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MAXIMAL ASYMPTOTIC NONBASES

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ABSTRACT. Let A be a set of nonnegative integers. If all but a finite number of positive integers can be written as a sum of h elements of A, then A is an asymptotic basis of order h. Otherwise, A is an asymptotic nonbasis of order h. A class of maximal asymptotic nonbases is constructed, and it is proved that any asymptotic nonbasis of order 2 that satisfies a certain finiteness condition is a subset of a maximal asymptotic nonbasis of order 2.

Let A be a set of nonnegative integers containing 0. The h-fold sum of A, denoted hA, is the set of all sums of h not necessarily distinct elements of A. If hA contains all but a finite number of positive integers, then A is an *asymptotic basis of order h*. The set A is a *minimal asymptotic basis of order h* if A is an asymptotic basis of order h, but $A \setminus \{a\}$ is not an asymptotic basis of order h for every $a \in A$. Examples of minimal asymptotic bases were constructed in [1], and also an example of an asymptotic basis which contains no subset that is a minimal asymptotic basis.

The set A is an *asymptotic nonbasis of order h* if A is not an asymptotic basis of order h. If A is an asymptotic nonbasis of order h, but $A \cup \{a\}$ is an asymptotic basis of order h for every nonnegative integer $a \notin A$, then A is a *maximal asymptotic nonbasis of order h*. Maximal asymptotic nonbases were constructed in [1] by taking finite unions of the nonnegative parts of congruence classes. In this paper we construct a new class of maximal asymptotic nonbases that are not unions of congruence classes, and we prove that every asymptotic nonbasis of order 2 that satisfies a certain finiteness condition is a subset of a maximal asymptotic nonbasis of order 2. We do not know whether every asymptotic nonbasis is a subset of a maximal asymptotic nonbasis, nor whether there exist maximal asymptotic nonbases with zero density.

Let $[a, b]$ denote the set of integers n such that $a \leq n \leq b$.

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Theorem 1. *Let $h \geq 2$, and let $n_1 < n_2 < \cdots$ be an increasing sequence of positive integers such that $h^2 n_t + 2h \leq n_{t+1}$. Let*

$$A = [0, n_1] \cup \bigcup_{t=1}^{\infty} [hn_t + 2, n_{t+1}].$$

Then there exists a maximal asymptotic nonbasis A^* of order h such that $A \subset A^*$ and $hA = hA^*$.

Proof. We shall construct an increasing sequence $A = A_0 \subset A_1 \subset A_2 \subset \cdots$ of asymptotic nonbases of order h and two increasing sequences of positive integers $m_1 < m_2 < \cdots$ and $q_1 < q_2 < \cdots$ such that

(i) $m_1 < m_2 < \cdots < m_k$ are the k smallest integers not in A_k;

(ii) $A_k \cup \{m_k\}$ is an asymptotic basis of order h;

(iii) $hA_k = hA$ for all k; and

(iv) $q_j \notin (h-1)A_k$ for all $j \in [1, k]$.

Let $A^* = \bigcup_{k=0}^{\infty} A_k$. Clearly, $hA \subset hA^*$, since $A = A_0 \subset A^*$. If $n \in hA^*$, then $n \in hA_k$ for some k, and so $n \in hA$ by (iii). Therefore, $hA^* = hA$, and A^* is an asymptotic nonbasis of order h. Let $m \notin A^*$. Then $m \leq m_k$ for some k, and $m \notin A_k$. It follows from (i) that $m = m_j$ for some $j \in [1, k]$, and from (ii) that $A^* \cup \{m\}$ is an asymptotic basis of order h. Therefore, A^* is a maximal asymptotic nonbasis of order h such that $hA = hA^*$.

