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ON THE MAGNITUDE OF GENERALIZED RAMSEY NUMBERS FOR GRAPHS

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1. INTRODUCTION

If G and H are graphs (which will mean finite, with no loops or parallel lines), define the *Ramsey number* $r(G, H)$ to be the least number p such that if the lines of the complete graph K_p are colored red and blue (say), either the red subgraph contains a copy of G or the blue subgraph contains H. The *diagonal Ramsey numbers* are given by $r(G) = r(G, G)$. These definitions follow those of Chvátal and Harary [1]. Other terminology will follow Harary [2]. These generalized Ramsey numbers have been much studied recently; see [3] for a survey.

Although most of the work done so far in this field has concerned exact evaluation of $r(G, H)$ for rather special cases, it is also natural to ask asymptotic questions. This paper is motivated primarily by the following definition and conjecture.

Definition. A set $\{G_1, G_2, \ldots\}$ of graphs is called an *L-set* if there is a constant c such that

$$r(G_i)\, c \cdot p(G_i)$$

for all i, where $p(G_i)$ denotes the number of poin s of G_i. Also, call a set of ordered pairs (G_i, H_i) of graphs an L-set if

$$r(G_i, H_i) \leqslant c \cdot (p(G_i) + p(H_i))\,.$$

It is often convenient to speak of *L-sequences* as well.

Conjecture. Any set of graphs or pairs or graphs having bounded arboricity is an L-set.

Several comments are necessary at this point. First, note that the arboricity of a graph may be written

$$\max_{F \subseteq G} \frac{q(F)}{p(F) - 1}\,,$$

where the maximum is over all subgraphs of G and where $q(F)$ is the number of lines of F. Note that we might as well have taken the maximum over all induced subgraphs. For our purposes, a possibly more natural parameter than the arboricity of G is the *edge-density,* given by

$$\rho(G) = \max_{F \subseteq G} \frac{q(F)}{p(F)}\,.$$

The conjecture could equally well have been stated for this parameter instead of arboricity. Later, yet another convenient parameter will be introduced. Note that the conjecture could have been stated in more universal terms, namely that for some function f,

$$r(G, H) \leqslant (p(G) + p(H)) \cdot f(\rho(G) + \rho(H))\,.$$

The above conjecture has not been settled, but it has passed several tests that have been proposed. Each of these leads to a theorem of interest. Often the theorems take the form of estimating the Ramsey number of graphs in terms of that of related, simpler graphs. In fact, estimations of this sort form an important secondary motivation for this paper.

Many sets of graphs are already known to be L-sets. In the following lemma, we summarize what we will need.

Lemma 1.1. *The Ramsey numbers* $r(K_{1,m}, K_{1,n})$, $r(P_m, P_n)$, *and* $r(C_m, C_n)$ *are all no greater than* $\max\{2m, 2n\}$. *Moreover, if* T *is any tree on* n *points,* $r(T) \leqslant 4n + 1$.

The first three Ramsey numbers mentioned above have been evaluated exactly, and the above bound is not sharp. See [3] for references and the exact results.

We also call attention here to [4], which will be a companion paper to this one. The basic differences are that [4] will consider primarily lower bounds, and will emphasize cases in which fairly precise results can be given.

2. UNIONS OF GRAPHS

One of the simplest operations that can be performed on graphs is the disjoint union of graphs. The following easy lemma is taken from [5]; much sharper results are proved there, but this result suffices for our purposes.

Lemma 2.1. *Let* F, G, *and* H *be graphs, with* $p(G) = k$, $\rho(H) = l$. Then, if $m, n \geqslant 1$,

$$r(F, G \cup H) \leqslant \max(r(F, G) + l, r(F, H)),$$

$$r(mG, nH) \leqslant r(G, H) + (m-1)k + (n-1)l.$$

From this lemma, three theorems follow immediately.

Theorem 2.1. *Let*

$$\{(F_1, G_1), (F_2, G_2), \ldots\} \quad \text{and} \quad \{(F_1, H_1), (F_2, H_2), \ldots\}$$

be L-sets; then so is

$$\{(F_1, G_1 \cup H_1), (F_2, G_2 \cup H_2), \ldots\}.$$

Theorem 2.2. *Let* $\{G_1, G_2, \ldots\}$, $\{H_1, H_2, \ldots\}$, *and* $\{(G_1, H_1), (G_2, H_2), \ldots\}$ *be L-sets. Then so is*

$$\{G_1 \cup H_1, G_2 \cup H_2, \ldots\}.$$

Theorem 2.3. *If G and H are any graphs, then*

$$\{G, 2G, 3G, \ldots\}$$

and

$$\{(G, H), (2G, 2H), \ldots\}$$

are L-sets.

Lemma 2.1 has another significant consequence, namely that the above conjecture does not tell the whole story about L-sets.

Theorem 2.4. *The set*

$$\{4K_1, 4^2 K_2, 4^3 K_3, \ldots\}$$

is an L-set.

Proof. It is known (see [6] or Lemma 4.2 below) that $r(K_i) \leqslant 4^i$, so by Lemma 2.1,

$$r(4^i K_i) \leqslant 4^i + 2i(4^i - 1).$$

Since $p(4^i K_i) = i \cdot 4^i$, the proof is complete.

In Section 4 we will give an example in which the graphs are connected.

It is natural to ask whether the conditions of Theorems 2.1 and 2.2 can be weakened. It is clear that in Theorem 2.1, one cannot omit either of the two conditions, although they might perhaps be weakened. We will show that the same is true of the three conditions of Theorem 2.2.

In the first place the condition that $\{G_i\}$ be an L-set cannot be removed, for let $G_i = K_i$ and $H_i = iK_2$. Then it is easy to see that $r(K_i, iK_2) = 3i - 2$ and it is not hard to prove (see [7]) that $r(iK_2) = 3i - 1$. On the other hand it is well-known that $r(K_i) \geqslant 2^{i/2}$ (in fact, see Lemma 2.4 below), so that $\{G_i\}$ and hence $\{(G_i \cup H_i\}$ is not an L-set.

We will see shortly that we cannot remove the condition that $\{(G_i, H_i)\}$ be an L-set. First we state a result taken from [8].

