = d$ (typ $X(\prec) = d$) $[X]^r \cap I_\nu \neq \varnothing$.

It is obvious that (i), (ii), (iii) imply that there is $S'' \subset S$, $|S''| = b_\nu$ (typ $S''(\prec) = b_\nu$) with $[S'']^r \subset I_\nu$. Hence $a \xrightarrow{R} (b_\nu)^r_c$ implies $a \to (b_\nu)^r_c$. We might say that $a \xrightarrow{R} (b_\nu)^r_c$ says that $a \to (b_\nu)^r_c$ holds as a consequence of a partition relation for two classes.

We did not investigate the general problem very closely but the following special case seems to be interesting.

Problem III. *Assume* G.C.H. *Does then* $\aleph_2 \xrightarrow{R} (\aleph_1)^2_{\aleph_0}$ *hold*?

We formulate this special case without using the general notation.

(1) Assume $|S| = \aleph_2$. Let $[S]^2 = \bigcup_{n<\omega} I_n$ be arbitrary. Does then there exist $n < \omega$, $S' \subset S$ such that

$$|S'| = \aleph_2 \quad \text{and} \quad [X]^2 \cap I_n \neq \varnothing$$

for all $X \subset S'$, $|X| = \aleph_2$?

We mention two more possible generalizations of the ordinary Ramsey problem. Let a, b_ν, $\nu < \Omega(c)$, r,c be given as above, $|S| = a$ or typ $S(<) = a$ respectively.

Let $T \subset [S]^r$ be fixed, and assume $a \to (b_\nu)^r_c$.

We formulate the partition property relative to T.

(2) Whenever $T = \bigcup_{\nu<\Omega(c)} I_\nu$ then there are $S' \subset S$, $\nu < \Omega(c)$ $|S'| = b_\nu$ (typ $S'(<) = b_\nu$) with $[S']^r \subset I_\nu$.

One can ask two sorts of problems.

(a) Does (2) hold for every T satisfying certain properties?

(b) Does (2) hold for some T satisfying a certain property?

Assume now G.C.H. Let $a = \aleph_2$, $b_\nu = \aleph_1$, $c = \aleph_0$, $r = 2$. It is obvious that (2) holds for every T satisfying the following property:

(3) There is $S' \subset S$, $|S'| = \aleph_2$ such that $[X]^2 \cap T \neq \varnothing$ for every $X \subset S'$, $|X| = \aleph_2$.

In fact this follows from $\aleph_2 \to (\aleph_2, (\aleph_1)_{\aleph_0})^2$.

So here a question of type (b) arises if there is a T which does not satisfy (3) but satisfies (2). Here we have an answer.

THEOREM. *Let* $|S| = (2^a)^+ = b$, $a \geqq \aleph_0$. *There is a* $T \subset [S]^2$ *satisfying the following conditions*:

(i) *For every* $S' \subset S$, $|S'| = b$, *there is* $S'' \subset S'$, $|S''| = b$ *such that* $[S'']^2 \cap T = \varnothing$.

(ii) *For every* $\bigcup_{\nu<\Omega(c)} I_\nu = T$ *there are* $\nu < \Omega(c)$ *and* $Z \subset S$ *with* $[Z]^2 \subset I_\nu$ *and* $|Z| = a^+$.

PROOF IN OUTLINE. Put $\kappa = \Omega(a^+)$. By a theorem of Baumgartner, there is an ordered set $\langle S, \prec\rangle$, typ $S(\prec) = \Phi$ such that $\Phi \to (\kappa)^1_{2^a}$, and for every $S' \subset S$, $|S'| = b$ there is an $S'' \subset S'$, $|S''| = b$, such that $\langle S'', \succ\rangle$ is well-ordered. Let $<_0$ be a well-ordering of S, and $T = \{\{x, y\} : x, y \in S \wedge x \prec y \wedge x <_0 y\}$, i.e. the Sierpinski class. Then T satisfies (i), and (ii) holds by the theorem mentioned on p. 282

A typical theorem of type (a) is Galvin's generalization of the Baumgartner-Hajnal theorem $\omega_1 \to (\alpha)^2_k$ mentioned on p. 271.

