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ON PARTITION THEOREMS FOR FINITE GRAPHS

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1. INTRODUCTION

For a given finite graph G and positive integer k, let $r(G; k)$ denote the least integer r such that if the edges of K_r, the complete graph on r vertices, are arbitrarily partitioned into k classes then some class contains a subgraph isomorphic to G. The existence of $r(G; k)$ follows at once from the well-known theorem of Ramsey [8] which asserts that $r(K_n; k) < \infty$ for all n and k. In this paper we investigate the behavior of $r(G; k)$ for large k as G ranges over various classes of graphs.

We shall usually refer to the k classes as "colors" and the copy of G in a single class as "monochromatic". Also, the notation $G(m, n)$ denotes a graph on m vertices and n edges.

2. TREES

Let T_n denote a tree on n edges.

Theorem 1.

$$(1)\quad \begin{array}{ll} \text{(i)} & r(T_n; k) > (n-1)k+1, \quad n \geqslant 1, \quad \text{for } k \text{ large and} \equiv 1 (\text{mod } n); \\ \text{(ii)} & r(T_n; k) \leqslant 2kn+1, \quad n \geqslant 1, \quad k \geqslant 1 . \end{array}$$

Proof. To prove (i), we use the result of Ray-Chaudhuri and Wilson [9] which guarantees the existence of a *resolvable* balanced incomplete block design $D_{k,n}$ having $(n-1)k+1$ points and $\frac{k(kn-k+1)}{n}$ blocks of size n provided only that k is sufficiently large and $\equiv 1$ (mod n). Identify the points $D_{k,n}$ with vertices of $K_{(n-1)k+1}$. Assign the color i to all edges of $K_{(n-1)k+1}$ which correspond to a pair of points occurring in the i-th parallel class of $D_{k,n}$. This is a k-coloring of $K_{(n-1)k+1}$ which contains no monochromatic connected subgraph on $n+1$ vertices and, hence, (i) follows.

To prove (ii), we apply the elementary fact that for all T_n,

$$(2)\qquad T_n \subseteq G(m, mn) .$$

In any k-coloring of K_{2kn+1}, at least $\frac{1}{k}\binom{2kn+1}{2}$ edges must have the same color. Thus, we have a monochromatic $G(2kn+1,\ n(2kn+1))$ which by (2) contains a copy of T_n.

If the conjecture

$$(3)\qquad T_n \subseteq G\left(m, \left[\tfrac{1}{2}(n-1)m\right]+1\right)$$

of Erdős and V.T. Sós [4] were known to hold, (1) could be replaced by

$$(1')\qquad r(T_n;\ k) < kn + O(1)$$

which may be asymptotically correct.

3. FORESTS

Let F_n denote a forest (i.e., an acyclic graph) with n edges and no isolated vertices. Let $u(F_n)$ denote the cardinality of a minimum set of vertices whose removal completely disconnects F_n.

Lemma 1.

$$r(F_n; k) > \left[\frac{k+1}{2}\right](u-1), \qquad k \geq 1, \qquad u \geq 1 . \tag{4}$$

Proof. Let t denote $\left[\frac{k+1}{2}\right]$. Consider $K_{t(u-1)}$ as a K_t with K_{u-1}'s for "vertices". Label these copies of K_{u-1} by $1, 2, \ldots, t$. Assign the color i to all edges between vertices i and j for $1 \leq i < j \leq t$. Assign the color $t - 1 + i$ to all edges within the "vertex" K_{u-1} labeled i. This is a $(2t-1)$-coloring of $K_{t(u-1)}$ which contains no monochromatic copy of F_n (by the definition of $u(F_n)$). Since $2t - 1 \leq k$ then (4) holds. ∎

Note that if F_n has a component with n' edges then it is easy to show (similar to (1)) that

$$r(F_n; k) > (k-1)\left[\frac{n'}{2}\right] . \tag{5}$$

However, any F_n either has a component with $\sqrt{n}$ edges or satisfies $u(F_n) \geq \sqrt{n}$. Thus, (4) and (5) can be combined to give

Theorem 2.

$$r(F_n; k) > \frac{k(\sqrt{n}-1)}{2}, \qquad k \geq 1, \qquad n \geq 1 . \tag{6}$$

On the other hand, there exist for all n examples of F_n for which $r(F_n; k)$ is bounded above by $ck\sqrt{n}$. To see this, we first require a lemma.