Lemma 2.2. *Let G and H be graphs without isolated points. Let χ be the chromatic number of G and let n be the number of points in the largest connected component of H. Then*

$$r(G, H) \geqslant (\chi - 1)(n - 1) + 1 .$$

Theorem 2.5. *Let $G_i = 4^i K_i$ and $H_i = K_{1,n}$, where $n = i \cdot 4^i$. Then $\{G_i\}$ and $\{H_i\}$ are L-sets, but $\{(G_i, H_i)\}$ and $\{G_i \cup H_i\}$ are not.*

Proof. By Theorem 2.4 and Lemma 1.1 respectively, $\{G_i\}$ and $\{H_i\}$ are L-sets, and in fact $r(G_i)$ and $r(H_i)$ are each no more than $2i \cdot 4^i$. But by Lemma 2.2, $r(G_i, H_i) \geqslant i(i - 1) \cdot 4^i$, so $\{(G_i, H_i)\}$ is not an L-set. Finally, it is obvious then that $\{G_i \cup H_i\}$ is not an L-set.

The above shows that the ratio

$$\frac{r(G, H)}{\max (r(G), r(H))}$$

can be made arbitrarily large for suitable G and H, and in fact can be as large as a constant times the logarithm of $\max (r(G), r(H))$. We will whow in Section 4 that

$$\frac{\min (r(G), r(H))}{r(G, H)}$$

can also be made as large a constant times the logarithm of $r(G, H)$. This will be more difficult. In neither case have we been able to show that a constant times the logarithm is the largest the ratio can be, and in fact we have no reasonable upper bound on either ratio. It seems reasonable to conjecture, however, that the logarithm is the correct bound.

3. GRAPHS OF THE FORM $G + K_1$

Note that if $\{G_i\}$ is a set of bounded arboricity, so is $\{G_i + K_i\}$. (Recall that $G + K_1$ is the graph formed by adjoining one point to G, connecting that point to every point of G.) This suggests a test of the conjecture; and indeed we will see that if, in addition, $\{G_i\}$ is an L-set, then so is $\{G_i + K_1\}$. We begin with the following two results.

Lemma 3.1. *If* $n = r(G, H)$, *then*

$$r(G + K_1, H) \leqslant r(K_{1,n}, H) .$$

Proof. Consider a two-colored graph on $r(K_{1,n}, H)$ points. If there is a blue H, we are done. If not, there is a red $K_{1,n}$ and it is clear that there must now exist a red $G + K_1$ or a blue H.

Lemma 3.2. *If* $m = r(G + K_1, H)$ *and* $n = r(G, H + K_1)$, *then*

$$r(G + K_1, H + K_1) \leqslant m + n .$$

Proof. By Lemma 1.1, such a graph contains either a blue $K_{1,m}$ or a red $K_{1,n}$, and it is clear that in either case we are done.

It turns out that the most useful parameter of graphs to consider in this section is neither the arboricity nor the edge density, but $\sigma(G)$, defined as follows:

$$\sigma(G) = \max_{F \subseteq G} \delta(G) ,$$

where $\delta(G)$ is the minimum degree of points in F. In [9], a graph with $\sigma(G) = k$ is called k-degenerate. The relationship between $\sigma(G)$ and $\rho(G)$ is given in the next lemma.

Lemma 3.3. *For any graph* G *ot consisting entirely of isolated points,*

$$\rho(G) < \sigma(G) \leqslant 2\rho(G) .$$

Proof. To see that $\sigma(G) \leqslant 2\rho(G)$, note than any graph F has a point of degree $\leqslant 2q(F)/p(F)$. To prove that $\rho(G) < \sigma(G)$, assume the contrary and let $F_0 \subseteq G$ be such that $\rho(G) = q(F_0)/p(F_0)$. Since by hypothesis $\sigma(G) \leqslant \rho(G)$, we may remove from F_0 points of degree

$\leqslant \rho(G)$ until we have reduced F_0 to a single point. But this is impossible, since the number of lines remaining will then be $q(F_0) - (p(F_0) - 1) \cdot$ $\cdot\, q(F_0)/p(F_0) > 0$, a contradiction.

We now proceed to prove two lemmas about $r(K_{1,n}, H)$ which will permit the use of Lemma 3.1.

Lemma 3.4. *Let G have m points, and let F be derived from G by removing a point of degree d. Then*

$$r(K_{1,n}, G) \leqslant \max\,(r(K_{1,n}, F), d(n-1) + m)\,.$$

Proof. Take p to be the right hand side of the above inequality and two-color K_p; now assume that there is neither a red $K_{1,n}$ nor a blue G. Certainly there is either a blue F or a red $K_{1,n}$, so the graph contains a blue F. Let D be a set of d points of F such that if a point outside F is connected to each point of D by a blue line, a blue graph isomorphic to G results. Then each of the $d(n-1)+1$ points not in F must have at least one red line connecting to D. Therefore at least one point of D must have a red degree at least

$$(d(n-1)+1)/d = n - 1 + 1/d\;;$$

since the degree is an integer, it is at least n, giving a red $K_{1,n}$ and the desired contradiction.

Lemma 3.5. *If $p(G) = m$, $\sigma(G) = d$, then*

$$r(K_{1,n}, G) \leqslant d \cdot (n-1) + m\,.$$

Proof. We use induction on m. The result is trivial for $m = 1$, so assume it to be true for some $m-1$, $m \geqslant 2$. Let F be formed from G by removing a point of degree $\leqslant d$, so by Lemma 3.4,

$$r(G, K_{1,n}) \leqslant \max\,(r(F, K_{1,n}), d(n-1) + m)\,.$$

It remains to show $r(F, K_{1,n}) \leqslant d(n-1) + m$. But clearly $\sigma(F) \leqslant \sigma(G)$, so this holds by the induction hypothesis, completing the proof.