We construct the sequences $\{A_k\}$, $\{m_k\}$, and $\{q_k\}$ inductively. Clearly, hA consists of all nonnegative integers except those of the form $hn_t + 1$. Let m_1 be the largest positive integer such that $(h-1)(A \cup [0, m_1 - 1]) = (h-1)A$. Then $(h-1)A \subsetneq (h-1)(A \cup [0, m_1])$. Let $A_1' = A \cup [0, m_1 - 1]$, and choose an integer q_1 in

$$(h-1)(A_1' \cup \{m_1\}) \setminus (h-1)A_1' = (h-1)(A \cup [0, m_1]) \setminus (h-1)A.$$

Let

$$B_1 = \{hn_t + 1 - q_1 \mid hn_t + 1 - q_1 > \max(n_t, m_1, q_1)\}$$

and let $A_1 = A_1' \cup B_1$. Since $[0, m_1 - 1] \subset A_1' \subset A_1$ and $m_1 \notin B_1$, it follows that m_1 is the smallest positive integer not in A_1. If $hn_t + 1 \in hA_1$, then $hn_t + 1$ is the sum of h elements of A_1, and at least one of these summands must be in the interval $[n_t + 1, hn_t + 1]$. But there is at most one element of A_1 in this interval, namely, $hn_t + 1 - q_1$, hence $hn_t + 1 - q_1$ must be one of the h summands of $hn_t + 1$. Then the sum of the $h - 1$ remaining summands must be q_1. Since all elements of B_1 are greater than q_1, these summands are all elements of A_1'. But $q_1 \notin (h-1)A_1'$. Therefore, $hn_t + 1 \notin hA_1$, and so $hA = hA_1$. But

$$q_1 \in (h-1)(A_1' \cup \{m_1\}) \subset (h-1)(A_1 \cup \{m_1\}),$$

and so $A_1 \cup \{m_1\}$ is an asymptotic basis of order h. Therefore, the integers m_1 and q_1 and the asymptotic nonbasis A_1 satisfy conditions (i)–(iv).

Now suppose that integers $m_1 < \cdots < m_{k-1}$ and $q_1 < \cdots < q_{k-1}$ and asymptotic nonbases $A = A_0 \subset A_1 \subset \cdots \subset A_{k-1}$ satisfy conditions (i)–(iv). If $(h-1)(A_{k-1} \cup \{m_{k-1}+1\}) \neq (h-1)A_{k-1}$, let $m_k = m_{k-1}+1$. Otherwise, let m_k be the largest integer such that $m_k > m_{k-1}$ and

$$(h-1)(A_{k-1} \cup [m_{k-1}+1, m_k - 1]) = (h-1)A_{k-1}.$$

Let $A_k' = A_{k-1} \cup [m_{k-1}+1, m_k-1]$. Then $(h-1)A_{k-1} = (h-1)A_k' \subsetneqq (h-1)(A_k' \cup \{m_k\})$. Choose an integer q_k in $(h-1)(A_k' \cup \{m_k\}) \setminus (h-1)A_k'$, and let

$$B_k = \{hn_t + 1 - q_k \mid hn_t - q_k > \max(n_t, m_k, q_1, \cdots, q_k)\}.$$

Now let $A_k = A_k' \cup B_k$. Since $A_k \setminus A_{k-1}$ consists of integers all greater than m_{k-1}, and since $[m_{k-1}+1, m_k-1] \subset A_k' \subset A_k$, it follows that $m_1 < \cdots < m_{k-1} < m_k$ are the k smallest integers not in A_k. If $hn_t + 1 \in hA_k$, then $hn_t + 1$ is the sum of h elements of A_k, at least one of which must be in the interval $[n_t+1, hn_t+1]$. But the only such elements of A_k are of the form $hn_t + 1 - q_j$ for $j \in [1, k]$. Since the elements of B_k are all larger than every q_j, it follows that $q_j \in (h-1)A_k'$ for some $j \in [1, k]$. But $q_k \notin (h-1)A_k'$, and, since $(h-1)A_k' = (h-1)A_{k-1}$, also $q_j \notin (h-1)A_k'$ for $j \in [1, k-1]$. Therefore, $hn_t + 1 \notin hA_k$, and so $hA_k = hA$. But $q_k \in (h-1)(A_k' \cup \{m_k\}) \subset (h-1)(A_k \cup \{m_k\})$, and so $A_k \cup \{m_k\}$ is an asymptotic basis of order h. Thus, the integers m_k and q_k and the set A_k satisfy conditions (i)–(iv). This completes the induction.