Let S_n denote a tree consisting of one vertex of degree n and n vertices of degree 1. Let mS_n denote the disjoint union of m S_n's.

Lemma 2.

(7) $$mS_n \subseteq G(t + m - 1, e)$$

for $e > \binom{m-1}{2} + \left(\frac{n-1}{2} + m - 1\right)t, \ t \geq m(n+1)^2, \ m \geq 1, \ n \geq 1.$

Proof. We proceed by induction on m. For $m = 1$, the lemma simply asserts that $G(t, e)$ has a vertex of degree $\geq n$ if $e > \left(\frac{n-1}{2}\right)t$ and this is certainly true. Assume, for some $m > 1$, the lemma holds for $1, \ldots, m - 1$.

(i) Suppose $G = G(t + m - 1, e)$ has at least m vertices $v_1, \ldots$ $\ldots, v_m$, each with degree $\geq m(n + 1)$. Then for each k, $1 \leq k \leq m$, a copy of S_n centered at v_k may be removed from G and thus, $mS_n \subseteq$ $\subseteq G$ in this case.

(ii) Suppose for some p, $0 \leq p < m$, G has exactly p vertices with degree $\geq m(n + 1)$, say $v_1, \ldots, v_p$. Let G' denote the subgraph of G induced by the remaining $t + m - 1 - p$ vertices. There are two possibilities.

(a) All vertices of G' have degree $\leq n - 1$. Thus G' has at most $(t + m - 1 - p)\left(\frac{n-1}{2}\right)$ edges and so G has at most

$$\binom{p}{2} + \left(p + \frac{n-1}{2}\right)(t + m - 1 - p)$$

edges. But for $p \leq m - 1$ this quantity does not exceed

$$\binom{m-1}{2} + \left(m - 1 + \frac{n-1}{2}\right)t$$

which contradicts the hypotheses on e.

(b) Some vertex v in G' has degree $\geq n$ in G'. We may delete a copy of S_n centered at v from G', causing a net loss of at most $m(n + 1)^2$ edges in G'. Replacing the vertices $v_1, \ldots, v_p$ we have left a graph $G_1 = G_1(t + m - 1 - n - 1, e_1) \subseteq G$ where

$$e_1 > \binom{m-1}{2} + \left(\frac{n-1}{2} + m - 1\right)t - m(n+1)^2 - p(n+1) \geqslant$$
$$\geqslant \binom{m-2}{2} + \left(\frac{n-1}{2} + m - 2\right)(t-n)$$

and

$$t - n + m - 2 \geqslant (m-1)(n+1)^2$$

for $t \geqslant m(n+1)^2$. Hence, by the induction hypothesis, $(m-1)S_n \subseteq G_1$ and so $mS_n \subseteq G$. This completes the proof of (7). ∎

Theorem 3.

(8) $$r(nS_n; k) \leqslant 3kn, \quad n \geqslant 1, \quad k \geqslant 3n^2 .$$

Proof. Let $t = 3kn$. Any k-coloring of K_t contains a monochromatic subgraph $G(t, e)$ where $e \geqslant \frac{1}{k}\binom{t}{2}$. By Lemma 2, $nS_n \subseteq G(t, e)$ provided

$$e > \binom{n-1}{2} + \left(\frac{n-1}{2} + n - 1\right)(t - n + 1)$$

and

$$t - n + 1 \geqslant n(n+1)^2 .$$

But these conditions are certainly satisfied for $t = 3kn$, $k \geqslant 3n^2$, $n \geqslant 1$. ∎

Thus, if n is a square and $k \geqslant 3n$ then

(9) $$r(\sqrt{n}\, S_{\sqrt{n}}; k) \leqslant 3k\sqrt{n} .$$

The following example shows that the bound on e in Lemma 2 is best possible when n is odd. Let H be a regular graph on t vertices of degree $n-1$. Form the graph $G = G\left(t + m - 1, \binom{m-1}{2} + \left(\frac{n-1}{2} + m - 1\right)t\right)$ by adjoining a copy of K_{m-1} and joining each vertex of K_{m-1} to each vertex of H. Clearly $mS_n \not\subseteq G$.

For k relatively small compared to n, the situation is somewhat different.

Theorem 4.