Lemma 3.6. *Let $r(G,H) = t$, $r(G) = u$, $p(G) = m$, $p(H) = n$,*

$\sigma(G) = c$, $\sigma(H) = d$. Then

$$(3.1) \qquad r(G + K_1, H) \leqslant d(t-1) + n\,,$$

$$(3.2) \qquad r(G + K_1, H + K_1) \leqslant (c+d)(t-1) + m + n\,,$$

$$(3.3) \qquad r(G + K_1) \leqslant 2c(u-1) + 2m\,.$$

Proof. Relation (3.1) follows direetly from Lemmas 3.1 and 3.5. Relation (3.2) follows from (3.1) and Lemma 3.2, and (3.3) follows from (3.2) upon setting $G = H$.

From Lemmas 3.6 and 3.3 we immediately deduce the following two results.

Theorem 3.1. *Suppose* $\{G_1, G_2, \ldots\}$ *is an* L*-set having bounded arboricity. Then* $\{G_1 + K_1, G_2 + K_1, \ldots\}$ *is an* L*-set.*

Theorem 3.2. *Suppose* $\{(G_1, H_1), (G_2, H_1), \ldots\}$ *is an* L*-set, with the* G_i *and* H_i *having bounded arboricity. Then* $\{(G_1 + K_1, H_1 + K_1), (G_2 + K_1, H_2 + K_1), \ldots\}$ *is an* L*-set.*

Lemma 3.6 can also be used to estimate $r(K_m, G)$. We give two theorems. Note first the obvious fact that if $p(G) = n$, $r(K_2, G) = n$.

Theorem 3.3. *If* F *is any forest on* n *points, then* $r(K_m, F) \leqslant (m-1)(n-1) + 1$. *If* F *is in fact a tree, then* $r(K_m, F) = (m-1)(n-1) + 1$.

Proof. It is clear that F is a forest if and only if $\sigma(F) = 1$. In this case, relation (3.1) of Lemma 3.6 yields

$$r(K_m, F) \leqslant r(K_{m-1}, F) + n - 1\,,$$

and starting with $r(K_2, F) = n$, we easily find that $r(K_m, F) \leqslant (m-1)(n-1) + 1$. The second part of the theorem now follows from Lemma 2.2.

The above theorem (or rather the second part; but the first part is a trivial consequence) has been proved by Chvátal [10], Stahl [11] has evaluated $r(K_m, F)$ for all forests F.

Theorem 3.4. *Let* $p(G) = n$, $\sigma(G) = d \geq 2$. *Then*

$$r(K_m, G) \leq \frac{(n-1)d^{m-1} - n + d}{d-1}.$$

Proof. By relation (3.1) of Lemma 3.6, an upper bound is given by $r(K_m, G) \leq u_m$, where the u_m satisfy the recurrence relation

$$u_m \leq du_{m-1} + (n-d),$$

subject to the initial condition $u_2 = n$. Observing that u_m must be of the form $xd^m + y$, it is easy to solve for x and y. Simplying the resulting formula for u_m, we find that

$$r(K_m, G) \leq u_m = \frac{(n-1)d^{m-1} - n + d}{d-1},$$

and the proof is complete.

A slightly weaker result than this has been proved by Stahl [11], using similar methods.

Lemmas 3.1 and 3.2 imply that $r(G + K_1) \leq 2m$, where $r(G) = n$ and $r(K_{1,n}, G) = m$. It is possible to give a result that is often sharper.

Lemma 3.7. *Let* $n = r(G)$ *and* $m = r(K_{1,n}, G)$. *Then*

$$r(G + K_1) \leq \frac{m^2 - 2n + 1}{m - n}.$$

Proof. Let $s = r(G + K_1) - 1$ and consider any two-coloring of K_s. We first observe that if the graph contains a red and a blue $K_{1,n}$ with all their endpoints in common, we are done, since the set of n endpoints must induce a monochromatic G. Moreover, if the graph contains a monochromatic $K_{1,m}$, we are also done, since we must than have either the configuration mentioned above or a monochromatic $G + K_1$ outright.

Consider now any point of the graph and assume that the point has k red lines emanating from it to a set A of points and $s - k - 1$ blue lines emanating to a set B, where $k \leq m - 1$ and $s - k - 1 \leq m - 1$. If any point of A has n red lines going to B, we are done, so we may assume the number of red lines running between A and B is no greater than $k(n-1)$. Similarly, the number of blue lines running between A and B is no greater than $(s-k-1)(n-1)$. But the total number of lines between A and B is $k(s-k-1)$. For this to be possible we must must have

$$k(s-k-1) \leq k(n-1) + (s-k-1)(n-1) = (s-1)(n-1)\,.$$

The left hand side of this is minimized when k is as large or as small as possible, that is, when $k = m-1$ or $s-k-1 = m-1$. In either case the left hand side becomes $(m-1)(s-m)$. Consequently, we must have $(m-1)(s-m) \leq (s-1)(n-1)$, which is equivalent to

$$s \leq \frac{m(m-1)-(n-1)}{m-n}\,.$$

From this the result follows directly.

Lemma 3.7 is strong enough to establish an interesting asymptotic result.

Theorem 3.5. *If* $n \geq r(K_k)$, $k \geq 2$, *and* G *denotes the graph* $K_k \cup (n-k)K_1$, *then for some absolute constant* c,

$$kn + 1 \leq r(G + K_1) \leq kn + cn/k\,.$$

Proof. The lower bound follows from Lemma 2.2. To prove the upper bound, observe that $r(G) = n$ whenever $n \geq r(K_k)$. From Theorem 3.3, $m = r(G, K_{1,n}) = (k-1)n + 1$, so by Lemma 3.7,

$$r(G + K_1) \leq \frac{((k-1)n+1)^2 - 2n + 1}{(k-2)n+1} =$$

$$= \frac{k^2n^2 - 2kn^2 + n^2 - 2(k-1)n + 1 - 2n + 1}{kn - 2n + 1} \leq kn + cn/k\,.$$

This completes the proof.

This result yields $r(G + K_1) = kn + O(n/k)$ and thus an asymptotic formula as k, and hence $n \geq r(K_k)$, becomes large, although not when k is fixed. In [4] it will be shown that under conditions similar to those of Theorem 3.5, it is in fact true that $r(G + K_1) = kn + 1$. We have included Theorem 3.5 to illustrate Lemma 3.7 and also for the sake of completeness, since this theorem will be used in the proof of Theorem 4.4.

