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Problems and Results on Finite and Infinite Graphs

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In this short note I discuss some of the problems which occupied my collaborators and myself for a very long time. I tried to select those problems which are striking and which are not too well known. I will also give a few proofs which I hope are new. I published many papers on problems in graph theory, e.g. [1, 2]; see also [3].

I. Let α be an ordinal number which has no immediate predecessor. $G(\alpha)$ denotes a graph whose vertices have order type α. Hajnal, Milner and I conjectured [4] that every $G(\alpha)$ either contains an infinite path or an independent set of type α. We proved our conjecture for $\alpha < \omega_1{}^{\omega+2}$ and our method breaks down completely for $\alpha = \omega_1{}^{\omega+2}$. Hajnal, Milner and I proved that every $G(\alpha)$ either contains a C_4 or an independent set of type α. In fact we proved that $G(\alpha)$ either contains a $K(n;\aleph_0)$ (i.e. a bipartite graph of n white and $\aleph_0$ black vertices) or an independent set of type α. Our proof was never published since Laver [5] proved our conjecture: Let ξ be an order type without fixed points. Then $G(\xi)$ either contains C_4 or an independent set of type ξ.

Hajnal and I proved that every graph which contains no infinite path has chromatic number $\leq \aleph_0$. Before closing I mention one more of our conjectures. Is it true that every $G(\omega_1{}^{\omega+1})$ either contains a pentagon or an independent set of type $\omega_1{}^{\omega+1}$?

II. The following beautiful conjecture is due to Walter Taylor: Let $K(G) = \aleph_1$ ($K(G)$ is the chromatic number of G). Then to every cardinal number m there is a graph G' with $K(G') = m$ and every finite subgraph of G' also occurs in G.

It would be very desirable to characterise the families F of finite graphs with the property that there are graphs of arbitrarily large chromatic number whose all finite subgraphs are in F.

Many generalisations and modifications are possible; e.g. "finite" can be replaced by "power $\leq m$"). Hajnal, Shelah and I [6] prove in a recent paper that every G with $K(G) \geq \aleph_1$ contains for some n_0 all C_n with $n > n_0$ (C_n is a circuit of n edges). Our simplest unsolved problem states: Is it true that to every G with $K(G) \geq \aleph_1$ there is an n_0 and an edge e so that for every $n > n_0$ G contains a C_n passing through e ?

Galvin [7] posed the following startling question:

Is it true that the function $K(G)$ has the Darboux property in the following sense: Let $m < n$, $K(G) = n$. Is it true that G has a spanned subgraph G_1 with $K(G_1) = m$? Galvin [7] showed that if g.c.h. (generalised hypothesis of the continuum) is not assumed it is consistent to assume that the answer is negative. The answer is perhaps positive if g.c.h. is assumed, and it very well may be positive even without g.c.h. if we consider all subgraphs of G (not only spanned subgraphs).

Hajnal and I [8] proved (assuming C.H.) that there is a graph G of $\aleph_1$ vertices with $K(G) = \aleph_1$ not containing a $K(\aleph_0, \aleph_0)$ (i.e. a complete bipartite graph of $\aleph_0$ white and $\aleph_0$ black vertices). Hajnal [9] later proved that there is such a graph which further does not contain a triangle. Is there such a graph the smallest odd circuit of which has size $2\ell + 1$? (By a result of Hajnal and myself [8] $K(G) = \aleph_1$ implies that G contains a $K(n; \aleph_1)$ for every finite n.)

In a forthcoming paper with Galvin and Hajnal [10] we systematically study set systems of large chromatic number not containing prescribed subsystems.

III. Hajnal and I conjectured that for every k and $\ell \geq 3$ there is a $G_{k,\ell}$ which contains no $K(\ell + 1)$ but if one colors the edges of $G_{k,\ell}$ by k colors in an arbitrary way there always is a monochromatic $K(\ell)$. Folkman [11] proved our conjecture for $k = 2$. Recently our general conjecture was proved by Nešetřil and Rödl. In fact they proved a much more general theorem. Nevertheless many finite and infinite problems remain. First of all some numerical problems. Let $f(k,\ell_1,\ell_2)$ be the smallest integer n for which there is a graph $G(n)$ not containing $K(\ell_2)$ but if we color the edges of $G(n)$ by k colors there always is a monochromatic $K(\ell_1)$. Graham [12] proved $f(2,3,6) = 8$ and Inving [13] proved $f(2,3,5) \leq 18$.

On the other hand Folkman's upper bound for $f(2,3,4)$ is enormous (it is much bigger than $10^{10^{10^{10^{10^{10^{10}}}}}}$, the same holds for the bound of Nešetřil and Rödl. I offer max (100 dollars, 300 Swiss francs) for a proof or disproof of $f(2,3,4) < 10^{10}$.

I now state some old problems of Hajnal and myself [14]. Is it true that for every infinite cardinal m there is a graph $G((2^m)^+)$ which does not contain a $K(4)$ but if one colors its edges by m colors there always is a monochromatic triangle ? The problem is open for all $m \geq \aleph_0$.

Is there a graph G which does not contain a $K(\aleph_1)$ but if one colors the edges of G by $\aleph_0$ colors there always is a monochromatic $K(\aleph_0)$? On the other hand there is a G which contains no $K(\aleph_2)$ but if we color its edges by $\aleph_0$ colors there always is a monochromatic $K(\aleph_1)$.