(10) $r(F_n; k) > c_1\sqrt{k}n, \quad 1 \leq k \leq n^2$

for some positive constant c_1 *(independent of* k *and* n*).*

Proof. From a finite projective plane $PP(r)$ of order r, we construct a covering of K_{r^2+r+1} by $r^2 + r + 1$ copies of K_{r+1} as follows. The vertices of K_{r^2+r+1} are the points of $PP(r)$. The vertices of the K_{r+1}'s are just the sets of $r + 1$ points which lie on each of the $r^2 + r + 1$ lines of $PP(r)$. The edges of the K_{r+1}'s cover the edges of K_{r^2+r+1} by the properties of $PP(r)$. Now, replace each point of $PP(r)$ by a copy of K_t where $t = [n/\sqrt{k}]$, keeping in mind the restriction $k \leq n^2$. This gives a covering of $K_{(r^2+r+1)t}$ by $r^2 + r + 1$ copies of $K_{(r+1)t}$. By choosing $r + 1$ to be the greatest prime power $< \sqrt{k} - 1$ (which guarantees the existence of $PP(r)$) and using the fact that $p_{m+1}/p_m \to 1$ for the primes p_m, we see that for a suitable constant $c_1 > 0$, we have covered $K_{c_1\sqrt{k}n}$ by $\leq k$ copies of K_n. Hence, assigning different colors to the edges of the different K_n's, no monochromatic copy of F_n has been formed and (10) follows. ∎

On the other hand, it follows from (7) that for a suitable universal constant c_2,

(11) $r(\sqrt{n}\, S_{\sqrt{n}}) < c_2\sqrt{k}n, \quad 1 \leq k \leq n\,,$

when n is a square. Thus, for both (6) and (10), the upper bound on $r(\sqrt{n}\, S_{\sqrt{n}}; k)$ comes to within a constant factor of the general lower bound.

4. EVEN CYCLES

As might be expected, the more highly structured a graph G is, the more difficult it is to obtain accurate bounds on $r(G; k)$. Still, even the rough bounds we derive for cycles C_m on m vertices point out the striking difference in the behavior of $r(C_m; k)$ for even and odd m. We first consider the case m even.

Theorem 5.

$$(12) \qquad r(C_{2n}; k) > c_3 k^{1+\frac{1}{2n}}, \qquad k \geq 1\,, \quad n \geq 1\,,$$

where $c_3 = c_3(n)$.

Proof. Set $\epsilon = \frac{1}{2n+1}$. For a large h, $h^{1-\epsilon}$-color the edges of K_h uniformly at random. Since there are $(h^{1-\epsilon})^{\binom{h}{2}}$ ways to color K_h and there are $< h^{2n} C_{2n}$'s in K_h then the total number of monochromatic C_{2n}'s in all colorings is $\leq h^{2n} h^{1-\epsilon} (h^{1-\epsilon})^{\binom{h}{2} - 2n}$. Thus, the expected number of monochromatic C_{2n}'s is no more than

$$\frac{h^{2n} (h^{1-\epsilon})^{\binom{h}{2} - 2n + 1}}{(h^{1-\epsilon})^{\binom{h}{2}}} = h^{1+\epsilon(2n-1)}\,.$$

This implies there exists an $h^{1-\epsilon}$-coloring of K_h in which there are $\leq h^{1+\epsilon(2n-1)}$ monochromatic C_{2n}'s formed. Form a graph $G = G(h, e)$ with $e \leq h^{1+\epsilon(2n-1)}$ by removing one edge from each of these monochromatic C_{2n}'s. By a theorem of Nash-Williams [7], G may be decomposed into no more than $\sqrt{e/2} + 1/2$ acyclic subgraphs. If we assign a new color to each of these subgraphs then we have shown the existence of an $\left(h^{1-\epsilon} + c h^{\frac{1}{2}(1+\epsilon(2n-1))}\right)$-coloring of K_h which contains no monochromatic C_{2n}. Replacing ϵ by $\frac{1}{2n+1}$ and letting $k = (1+c) h^{\frac{2n}{2n+1}}$ we see that for a suitable * $c_3 = c_3(n)$,

$$r(C_{2n}; k) > c_3 k^{1+\frac{1}{2n}}, \qquad k \geq 1\,, \quad n \geq 1\,,$$

and (12) is proved. ∎

In the other direction we have the following result.

*Since we must have $h \geq h(n)$ for the preceding arguments to be valid.