As has been mentioned in the proof of Theorem 2.4, $r(K_k) \leqslant 4^k$, so that we may take n this small for each k. We thus have a set of graphs H_k for which $r(H_k) \sim cp(H_k) \log p(H_k)$ for some c. This, and some similar sets of graphs, are the only sets of graphs not L-sets for which we can prove even as precise a result as Theorem 3.5. It is interesting, then, that for some such graphs the exact result mentioned above can be proved.

4. SOME SPECIAL RESULTS

In this section we prove three main results. Each asserts that certain Ramsey numbers can be smaller than other related ones by any arbitrary factor. The first of these results is that promised in Section 2, namely that $r(G, H)$ can be much smaller than both $r(G)$ and $r(H)$. It is first necessary to give three more lemmas.

Lemma 4.1. *Let G be a bipartite graph with two maximal sets of independent points A and B having a and b points respectively. Let each point of A have degree at least d. Then there are s points in A and t points in B such that these $s + t$ points induce a copy of $K_{s,t}$ in G, provided that*

$$s < a \binom{d}{t} \Big/ \binom{b}{t} + 1 .$$

Proof. The number of subsets of B having t points is $\binom{b}{t}$. Since each point of A has degree at least d, each point of a is the center of at least $\binom{d}{t}$ t-stars $K_{1,t}$. Hence, if s and t satisfy the above inequality, at least s of the t-stars meet the same t points of B, yielding the desired $K_{s,t}$.

The next result is well-known; it was first proved in [6].

Lemma 4.2.

$$r(K_m, K_n) \leqslant \binom{m+n-2}{m-1} .$$

Note that this implies $r(K_n) \leq 4^n$, as stated in Section 2.

The following lemma is taken from [12], which is based on a method of Erdős [13].

Lemma 4.3. *Let* p, q *and* s *be the number of points, lines, and symmetries of* G *respectively. Then*

$$r(G) > (s2^{q-1})^{1/p} .$$

Hence $r(K_m) > 2^{m/2}$ and $r(K_{m,m}) > 2^m$.

We now proceed with the following result.

Theorem 4.1. *There exist graphs* $G_1, G_2, \ldots$ *and* $H_1, H_2, \ldots$ *such that* $\{(G_i, H_i)\}$ *is an L-set, but* $\{G_i\}$, $\{H_i\}$ *are not.*

Proof. Choose any $k \geq 3$ and then choose an n and an $m > n$ such that

$$(4.1) \qquad 2m\binom{m}{n} > kn\binom{2m}{n} ,$$

$$(4.2) \qquad km > r(K_k, K_{kn,kn}), \quad \text{and}$$

$$(4.3) \qquad 2k^2m < (K_{kn,kn}) .$$

To see that this is possible, observe that

$$\binom{2m}{n}\Big/\binom{m}{n} < 3^n \qquad \text{if} \qquad m \geq 2n ,$$

$$r(K_k, K_{kn,kn}) \leq r(K_k, K_{2kn}) \leq \binom{(2n+1)k-2}{k-1} \leq (2kn)^k$$

if $n \geq 3$, and

$$(4.4) \qquad r(K_{kn,kn}) > 2^{kn} ,$$

where the last two inequalities come from Lemmas 4.2 and 4.3 respectively. Hence, if n is chosen large enough, an m can be chosen satisfying (4.1), (4.2), and (4.3).

Now set $H_k = K_{kn,kn}$, and let G_k be the graph on $km + 1$ points consisting of a point connected by one line each to m copies of K_k. We first note that by Lemma 2.2,

$$r(G_k) > (k-1)km ,$$

so that $\{G_k\}$ is not an L-set. Also, by (4.4), $\{H_k\}$ is not an L-set.

It remains to show that $\{(G_k, H_k)\}$ is an L-set; we will prove that $r(G_k, H_k) \leqslant 3km$, which clearly suffices. Color the edges of a K_{3km} red and blue, and assume there is neither a red G_k nor a blue $H_k = K_{kn,kn}$. By (4.2) and Lemma 2.1, $r(2mkK_k, H_k) < 3km$, so we may assume that the graph contains a red $2mK_k$. There remain at least $2m$ points not in this red $2mK_k$; we will show that the two-colored complete graph on the points of the $2mK_k$, together with $2m$ remaining points, contains a red G_k or a blue H_k.

Map this graph to a two-colored $K_{2m,2m}$ with maximal independent sets A and B as follows: To each of the $2m$ red copies of K_k, assign one point of B and to each of the remaining $2m$ points, assign one point of A. Color a line joining a point of A to one of B red if the corresponding star joining a point to a K_k has any of its lines red; color it blue in the contrary case, that is, if all the lines of the star are blue. Observe that if the new graph contains a red $K_{1,m}$ with one point in A and m in B, then the old graph contains a red G_k. If on the other hand the new graph contains a blue $K_{kn,n}$ with kn points in A and n in B, the old graph contains a blue $K_{kn,kn} = H_k$. Hence, the proof will be complete if it can be shown that one of these two cases must always hold.

To see this, we use Lemma 4.1 with $a = b = 2m$, $d = m$, $s = kn$, and $t = n$. Note that since any point of A with m lines emanating from it yields a red G_k, we may assume each point of A has at least m blue lines emanating from it. Since 4.1 asserts that

$$kn < 2m\binom{m}{n} \Big/ \binom{2m}{n} ,$$

Lemma 4.1 applies to the blue graph, and the proof is complete.

Note that this proof has shown in fact that the ratio

$$\frac{\min(r(G), r(H))}{r(G, H)}$$

can be as large as $\log r(G, H)$.

The next theorem makes good a claim made in Section 2, namely that connected graphs can be given that behave like those of Theorem 2.4.