Here is a further possible generalization of type (a).

Problem IV. *Let* typ $S(\prec) = \omega_1$, *and let* $T \subset [S]^2$ *be such that for all* $S' \subset S$, $|S'| = \aleph_1$, *and* $\alpha < \omega_1$ *there is* $S'' \subset S'$, $[S'']^2 \subset T$, typ $S''(\prec) = \alpha$.

Let $T = \bigcup_{n<k} T_n$, $k < \omega$.

Does there exist an $n < k$ *such that for all* $\alpha < \omega$, *there is* $Z \subset S$, $[Z]^2 \subset I_n$, typ $Z(\prec) = \alpha$?

The following problem was formulated by Erdös and Prikry.

Let $|S| = \aleph_0$, $[S]^{\aleph_0} = \bigcup_{n<\omega} I_n$ be arbitrary. Does then there exist $n < \omega$ and $A, B, C \in I_n$ such that A, B, C are different and $A \cup B = C$?

This was answered by G. Elekes who proved that the answer is affirmative. His proof does not work for the general case when $\aleph_0$ is replaced by an arbitrary cardinal a.

Problem V (Erdös-Prikry). *Let* $|S| = \aleph_1$, $[S]^{\aleph_1} = \bigcup_{\xi<\omega_1} I_\xi$. *Does there exist* $\xi < \omega_1$ *and three sets* $A, B, C \in I_\xi$ *such that* $A \cup B = C$?

We do not know the answer even if we assume $2^{\aleph_0} = \aleph_1$.

Galvin remarked that if we partition to countably many classes only and assume C.H., then we have a positive answer.

It is worth remarking that another similar question of Erdös and Prikry can be solved with Elekes' idea.

A family of sets $\mathscr{F}$ is said to be a Δ-system if there is a set D such that

$$A \cap B = D \quad \text{for all } A \neq B \in \mathscr{F}.$$

THEOREM. *Assume* G.C.H. *Let* $a \geqq \aleph_0$, $|S| = a$*-regular*

$$[S]^a = \bigcup_{\xi<\Omega(a)} I_\xi.$$

Then there is $\xi < \Omega(a)$, *and a* Δ*-system* $\mathscr{F}$, *with* $\mathscr{F} \subset I_\xi$ *and* $|\mathscr{F}| = a$.

Problem VI. *Assume* G.C.H. *Let* $|S| = \aleph_2$. *Does there exist a disjoint partition* $[S]^2 = \bigcup_{\nu<\omega_1} I_\nu$ *satisfying the following condition*:

For all $S' \subset S$, $|S'| = \aleph_2$ *there is* $Z \subset S$, $|Z| = \aleph_1$ *such that all different pairs of* Z *belong to different* I_ν?

The following partition relation was defined in [**10**, 20.1] (for cardinals) as a special case of a more general relation we do not introduce here.

DEFINITION 2. Let a, b, c, d be types, $r < \omega$. $(a, b) \xrightarrow{0} (c, d)^r$ is said to hold if for every ordered set $\langle S, <\rangle$ of typ $S(<) = a$ and for every $I \subset [S]^r$, one of the following conditions hold:

(i) There is $X \subset S$, $[X]^r \subset I$, typ $X(<) = c$.

(ii) There is $Y \subset S$, typ $Y(<) = d$ such that for all $Z \subset Y$, $[Z]^r \subset I$ we have $|Z| < b$.

Note that $(a, r) \xrightarrow{0} (c, d)^r$ iff $a \to (c, d)^r$. Galvin and Shelah rediscovered this relation for types and with Erdös they gave a number of remarks concerning the simplest cases.

THEOREM. (a) $(\omega_1, n + 2) \xrightarrow{0} (\omega_1, \omega(n + 1) + n + 1)^2$ *for* $n < \omega$.