Theorem 6. *For all* $\epsilon > 0$, $n \geqslant 2$, *there exists* $c_4 = c_4(\epsilon, n)$ *such that*

$$r(C_{2n}; k) < c_4 k^{1 + \frac{1+\epsilon}{n-1}}, \qquad k \geqslant 1 . \tag{13}$$

Proof. Choose $c > 0$ and for a large k (to be determined later) let $K_{ck^{1+\epsilon}}$ be arbitrarily k-colored. Hence, $K_{ck^{1+\epsilon}}$ must contain a monochromatic subgraph $G = G(ck^{1+\epsilon}, e)$ where $e \geqslant \frac{1}{3} c^2 k^{1+2\epsilon}$.

By a recent result of Bondy and Simonovits [2], G contains a copy of C_{2n} provided the following two inequalities hold:

(i) $\qquad n \leqslant \dfrac{e}{100\, ck^{1+\epsilon}}$,

(ii) $\qquad n(ck^{1+\epsilon})^{1/n} \leqslant \dfrac{e}{10\, ck^{1+\epsilon}}$.

However, it is easily checked that for any $\delta > 0$, if ϵ is taken to be $\frac{1+\delta}{n-1}$ then for sufficiently large c and k, (i) and (ii) both hold. Thus, for suitable $c_4 = c_4(\delta, n)$,

$$r(C_{2n}; k) < c_4 k^{1 + \frac{1+\delta}{n-1}}, \qquad k \geqslant 1$$

and (13) is proved. ∎

Of course, since C_{2n} contains a subtree on $2n - 1$ edges then by (5)

$$r(C_{2n}; k) > (k-1)(n-1), \qquad k \geqslant 1, \quad n \geqslant 1 . \tag{14}$$

It is interesting to note that initially for k, $r(C_{2n}; k)$ is bounded *above* by ckn.

In particular, the argument of Theorem 6 can be suitably modified to establish

$$r(C_{2n}; k) \leqslant 201\, kn, \qquad 1 \leqslant k \leqslant \frac{10^n}{201\, n}, \qquad n > 1 . \tag{15}$$

It has recently been shown [3] for C_4 that

$$r(C_4; k) \leqslant k^2 + k + 1 \quad \text{for all } k ,$$

$$r(C_4; k) > k^2 - k + 1 \quad \text{for } k = \text{prime power.}$$

Hajnal and Szemerédi had previously shown (unpublished) that

$$r(C_4; k) > ck^2 \quad \text{for some} \quad c > 0 .$$

5. ODD CYCLES

Theorem 7.

$$(16) \qquad 2^k n < r(C_{2n+1}; k) < 2(k+2)!n, \qquad k \geqslant 1 , \quad n \geqslant 1 .$$

Proof. The lower bound follows easily by induction on k. For $k = 1$, $C_{2n+1} \not\subseteq K_{2n}$. If there exists a k-coloring of $K_{2^k n}$ with no monochromatic C_{2n+1} then by joining two such copies of $K_{2^k n}$ by edges of color $k + 1$ we have a $(k + 1)$-coloring of $K_{2^{k+1} n}$ with no monochromatic C_{2n+1}.

We now prove the upper bound. Let $t_0 = 2(k + 2)!n$ and suppose K_{t_0} is arbitrarily k-colored. Then for some color, say color c_1, some vertex v_1 has at least $t_1 \geqslant \frac{t_0 - 1}{k}$ edges of color c_1 leaving it. Let G_1 be the complete subgraph spanned by the t_1 vertices connected to v_1 by these edges of color c_1. If G_1 contained a subset of m vertices which spanned a subgraph G_1' containing $\geqslant mn$ edges of color c_1, then by a theorem of Erdős and Gallai [5] G_1' would contain a path P_{2n-1} of $2n - 1$ edges of color c_1. This, together with the two edges of color c_1 to v_1, would form a monochromatic C_{2n+1}. Hence we may assume all subsets of m vertices of G_1 span $< mn$ edges of color c_1. Thus, some vertex v_2 in G_1 has $\leqslant 2n - 1$ edges in G_1 of color c_1. Therefore, for some new color $c_2 \neq c_1$, v_2 has a least

$$t_2 \geqslant \frac{t_1 - 1 - (2n - 1)}{k - 1}$$

edges of color c_2, etc.

Continuing this argument recursively, we find that some monochromatic C_{2n+1} must occur provided $t_k \geqslant 1 + 2kn$. A brief calculation shows that for $t_0 \geqslant 2(k+2)!n$, this is indeed the case and so (16) is established. ∎

Another upper bound on $r(C_{2n+1}; k)$ which is probably better than that in (16) is given by the following result.