Theorem 4.2. *If* $G_i = 4^{2i}K_{i,i} + K_1$, *then* $\{G_i\}$ *is an* L*-set.*

Proof. We will show that in fact $r(G_i) \leqslant 2(18i + 2)4^{2i}$. We begin by observing that by Lemma 4.2, $r(K_{i,i}) \leqslant r(K_{2i}) \leqslant 4^{2i}$. Letting $n = (4i + 1)4^{2i}$, we see that $r(K_{1,n}, K_{i,i}) \leqslant 4n$, since by Lemma 4.1, for this to be true it is sufficient that

$$i < 2n\binom{n}{i} \Big/ \binom{2n}{i} + 1 ,$$

which is certainly the case. We now apply Lemma 2.1 to find $r(4^{2i}K_{i,i}) \leqslant 4^{2i} + 4i \cdot 4^{2i} = (4i + 1)4^{2i} = n$ and $r(K_{1,n}, 4^{2i}K_{i,i}) \leqslant 4(4i + 1)4^{2i} + 2i \cdot 4^{2i} = (18i + 1)4^{2i}$. Finally, from Lemmas 3.1 and 3.2,

$$r(G_i) \leqslant 2r(G_i, 4^{2i}K_{i,i}) \leqslant 2r(K_{1,n}, 4^{2i}K_{i,i}) \leqslant 2(18i + 1)4^{2i} ,$$

and the proof is complete.

Note that by Lemma 4.3, a necessary condition for $\{G_i\}$ to be an L-set is that there be a constant c such that $q(G_i)/p(G_i) \leqslant c \log p(G_i)$ for every i. Thus Theorem 4.2 gives the best possible order of growth. Note that Theorem 3.5 gives a set of graphs for which the above inequality holds, but which is not an L-set.

The final result of this section requires a generalization, taken from [1], of the Ramsey numbers defined in Section 1. If $\mathscr{G}$ is a set of graphs, let $r(\mathscr{G})$ be the least number p such that if the lines of K_p are two-colored, the graph contains a monochromatic G for some $G \in \mathscr{G}$. If $\mathscr{G} = \{G_1, G_2\}$ is natural to ask about the ratio $\min(r(G_1), r(G_2))/r(\mathscr{G})$. The ratio is clearly at least unity; we will show that it can be at least a constant times the square root of the logarithm of $r(\mathscr{G})$. It will be convenient to prove the result in two stages.

Theorem 4.3. *Let* $k_2 \geqslant k_1 \geqslant 2$ *and* $n_1 > n_2 > r(K_{k_2})$. *Let* $G_i = K_1 + (K_{k_i} \cup (n_i - k_i)K_1)$, $i = 1, 2$, *and let* $\mathcal{G} = \{G_1, G_2\}$. *Then*

$$k_1 n_2 + 1 \leqslant r(\mathcal{G}) \leqslant 2(k_1 - 1)n_2 + 2 .$$

Proof. The lower bound is established by considering a red $k_1 K_{n_2}$ and its completementary (blue) graph. The components of the red graph are too small to contain either G_1 or G_2, and the chromatic number of the blue graph is $k_1 \leqslant k_1 + 1 < k_2 + 1$, so that it cannot contain G_1 or G_2 either.

To establish the upper bound, consider a two-colored complete graph on $2(k_1 - 1)n_2 + 2$ points. By Lemma 1.1, there is a monochromatic, say red, star of degree $(k_1 - 1)n_2 + 1$. Denote the set of endpoints of this star by A. By Theorem 3.3, either the points of A induce either a red K_{k_1} or a blue K_{1,n_2}. In the former case, we have a red G_1. In the latter case, we have a red and a blue K_{1,n_2} with all n_2 endpoints in common. Since $n_2 > r(K_{k_2})$, we must have a monochromatic G_2. Thic ompletes the proof.

The next result indicates the significance of Theorem 4.3.

Theorem 4.4. *Let the assumptions of Theorem* 4.3 *hold, but assume in addition that* $k_1 n_1 = k_2 n_2$, $k_2 = k_1^2$ *and* $n_2 = 4^{k_2}$. *Then for some constant* c,

$$\min (r(G_1), r(G_2))/r(\mathcal{G}) \geqslant \sqrt{c \log r(\mathcal{G})} .$$

Proof. By Theorem 3.5, $\min (r(G_1), r(G_2)) \geqslant k_1 n_1$. By Theorem 4.3, $r(\mathcal{G}) \leqslant 2k_1 n_2$. Hence the ratio in question is at least $n_1/2n_2 = k_2/2k_1 = \sqrt{k_2/2}$. But $k_2 = \log n_2/\log 4 \geqslant c_1 \log r(\mathcal{G})$ for some c_1, so the result follows.

Note that this result is the best that can be given using graphs of this type.

5. POWERS OF CYCLES AND MORE GENERAL GRAPHS

A significant test of the conjecture of Section 1 are the squares, or higher powers, of cycles. In fact, we will prove a rather general result. Let $\mathscr{L}_k(G) = L_2(G)[K_k]$, in the notation of [2]. This graph may be described as follows. Replace each point of degree d with a K_{dk}. If two points are adjacent in G, connect the corresponding complete graphs with a $K_{k,k}$, in such a way that all such copies of $K_{k,k}$ are disjoint. An example of a graph G, $\mathscr{L}_1(G)$, $\mathscr{L}_2(G)$ is given in Figure 1 below.

We will show that there is a function f such that if $\Delta(G) = d$ (the maximum degree of G), then

$$r(\mathscr{L}_k(G)) \leqslant f(d, k) \cdot r(G) .$$

It is easy to see that $C_{kn}^k \subseteq \mathscr{L}_k(C_n)$ and that powers of paths and cycles of other orders can also be obtained easily as subgraphs of graphs of the same form. This is the connection between the above result and the title of this section. Before proving that result several lemmas will be necessary.

Lemma 5.1. *Let* $p(G) = n, \sigma(G) = d$. *Then*

$$r(mK_l, G) \leqslant nd^l + ml .$$

Proof. The above inequality follows from Theorem 3.3 and 3.4 and Lemma 2.1.

Lemma 5.2. *Let G and H be graphs related in the following way: For every point of G there corresponds a complete graph $K_l \subseteq H$, and for every pair of points that are adjacent in G, the corresponding complete graphs are connected by a complete bipartite graph $K_{kd,kd}$, where d is the maximum degree of G. Then $\mathscr{L}_k(G) \subseteq H$.*

Proof. (Note that the conditions implicitly entail $kd \leqslant l$.) For each point of G of degree $d' \leqslant d$, we may pick d' sets of k points each from the d' copies of $K_{kd,kd}$ entering the K_l corresponding to that point of G, and such that all d' sets are disjoint. If we do this for every point of G, the set of all points of chosen induce a copy of $\mathscr{L}_k(G)$ in H.