(b) C.H. $\Rightarrow (\omega_1, n + 2) \not\xrightarrow{0} (\omega_1, \omega(n + 1) + n + 2)^2$ *for* $n < \omega$.

In case $n = 0$, (a) is $\omega_1 \to (\omega_1, \omega + 1)^2$, an old Erdös-Rado theorem; (b) is $\omega_1 \not\to (\omega_1, \omega + 2)^2$, an old result of Hajnal. The following is the simplest unsolved case.

Problem VII. *Assume* C.H. *Does* $(\omega_1, \omega) \xrightarrow{0} (\omega_1, \omega^2)^2$ *hold*?

In case $a > \omega_1$ the problem becomes more involved because of the independence results. Since $\omega_2 \nrightarrow (\omega_1 + \omega)^2_2$ is consistent with G.C.H., one can ask problems where c is less than a and assuming $\omega_2 \nrightarrow (\omega_1 + \omega)^2_2$ one can get a partial result similar to the above theorem. The first problem which remains is the following:

Problem VIII. *Is* $(\omega_2, \omega) \xrightarrow{0} (\omega_1 + \omega, \omega_1 \cdot \omega)^2$ *provable in* ZFC + G.C.H. *or is the* $\overset{0}{\nrightarrow}$ *consistent with* ZFC + G.C.H.?

To make clear the next problems we state a number of results. First we need some definitions.

DEFINITION 3. (a) We will consider pairs $\langle S, I\rangle$ where $I \subset [S]^2$. As need will be we will consider them partitions (meaning the 2-partition of length 2 of S: $[S]^2 = I \cup ([S]^2 - I)$) or relational structures with one binary, symmetric, irreflexive relation.

(b) We briefly write $\langle Z, \mathscr{J}\rangle \to \langle S, I\rangle$ if $\langle Z, \mathscr{J}\rangle$ is isomorphic to a substructure of $\langle S, I\rangle$. We put $\langle Z, \mathscr{J}\rangle \xrightarrow{W} \langle S, I\rangle$ iff

$$\begin{array}{ll} \text{either} & \langle Z, \mathscr{J}\rangle \to \langle S, I\rangle \\ \text{or} & \langle Z, \mathscr{J}\rangle \to \langle S, \neg I\rangle = \langle S, [S]^2 - I\rangle. \end{array}$$

(c) Let a, b, c be cardinals. We say that $\langle S, I\rangle$ establishes $a \nrightarrow (b, c)^2$ if $|S| = a$ and

$$\begin{array}{ll} [X]^2 \not\subset I & \text{for all } X \in [S]^b, \\ [Y]^2 \cap I \neq \varnothing & \text{for all } Y \in [S]^c. \end{array}$$

The definition extends to partition relations with more complicated entries in a self-explanatory way.

THEOREMS. (1) *If* $\langle S, I\rangle$ *establishes* $\aleph_1 \nrightarrow ([\aleph_0, \aleph_1])^2_2$ *then*

$$\langle Z, \mathscr{J}\rangle \to \langle S, I\rangle \quad \textit{for all } \langle Z, \mathscr{J}\rangle$$

with $|Z| \leqq \aleph_0$.

(2) *Let* $\langle Z_0, \mathscr{J}_0\rangle$ *be a "quadrilateral without diagonals", i.e.*

$$Z_0 = [0, 4), \qquad \mathscr{J}_0 = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{0, 3\}\}.$$

Then the following two conditions are equivalent:

(i) $\langle Z_0, \mathscr{J}_0\rangle \to \langle S, I\rangle$ *for all* $\langle S, I\rangle$ *establishing* $\aleph_1 \nrightarrow (\aleph_1)^2_2$.

(ii) *Suslin's Hypothesis* = S.H.

(*Part of this equivalence is implicitly contained in the old literature.*)

(3) *Let* $\langle Z_1, \mathscr{J}_1\rangle$ *be defined as follows*:

$Z_1 = \bigcup_{n<\omega} T_n$, $|T_n| = \aleph_0$ *for* $n < \omega$, *the* T_n *are disjoint*;

$\mathscr{J}_1 = \bigcup_{n<\omega}[T_n]^2$.