Theorem 8. *For a suitable constant* c,

$$r(C_{2n+1}; k) < ck^3 n r^2(C_3; k), \quad n \geqslant 1 .$$

Proof. Let m_3 denote $r(C_3; k)$ and let s denote $3km_3$. From the definition of m_3 it follows that for some $c_1 > 0$, any k-colored K_s contains at least $c_1 km_3$ monochromatic C_3's. Hence for t large, if K_t is k-colored then each choice of s vertices of K_t spans at least $c_1 km_3$ monochromatic C_3's. If we sum this over all $\binom{t}{s}$ choices of s vertices in K_t, we see that each monochromatic C_3 has been counted at most $\binom{t-3}{s-3}$ times. Hence, there are at least

$$\frac{c_1 km_3 \binom{t}{s}}{\binom{t-3}{s-3}}$$

monochromatic C_3's in K_t and so at least

$$\frac{c_1 m_3 \binom{t}{s}}{\binom{t-3}{s-3}} > \frac{c_2 m_3 t^3}{s^3}$$

monochromatic C_3's all having the same color, say, color c'. For $t = ck^3 nm_3^2$ this number is at least $c_3 nt^2$. Thus, some vertex v in K_t has at least $c_4 nt$ of the edges of these triangles incident to it. The corresponding vertices of these edges span a graph G which contains all the third edges of the triangles, i.e., at least $\frac{1}{2} c_4 nt$ edges of color c'. By the previously mentioned theorem of Erdős and Gallai, if $\frac{1}{2}c_4 \geq 1$ then G must contain a path P_{2n-1} consisting of $2n-1$ edges of color c'. This, together with v now forms a monochromatic C_{2n+1}. By choosing c sufficiently large, we can force $c_4 \geq 2$ and the argument is complete.

It is probably true that

$$\lim_{k\to\infty} \frac{r(C_{2n+1}; k)}{r(C_3; k)} = 0 \qquad \text{for} \qquad n \geq 2,$$

but this is not known at present.

We note here that for the complete bipartite graph $K_{n,n}$, the inclusion

$$(17) \qquad K_{n,n} \subseteq G(m, c_1 m^{2-1/n})$$

due to Kővári, Sós and Turán [6] implies that $r(K_{n,n}; k) < (c_2 k)^n$ for suitable constants $c_i > 0$. The determination of $r(K_n; k)$ is a well-known classical problem. It is known [1] that

$$e^{c_1 kn} < r(K_n; k) < k^{c_2 kn}$$

for suitable constants $c_i > 0$.

6. CONCLUDING REMARKS

A number of questions remain open, several of which we mention here.

(i) Is it true for trees T_n that

$$r(T_n; k) = kn + O(1)?$$

As mentioned before, this would follow from the conjecture

$$T_n \subseteq G\left(m, \left[\tfrac{1}{2}(n-1)m\right] + 1\right). \quad m \geq n+1.$$

(ii) It follows from Lemma 1 that if T is a maximum component of a forest F and $u(F)$, as before, denotes the cardinality of a minimum set of vertices whose removal completely disconnects F, then

$$r(F;k) > \max\left\{\left[\frac{k+1}{2}\right](u-1),\ r(T;k)\right\}.$$

Is this essentially the correct behavior of $r(F;k)$?

(iii) It is known that K_{2^n} can be decomposed into n bipartite graphs while K_{2^n+1} can not be so decomposed. What is the least odd circuit which must occur in any decomposition of K_{2^n+1} into n subgraphs?

(iv) It follows from what we have proved that for any graph G_n with n edges

$$r(G_n;k) > ck\sqrt{n}$$

for a suitable constant c. Among all such graphs, which have the fastest growing values of $r(G_n;k)$? For example, is it true that

$$r(K_n;k) \geqslant r(G_{\binom{n}{2}};k), \qquad k \geqslant 1, \qquad n \geqslant 1,$$

for any graph $G_{\binom{n}{2}}$ with $\binom{n}{2}$ edges?

(v) Is it true that

$$\lim_{k\to\infty} \frac{r(C_{2n+1};k)}{r(C_3;k)} \to 0 \quad \text{for} \quad n \geqslant 2.$$

It is not even known at present that

$$\frac{\log r(C_{2n+1};k)}{k} = O(1), \qquad n \geqslant 2.$$

Trivially,

$$r(K_n;k) < k^{kn}$$

but perhaps

$$r(K_n; k) < c_n^k .$$

It would be of interest to investigate $r(G; k)$ when both $|G|$ and k tend to infinity, but we do not do this here.

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