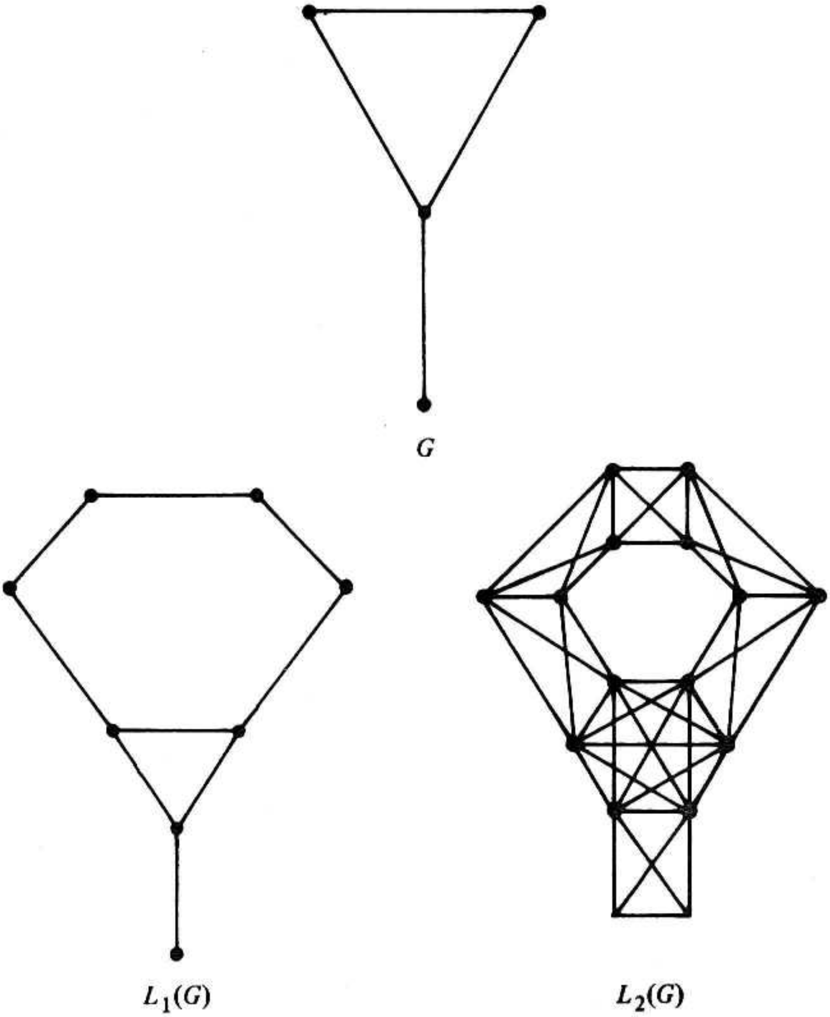

Figure 1

Lemma 5.3. *Let G and H be graphs related in the following way: For every point of G there corresponds a set of l points of H, and for every line of G there corresponds a set of lines joining points of the two corresponding sets, with the property that at least t points of either* *set are each joined to at least* t *points of the other set. Suppose further that* $t > l - l/d$, *where* d *is the maximum degree of points of* G. *Then* $G \subseteq H$.

Proof. We build up a copy of G in H a point at a time. We must do this in such a way that each point is joined to those points already chosen which are adjacent to it in G, and also so that it is joined to at least t points each of the adjacent sets of l points from which no point has yet been chosen. Let V be any of the sets of l points from which no point has yet been chosen. Suppose that x points adjacent to v have already been chosen any y sets adjacent to v have not yet had points chosen from them. Note that $x \leqslant d - y$ and that either x or y may be zero.

We see that at least $l - lx/d \geqslant ly/d$ points of V are adjacent to the x adjacent points already chosen. Denote the subset of points of V for which this is true by U. Each point of U is adjacent to at least $l - l/d$ of the sets of l points to which V is adjacent and from which no point has yet been chosen. Hence there must be at least one point of U of the sort we seek. Thus we may continue to choose points until G has finally been built up. This completes the proof.

Lemma 5.4. *Let* d *and* k *be given. Then there exists an* l *such that if the complete bipartite graph* $K_{l,l}$ *is two colored, either the red subgraph contains a* $K_{kd,kd}$ *or there is a* $t > l - l/d$ *such that at least* t *of the points of each maximal independent set of the* $K_{l,l}$ *are connected by blue lines to at least* t *points of the other.*

Proof. Note that the result is trivial if $d = 0$ or 1, so we assume $d \geqslant 2$. Suppose that the blue subgraph does not have the desired property. Then at least $t > l/d$ of the points of one (in fact, each) of the maximal independent sets of the $K_{l,l}$ have at least l/d red lines going to the other. We now apply Lemma 4.1 and find that the red subgraph will contain a $K_{kd,kd}$ provided that

$$kd < (l/d)\binom{l/d}{kd} \Big/ \binom{l}{kd} + 1\,.$$

But as l becomes large the right hand side is asymptotic to

$$(l/2)(l/d)^{kd}/l^{kd} = l/2d^{kd}$$

Hence, it is sufficient to choose l about as large as $2d^{kd}$. This completes the proof.

We are now in a position to prove the principal theorem of this section.

Theorem 5.1. *There exist functions* f_1 *and* f_2 *such that if* $\max(\Delta(G), \Delta(H)) = d$, *then*

$$r(\mathscr{L}_k(G), H) \leqslant f_1(d, n) r(G, H) ,$$

$$r(\mathscr{L}_k(G), \mathscr{L}_k(H)) \leqslant f_2(d, n) r(G, H) .$$

Proof. Note that it is necessary to prove only the first inequality, since the second follows by applying the first to itself.