Then $\langle Z_1, \mathscr{J}_1\rangle \xrightarrow{W} \langle S, I\rangle$ *for all* $\langle S, I\rangle$ *establishing* $\aleph_1 \nrightarrow (\aleph_1)^2_2$.

We say $\langle S, I\rangle$ *is a tree if there is a partial order such that* $\langle S, \leqslant\rangle$ *is a tree and* I *consists of the comparable pairs.*

(4) *Assume* $\neg$S.H. *and* $\langle Z, \mathscr{J}\rangle$ *is finite. Then*

(a) $\langle Z, \mathscr{J}\rangle \xrightarrow{W} \langle S, I\rangle$ *for all* $\langle S, I\rangle$ *establishing* $\aleph_1 \nrightarrow (\aleph_1)^2_2$ *iff* $\langle Z, \mathscr{J}\rangle$ *is a tree.*

(b) $\langle Z, \mathscr{J}\rangle \to \langle S, I\rangle$ *for all* $\langle S, I\rangle$ *establishing* $\aleph_1 \nrightarrow (\aleph_1)^2_2$ *iff* $\langle Z, \mathscr{J}\rangle$ *is a tree of the following form*:

$$Z = \{x_i : i < n\} \cup \{y_i : i \in M \subset [0, n)\},$$

$x_0 \prec \cdots \prec x_{n-1}$; $x_i \prec y_i$ *for* $i \in M$, *the* y_i *are incomparable.* ($\neg$S.H. *is used only to show that the conditions are necessary.*)

(5) *Let* $\langle Z_2, \mathscr{J}_2\rangle$ *belong to a normal tree of rank* ω. *Then* $\langle Z_2, \mathscr{J}_2\rangle \overset{W}{\nrightarrow} \langle R, I_0\rangle$, *where R is the set of reals and* I_0 *is the Sierpinski partition.*

Problem IX. *Assume* G.C.H. *and let* $\langle S, I\rangle$ *establish* $\aleph_{\alpha+1} \nrightarrow ([\aleph_\alpha, \aleph_{\alpha+1}])^2_2$. *For what* $\beta \leqq \alpha$ *does*

$$\langle Z, \mathscr{J}\rangle \to \langle S, I\rangle \tag{0}$$

hold for $|Z| \leqq \aleph_\beta$?

Note, that as a generalization of Theorem (1), (0) holds with $\beta = 0$ for every α. For $\alpha = 1$, $\beta = 1$, we think that a negative answer will be consistent.

For $\alpha = 2$, $\beta = 1$, we have no guess.

We now list some questions concerning the case $|S| = \aleph_1$, not covered by the given theorems. Some of them might be quite easy to answer.

Problem X. *We have no counterexample to the following*:

(1) *Assume* $\langle Z, \mathscr{J}\rangle \to \langle R, I_0\rangle$ (*where* $\langle R, I_0\rangle$ *is the Sierpinski partition defined in Theorem* (5) *and* $|Z| < \aleph_0$ (*or even* $|Z| \leqq \aleph_0$). *Is it true then that* $\langle Z, \mathscr{J}\rangle \to \langle S, I\rangle$ *holds for every* $\langle S, I\rangle$ *establishing* $\aleph_1 \nrightarrow ([\aleph_1, \aleph_1])^2_2$?

(2) *Give the characterization of Theorem* (4) *assuming* S.H. (*The characterization is different because of Theorem* (2).)

(3) *Assume* $\neg$S.H. *Give a characterization of countable structures* $\langle Z, \mathscr{J}\rangle \xrightarrow{W} \langle S, I\rangle$ *for all* $\langle S, I\rangle$ *establishing* $\aleph_1 \nrightarrow (\aleph_1)^2_2$.

