COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
10. INFINITE AND FINITE SETS, KESZTHELY (HUNGARY), 1973.

A NON-NORMAL BOX PRODUCT

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We use the convention that a cardinal is the smallest ordinal of that cardinality, and an ordinal is the set of ordinals less than it is. The topology on an ordinal is the order topology.

If $\{X_n\}_{n \in \omega_0}$ is a collection of topological spaces, then the *box product* of $\{X_n\}_{n \in \omega_0}$ is $\prod_{n \in \omega_0} X_n$ with the topology induced by using $\left\{ \prod_{n \in \omega_0} U_n \subset \prod_{n \in \omega_0} X_n \mid U_n \text{ is open in } X_n \text{ for all } n \right\}$ as a basis.

Suppose X is the box product of $\{\alpha_n\}_{n \in \omega_0}$ where each α_n is an ordinal; if $f \in X$, then $f(n)$ will denote the n-th coordinate of f.

We say F is a *scale* (of cardinality κ) provided F is a family $\{f_\alpha\}_{\alpha < \kappa}$ of members of $\omega_0^{\omega_0}$, such that:

(1) $\alpha < \beta < \kappa$ implies $f_\alpha(n) < f_\beta(n)$ for all but finitely many n,

(2) $f \in \omega_0^{\omega_0}$ implies there is an $\alpha < \kappa$ and an $m < \omega$ with $f(m) < f_\alpha(m)$ for all $m > n$.

Suppose κ is cardinal for which there is a scale. Clearly $\omega_1 \leqslant \kappa \leqslant \leqslant 2^{\omega_0}$; so the Continuum Hypothesis [CH] yields $\kappa = \omega_1$. But it is consistent with the usual axioms of set theory that κ be ω_1 or ω_2 or $\omega_3, \ldots$.

In [1] it is proved among other things that:

(a) [CH] implies X is paracompact if $\alpha_n = \omega_0 + 1$ for all n.

(b) [CH] implies X is paracompact if $\alpha_n = \omega_n + 1$ for all n.

(c) [CH] implies X is normal (but not paracompact) if $1 < k \in \omega_0$ and $\alpha_0 = \omega_k$ but $\alpha_n = \omega_0 + 1$ for all $n > 1$.

(d) No conclusion is reached if $\alpha_0 = \omega_1$ and $\alpha_n = \omega_0 + 1$ for $n > 0$.

Consider these facts in the light of the theorem proved in this paper:

Theorem (Erdős): *If $\kappa \neq \omega_1$ is the minimal cardinality of a scale, then X is not normal where $\alpha_0 = \kappa$ and $\alpha_n = \omega_0 + 1$ for all $n > 0$.*

Thus it is consistent with the usual axioms of set theory that the box product $\omega_k \times (\omega_0 + 1) \times (\omega_0 + 1) \times \ldots$ be either normal or not normal for all integers $k > 1$. But the problem with $k = 1$ is still untouched and seems harder than ever. Also the conjecture of Rudin that (a) is true without [CH] in the hypotheses seems more interesting.

Proof of the Theorem. Assume $\kappa > \omega_1$ is the cardinality of a scale $\{f_\alpha\}_{\alpha<\kappa}$ and there is no shorter scale. Also assume X is the box product $(\kappa \times (\omega_0 + 1) \times (\omega_0 + 1) \times \ldots)$. For each $\alpha < \kappa$ and $i < \omega_0$, define $h_{\alpha i} \in X$ by $h_{\alpha i}(0) = \alpha$ and $h_{\alpha i}(n) = f_\alpha(n-1) + i$ for $n > 0$. Let $H = \{h_{\alpha i} \mid \alpha < \kappa$ and $i < \omega_0\}$. For each $\alpha < \kappa$, define $k_\alpha \in X$ by $k_\alpha(0) = \alpha$ and $k_\alpha(n) = \omega_0$ for all $n > 0$. Let $K = \{k_\alpha \mid \alpha < \kappa\}$.

Observe that K is closed and disjoint from $\bar{H}$. Assume open sets $U \supset H$ and $V \supset K$. We prove $U \cap V \neq \phi$ and thus X is not normal.

For $0 < \alpha < \kappa$ we assume without loss of generality that $k_\alpha(n) > 0$ and $h_{\alpha i}(n) > 0$. Thus, since U and V are open, there are $u_{\alpha i}$ and

v_α of X such that $u_{\alpha i}(n) < h_{\alpha i}(n)$ and $v_\alpha(n) < k_\alpha(n)$ for all n, and $\{g \in X \mid u_{\alpha i}(n) < g(n) \leqslant h_{\alpha i}(n)\} \subset U$ and $\{g \in X \mid v_\alpha(n) < g(n) \leqslant k_\alpha(n)\} \subset V$.

For each $0 < \beta < \kappa$, $v_\beta(0) < \beta$. Thus there is a $\delta < \kappa$ such that $\gamma < \kappa$ implies $\gamma < \beta$ for some β with $v_\beta(0) < \delta$. Let $\Delta = \{\beta < < \kappa \mid v_\beta(0) < \delta\}$. Let $\theta = \{\alpha < \kappa \mid \alpha$ has uncountable cofinality$\}$. Since $\{u_{\alpha i}(0)\}_{i \in \omega_0}$ is countable, for each $\alpha \in \theta$ there is $\beta_\alpha < \alpha$ such that $u_{\alpha i}(0) < \beta_\alpha$ for all $i \in \omega_0$. Again, since $\beta_\alpha < \alpha$ for all $\alpha \in \theta$ and the cofinality of κ is greater than ω_1, there is $\lambda < \kappa$ implies $\gamma < \alpha$ for some $\alpha \in \theta$ with $u_{\alpha i}(0) < \lambda$ for all $i \in \omega_0$. Let $\Lambda = \{\alpha < \kappa \mid u_{\alpha i}(0) < \lambda$ for all $i \in \omega_0\}$.

Choose $\mu < \kappa$ with $\lambda < \mu$ and $\delta < \mu$. Choose $\beta \in \Delta$ with $\mu < \beta$. There is $\eta < \kappa$ with $f_\eta(n) > v_\beta(n+1)$ for all $n > 0$. Choose $\alpha \in \Lambda$ with $\alpha > \eta$ and $\alpha > \mu$. Then $f_\alpha(n) > v_\beta(n+1)$ for all but finitely many n. Thus there exists a positive integer i such that $f_\alpha(n) + i > v_\beta(n+1)$ for all n_i hence $h_{\alpha i}(n+1) > v_\beta(n+1)$ for all $n \in \omega_0$. Since $\alpha \in \Lambda$ and $\lambda < \mu < \alpha$, $(\mu, h_{\alpha i}(1), h_{\alpha i}(2), \ldots) \in U$. Since $\beta \in \Delta$ and $\delta < \mu < \alpha$ and $v_\beta(n+1) < h_{\alpha i}(n+1) < \omega_0$ for all n, $(\mu, h_{\alpha i}(1), h_{\alpha i}(2), \ldots) \in V$. Thus $U \cap V \neq \phi$.

REFERENCE

[1] M.E. Rudin, Countable box products of ordinals, *Transactions of the A.M.S.*, (to appear).