Let $m = r(G, H)$, $n = p(H)$, and let l be a number large enough that Lemma 5.4 is applicable. By Lemma 5.1, $r(mK_l, H) \leqslant nd^l + ml$. Consider a two-colored complete graph on that many points; we will show that such a graph contains a red $\;_k(G)$ or a blue H. Since we are done if there is a blue H, we may assume that the graph contains a red mK_l. We focus our attention on the graph induced by the points of these m red K_l.

Let us form a new two-colored complete graph of order m from this graph. Let each red X_l correspond to a single point of the new graph. Consider the two-colored $K_{l,l}$ connecting any two of the red K_l in the original graph. By Lemma 5.4, at least one of the two alternatives of that lemma must apply. If the first applies, namely that the two K_l are connected by a red $K_{kd,kd}$, color the corresponding line in the new graph red; otherwise color it blue. Since the new graph has $m = r(G, H)$ points, it contains either a red G or a blue H. In the first case, Lemma 5.2 applies and the original graph contains a red $\mathscr{L}_k(G)$; in the second, Lemma 5.3 applies and the original graph contains a blue H. This completes the proof.

Various consequences follow from the above theorem. For instance, if $S_m(G)$ is the m-th subdivision graph of G, then $(S_m(G))^m \subseteq \mathscr{L}_k(G)$, provided $k \geqslant m/2 + 1$. In fact, if H is formed from G by inserting at least m and no more than n points into each line of G, then it can be seen that $H^m \subseteq \mathscr{L}_k(G)$, provided $k \geqslant m/2 + 1$. From this, the next result follows, announced at the beginning of this section.

Theorem 5.2. *If m is fixed, then $\{P_i^m\}$* and *$\{C_i^m\}$ are L-sets.*

6. SUBDIVISION GRAPHS AND RELATED GRAPHS

From Theorem 5.1 and the above discussion it follows immediately that if $\{G_i\}$ is an L-set with bounded degree, then for m fixed, $\{S_m(G_i)\}$ is an L-set. It is also easy to prove from Theorem 5.1 that if G is any graph, then $\{S_i(G)\}$ is an L-set. In this section we will give direct proofs of stronger results.

Theorem 6.1. *Let G be any graph on n points with the property that any two points of degree $\geqslant 3$ are at distance at least three. Then $r(G) \leqslant 18n$.*

Proof. Consider a two-colored K_{18n}. At least $\dfrac{18n(18n-1)}{4}$ lines are the same color, say red. We show that the red graph contains a graph in which every point has degree at least $5n$. Remove a point with degree $< 5n$, if any. Now remove a point with degree $< 5n$ from the new graph, if any. Continue this process until we obtain a graph of the desired type, or until all points have been removed. But the latter case cannot happen, since after $13n$ points have been removed we are left with a graph with $5n$ points and at least

$$\frac{18n(18n-1)}{4} - 13n(5n-1) = \frac{32n^2 + 17n}{2} > \frac{5n(5n-1)}{2}$$

lines, a contradiction.

Thus we have a red graph H with every point having degree $\geqslant 5n$, and its complementary (blue) graph. We will show that either the red graph contains G or the blue graph contains $K_{2n,2n}$. We attempt to build up a red G in a straightforward manner. First, choose arbitrarily a set S of points to be used as those points of G which have degree $\geqslant 3$. Examining the remaining points of G, we see that they and the edges incident to them, must belong to certain paths or cycles. If such a path is connected at only one end to a point of S, or to no point of S, we may clearly build it up in the red graph with no difficulty. In the remaining cases we have either a path connected at both ends to points of S, or a cycle containing one or no point of S. In the case of a cycle containing no point of S, choose any point arbitrarily to serve as a point of the cycle and adjoin it so S; designate by T the set S to which such points have been adjoined.

To build up G, it is necessary to form paths or cycles $t_0 t_1 \ldots t_k$, where $t_1, t_k \in T$ and $k \geqslant 3$. We will call these paths or cycles *links*. Again, we have no difficulty in forming a path $t_0 \ldots t_{k-3}$ in H. It remains to choose t_{k-2} and t_{k-1} so that the edges $t_{k-3}t_{k-2}$, $t_{k-2}t_{k-1}$, and $t_{k-1}t_k$ are in H. But by hypothesis, t_{k-3} and t_k each have degree at least $5n$. From this fact we see that we can find $4n$ distinct points $u_1, \ldots, u_{2n}, v_1, \ldots, v_{2n}$ such that $t_{k-3}u_i$ and $v_j t_k$ are in H for each i and j and such that no u_i and v_j have been used already in building G. If any edge $u_i v_j$ is in H, the desired link can be completed. If not, the blue graph contains a $K_{2n,2n}$.

It remains to show that the existence of a blue $K_{2n,2n}$ guarantees a monochromatic G. Let us denote by U and V the two maximal independent sets of $2n$ points in the $K_{2n,2n}$: By Lemma 2.1, $r(G, nK_2) \leqslant$ $\leqslant 2n$. Consequently, fixing our attention on U, we see that either U induces a red G or a blue $[n/2]K_2$. Thus the proof will be complete if we show that in the latter case, $U \cup V$ induces a blue G. Choose a set of points in V to correspond to the points of T. There is no difficulty in building up any necessary paths for the desired blue copy of G that are not links, nor in building up links with even length (and therefore an odd number of points not in T). But building up a link of odd length is not difficult either; simply incorporate one of the edges of the blue $[n/2]K_2$ of U. This completes the proof.

This theorem is of interest in that it shows a rather large set of graphs to be an L-set. It is probably true that the theorem holds with the property of G weakened to that of having no two adjacent points of degree $\geqslant 3$. The authors have not succeeded in showing this, but it has been possible to deal with the case in which G is the subdivision graph of K_n, that is when $G = S_1(K_n) = S(K_n)$.

Theorem 6.2.

$$r(S(K_n)) \leqslant 3n^2 + 3n \, .$$

Proof. Set $N = 3n^2 + 3n$ and consider a two-colored K_N. Note that $p(S(K_n)) = n(n+1)/2 = N/6$. At least half the lines of this graph are of one color, say red. Call the red graph H. In what follows, only H will be considered. Since N is even, at least $N/2$ points of H have degree $\geqslant N/2$; call the set of such points S. Observe that if S contains n points such that each pair of such points are mutually adjacent to $n(n+1)/2$ points of S, it is easy to construct a copy of $S(K_n)$ in H. To do this, choose the n given points to be the n points of degree $n-1$ in the $S(K_n)$; it is clear that the rest of the points can be chosen without difficulty.