REFERENCES

1. J. Baumgartner, *A new class of order types* (preprint).

2. J. Baumgartner and A. Hajnal, *A proof (involving Martin's axiom) of a partition relation*, Yellow Series of the University of Calgary, Research Paper 122, April 1971; Fund. Math. (in print).

3. C. C. Chang, *A partition theorem for the complete graph on* ω^ω, J. Combinatorial Theory **12** (1972), 396–452.

4. K. J. Devlin, *On a problem of B. Jonsson*, Notices Amer. Math. Soc. **18** (1971), 664. Abstract #71T-E49; *Some weak versions of large cardinal axioms*, Ph.D. Thesis, Bristol, 1971.

5. K. J. Devlin, J. B. Paris, *More on the free subset problem* (preprint).

6. P. Erdös and A. Hajnal, *Unsolved problems in set theory*, Proc. Sympos. Pure Math., vol. 13, part 1, Amer. Math. Soc., Providence, R.I., 1971, pp. 17–48.

7. ———, *Some results and problems for certain polarized partitions*, Acta Math. Acad. Sci. Hungar. **21** (1970), 369–392.

8. P. Erdös, A. Hajnal and E. C. Milner, *Set mappings and polarized partition relations*, Combinatorial Theory and its Applications, Balatonfüred, Hungary, 1969.

9. P. Erdös, A. Hajnal and R. Rado, *Partition relations for cardinals*, Hungarian Academy of Sciences (to appear).

10.———, *Partition relations for cardinal numbers*, Acta Math. Acad. Sci. Hungar. **16** (1965), 93–196. MR **34** #2475.

11. J. Folkman, *Graphs with monochromatic complete subgraphs in every edge coloring*, SIAM J. Appl. Math. **18** (1970), 19–24. MR **42** #2979.

12. F. Galvin and K. Prikry, *Borel sets and Ramsey's theorem* (preprint).

13. F. Galvin and S. Shelah, *Some counterexamples in the partition calculus*, J. Combinatorial Theory Ser. A (to appear).

14. L. Haddad and G. Sabbagh, *Calcul de certains nombres de Ramsey généralisés*, C. R. Acad. Sci. Paris Sér A-B **268** (1969), A1233–A1234. MR **40** #1285.

15. A. Hajnal, *A negative partition relation*, Proc. Nat. Acad. Sci. U.S.A. **68** (1971), 142–144. MR **42** #5835.

16.———, *On some combinatorial problems involving large cardinals*, Fund. Math. **69** (1970), 39–53.

17. ———, *Ulam-matrices for inaccessible cardinals*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **17** (1969), 683–688. MR **41** #5221.

18. S. Hechler, *A note on some topological problems of Erdös* (preprint).

19. K. Kunen, *Inaccessibility properties of cardinals*, University Microfilms, 1969.

19.1. ———, *A partition theorem*, Notices Amer. Math. Soc. **19** (1971), 425. Abstract #71T–E17.

20. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic **2** (1970), 143–148.

21. E. C. Milner, *A finite algorithm for partition relations*, Lecture Delivered at the Canadian Math. Congress, 1971.

22. K. Prikry, *On a problem of Erdös, Hajnal and Rado*, Discrete Math. **2** (1972), 51–59.

23. ———, *On a problem of Gillman and Keisler*, Ann. Math. Logic **2** (1970), no. 2, 179–187. MR **42** #4408.

24. S. Shelah, *A combinatorial problem, stability and order for models and theories of infinitary, languages* (preprint).

25, ———, *Notes in combinatorial set theory* (preprint).

26. J. Silver, *The independence of Kurepa's conjecture and two-cardinal conjectures in model theory*, Proc. Sympos. Pure Math., vol. 13, part 1, Amer. Math. Soc., Providence, R.I., 1971, pp. 383–390.

27. R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, Ann. of Math. **94** (1971), 201–245.

28. W. Taylor, *Atomic compactness and elementary equivalence*, Fund. Math. **71** (1971), 103–112.

MAGYAR TUDOMANYOS AKADEMIA, BUDAPEST, HUNGARY