Remark. Since A contains arbitrarily long sequences of consecutive integers, and $A \subset A^*$, the maximal asymptotic nonbasis A^* is not a finite union of the nonnegative parts of congruence classes.

Theorem 2. *Let A be an asymptotic nonbasis of order h such that $A \cup F$ is an asymptotic nonbasis of order h for any finite set F of nonnegative integers. Then $A \subset A^*$, where A^* is an asymptotic nonbasis of order h such that, for every integer $x \notin (h-1)A^*$, the set $A^* \cup \{x\}$ is an asymptotic basis of order h.*

Proof. We shall construct a sequence $A = A_0 \subset A_1 \subset A_2 \subset \cdots$ of asymptotic nonbases of order h, and an increasing sequence of positive integers $n_1 < n_2 < \cdots$ such that

(i) $A_k \setminus A_{k-1}$ is a finite set of positive integers all larger than n_{k-1};

(ii) $n_1 < n_2 < \cdots < n_k$ are the k smallest integers not in hA_k; and

(iii) if $0 < x < n_k/2$ and $x \notin (h-1)A_k$, then $n_k - x \in A_k$.

Let $A^* = \bigcup_{k=0}^{\infty} A_k$. By (i) and (ii), the set hA^* does not contain the numbers $n_1, n_2, \cdots$, and so A^* is an asymptotic nonbasis of order h. If $x \notin (h-1)A^*$, then $x \notin (h-1)A_k$ for all k. Choose $n_k > 2x$. Then $n_k - x \in A_k \subset A^*$ by (iii), and so $n_k \in 2(A^* \cup \{x\}) \subset h(A^* \cup \{x\})$, since $0 \in A \subset A^*$. Therefore, $A^* \cup \{x\}$ is an asymptotic basis of order h for every positive integer $x \notin (h-1)A^*$.

We construct the sequences $\{A_k\}$ and $\{n_k\}$ inductively. Suppose that integers $n_1 < \cdots < n_{k-1}$ and asymptotic nonbases $A = A_0 \subset A_1 \subset \cdots \subset A_{k-1}$ satisfy conditions (i)–(iii). Let $A_k' = A_{k-1} \cup [n_{k-1}+1, 2n_{k-1}]$. By (i), $A_k' \setminus A$ is finite, and so the set A_k' is an asymptotic nonbasis of order h. Let n_k be the smallest integer such that $n_k > n_{k-1}$ and $n_k \notin hA_k'$. Then $n_k > 2n_{k-1}$. Let F_k be a maximal subset of the interval $[n_k/2, n_k]$ such that $n_k \notin h(A_k' \cup F_k)$. Let $A_k = A_k' \cup F_k$. Clearly, the set A_k satisfies conditions (i) and (ii). If $0 < x < n_k/2$ and $x \notin (h-1)A_k$, then $n_k - x \in [n_k/2, n_k]$, and so $F_k \cup \{n_k - x\} \subset [n_k/2, n_k]$ and $n_k \notin h(A_k' \cup F_k \cup \{n_k - x\})$. It follows from the maximality of F_k that $n_k - x \in F_k \subset A_k$. Therefore, A_k satisfies condition (iii), and the induction is complete.

Corollary. *Let A be an asymptotic nonbasis of order 2 such that $A \cup F$ is an asymptotic nonbasis of order 2 for every finite set F of nonnegative integers. Then A is a subset of a maximal asymptotic nonbasis of order 2.*

Remark. The Corollary suggests the following problem. If A is an asymptotic basis of order 2 such that $A \setminus F$ is also an asymptotic basis of order 2 for every finite subset F of A, then does A contain a subset that is a minimal asymptotic basis of order 2?

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