Suppose that some point x of S has the property that there exist points $y_1, y_2, \ldots, y_n$ of S such that for each i, x and y_i are mutually adjacent to fewer than $n(n+1)/2$ points of S. Then any pair of y's are mutually adjacent to at least $n(n+1)/2$ points of S, and by the above observation we are done.

Now suppose that there is no point x as above. If any point z_1 is chosen from S, then with fewer than n exceptions, each other point of S in mutually adjacent with z_1 to at least $n(n+1)/2$ points of S. Discard the exceptional points and choose z_2 from the remainder of S. Again there will be fewer than n exceptional points to be discarded. In this fashion, choose points $z_1, z_2, \ldots, z_n$. This can clearly be done, since fewer than n^2 of the $\geqslant N/2 = 3n(n+1)/2$ points of S will be discarded. For each pair of z's there are at least $n(n+1)/2$ points adjacent to both of them. Hence by the observation made earlier, a copy of $S(K_n)$

can be constructed. This completes the proof.

Note that we have actually shown an extremal result, namely that any graph with $N = 3n^2 + 3n$ points and at least $N(N-1)/4$ lines contains $S(K_n)$. Using the same methods, Theorem 6.2 can be improved, but only slightly. We conjecture that the true Ramsey number of $S(K_n)$ is about $2n^2$. Theorem 6.2 suggests a possible generalization. Let k be fixed and for each n define G_n as follows. Choose a set A of n points and a set B of $\binom{n}{k}$ points. For every k-tuple $\{a_1, \ldots, a_k\}$ of points of A, let there correspond a unique point of B and connect this point to all points of the k-tuple. For $k = 2$, G_n is just $S(K_n)$. By the conjecture of Section 1, for each k the set $\{G_n\}$ should be an L-set, but we have not been able to prove this.

Theorem 6.3. *There is a constant c such that if G is any graph on n points formed from K_m by inserting at least one point into each line, then $r(G) \leqslant cn$.*

Proof. We will only sketch the proof, which is based on the ideas of Theorems 6.1 and 6.2. Consider a two-colored K_{cn}, where c remains to be chosen. Without loss of generality we may assume that at least half the lines are red. In similar fashion to the proof of Theorem 6.1, we remove points until we have a graph H, every point of which has degree at least $5n$, but this time we also want every point to have degree at least $2p(H)/5$. If c is sufficiently large, it is easy to see that this can be done, and this determines c.

By the argument of Theorem 6.2 we can again find a red $S(K_m)$. Now some paths of length two must be replaced by longer paths which we call links, as in the proof of Theorem 6.1. As in that proof, we attempt to build up the desired links, and again we either succeed in forming a red G or we find a $K_{2n,\,2n}$ in the complementary (blue) graph of H. Proceeding as before this leads to a monochromatic G, completing the proof.

7. PROBLEMS AND CONJECTURES

Many interesting questions are suggested by the results proved here. In the first place, the conjecture of Section 1 remains unsettled. The authors offer a total of \$25 for settling it. However, it seems to be quite difficult, and probably further work must continue to be in the direction of partial results. One significant question is the following: If $\{G_i\}$ is an L-set with bounded arboricity, is $\{G_i \times K_2\}$ necessarily an L-set?

It is not completely clear what sets of graphs (or pairs of graphs) can be seen to be L-sets using the results of this paper, and to characterize such sets is an interesting problem in itself. Certainly there exist sets of graphs of bounded arboricity which have an unbounded number of points of unbounded degree, and except for those of Section 6, the results of this paper do not apply to these graphs. Another set of graphs beyond the reach of our results is that of graphs consisting of portions of a square lattice work.

Most of the results of this paper leave some question open. For example, Theorems 2.5 and 4.1 show that the ratios of certain Ramsey numbers can be as large as a constant times the logarithms of those Ramsey numbers, and Theorem 4.3 shows that certain ratios can be as large as a constant times the square roots of the logarithms. A natural question is whether or not these ratios can be made larger; we have not been able to give any useful upper bound to these ratios. It seems reasonable to conjecture that the bounds given here are in fact of best possible order. Indeed each result is of best order in the restricted sense that one cannot do better with graphs of the same type as used there. Another question left open by Theorem 2.5 is whether or not such a result can also be proved with all the G_i and H_i connected.

Theorems 2.4 and 4.2 give examples of L-sets $\{G_i\}$ for which $q(G_i)/p(G_i) \sim c \log p(G_i)$ for some constant c. Lemma 4.3 shows that an order of growth no greater than this is a necessary condition that a set of graphs be an L-set. Moreover, Lemma 2.2 shows that for such an order of growth to hold, it is necessary that the chromatic number of the graphs be bounded. Perhaps any set of graphs satisfying the above two conditions is an L-set. An interesting test case is the set $\{Q_i\}$ of cubes. The authors offer a total of $25 for deciding whether the set of cubes is an L-set.

Most of the results of Section 3 are easily seen to be relatively sharp, at least in some instances. An exception is Theorem 3.4, which for fixed G gives a bound on $r(K_m, G)$ which is exponential in m. Probably the true behavior is like a power of m. When $G = K_n$, this is an old problem in classical Ramsey theory.

The results of Section 5 suggest several questions. If $\{G_i\}$ is a set of graphs of bounded degree, the same is true of the set of line graphs $\{L(G_i)\}$, the set of total graphs $\{T(G_i)\}$, and the set of k-th powers $\{G_i^k\}$ for a fixed k. If $\{G_i\}$ is known in addition to be an L-set, what about the other three sets? The results of Section 5 bear on this question, but fall far short of answering it.

The bounds that come out of the proof of Theorem 5.1 are extremely large; can they be improved? The bounds amount to arithmetic progressions in m with the constant term a double exponential in kd and the coefficient of m a single exponential. While the true bounds must certainly be fairly large, one might expect constants not quite so huge.

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