Is it true that the class of bipartite graphs not containing C_4 does not have the Galvin-Ramsey property ? In other words, is there a bipartite graph G_1 not containing C_4 such that if G_2 is any graph not containing C_4 we can color the edges of G_2 by two colors so that there is no monochromatic subgraph isomorphic to G_1 ?

$G_1 \rightarrow (G,G)$ denotes the fact that if we color the edges of G_1 by two colors at least one color contains a monochromatic G. Moreover, $G_1 \longleftrightarrow (G,G)$ denotes the fact that G can be faithfully imbedded into one of the colors, i.e. G is monochromatic and there are no other edges in either color in the graph spanning G. The fact that to every finite G there is a finite G_1 satisfying $G_1 \longleftrightarrow (G,G)$ was raised by Hansen and proved by Deuber, Rödl and Hajnal, Pósa and myself. Recently much work has been done in determining or estimating the size of the smallest n for which $K(n) \longrightarrow (G,G)$ (see the survey paper of Burr [15]). Clearly if the number of vertices of G is fixed n is maximal if G is complete.

As far as I know no work has been done on finding the "smallest" G_1 for which $G_1 \longleftrightarrow (G,G)$. Denote by $f(G)$ the smallest integer for which there is a graph G_1 of $f(G)$ vertices satisfying $G_1 \longleftrightarrow (G,G)$. Determine or estimate $f(G)$. Further determine or estimate $\max f(G)$ where the maximum is to be extended over all graphs having m vertices. It is not at all clear that the maximum is reached if G is $K(m)$. The same questions can be put for the smallest integer $F(G)$ for which there is a G_1 of $F(G)$ edges with $G_1 \longleftrightarrow (G,G)$.

Let G be any graph. Let G_2 be a graph with the smallest number of edges for which $G_2 \longrightarrow (G,G)$. It is not at all clear to me that G_2 must be the complete graph (in fact I would expect this to be false).

The following beautiful question is due to Nešetřil and Rödl: A graph satisfying $G_1 \longleftrightarrow (G,G)$ is said to be irreducible if $G_2 \not\longleftrightarrow (G,G)$ for every proper subgraph (or alternatively for every proper spanned subgraph) of G_1. Is it true that for a large class of graphs G (e.g. for all $K(n)$, $n > 2$) the class of irreducible graphs with respect to G is infinite ? This is not even known if G is a triangle.

It is easy to see that if G contains no C_4 then $G \not\longrightarrow (C_3,C_3)$ since any two triangles of G can have at most one vertex in common. Is there an r for which there is a G which does not contain a C_4 and for which $G \longrightarrow (C_{2r+1}, C_{2r+1})$? In fact is there such a G with $G \longrightarrow (C_5,C_5)$?

Nešetřil, Rödl and I formulated the following problem which seems very interesting to me. Let G be a graph with $G \longrightarrow (K(n),K(n))$. Must G contain two $K(n)$'s which have at least three (or perhaps even $n-1$) vertices in common ? The first interesting case is $n=4$.

IV. A graph G is said to have property $P(\varepsilon)$ if any subgraph of G of n vertices contains an independent set of $(\frac{1}{2}-\varepsilon)n$ vertices. Hajnal and I [16] proved that for every k there is a graph G with property $P(\varepsilon)$ and chromatic number $\geq k$.

Our proof in fact gives a graph $G(n)$ with $K(G(n)) > C_\varepsilon \log n$ and property $P(\varepsilon)$. Is it true that $K(G(n)) < C'_\varepsilon \log n$ where $C'_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$?

G is said to have property $P'(\varepsilon)$ if any subgraph of n vertices contains a spanned bipartite subgraph of $(1-\varepsilon)n$ vertices. Hajnal, Szemerédi and I proved that for every k there is a graph G with $K(G) > k$ having property $P'(\varepsilon)$ - further we proved that for every infinite cardinal m there is a graph having property $P'(\varepsilon)$ and chromatic number m. For the many further problems and results in this direction I refer to our forthcoming paper with Hajnal and Szemerédi which I hope will be written soon.

V. A problem of Shelah states as follows: Let $g(k)$ be the smallest integer with the property that if we color the edges of a $K(g(k))$ by k colors there always is a monochromatic C_4 the two diagonals of which use at most one different color. Shelah conjectures that $g(k)$ tends to infinity exponentially, he has an analogous conjecture for r-graphs.

Last December when we discussed this problem we stated the following new problem: Denote by $f_r(k,\ell,t)$ the smallest integer m so that if the edges of a $K_r(m)$ are colored by k colors there always is a $K(t)$ whose edges are colored by at most ℓ colors. ($K_r(m)$ is the complete r-graph of m vertices, an edge of an r-graph is one of its r-tuples.)

Let r and t be fixed as ℓ decreases from $\binom{t}{2}$ to 1, $f_r(k,\ell,t)$ increases with decreasing ℓ. Perhaps there are integers $\binom{t}{2} \geq \ell_0^{(r)} > \ell_1^{(r)} > \ldots > \ell_{r-1}^{(r)} \geq 1$ so that $f_r(k,\ell,t)$ is linear in k for $\ell > \ell_0^{(r)}$, for $\ell_0^{(r)} \geq \ell > \ell_1^{(r)}$ $f_r(k,\ell,t)$ is of polynomial growth in k, for $\ell_i^{(r)} \geq \ell > \ell_{i+1}^{(r)}$, $1 \leq i \leq r-1$ $(\ell_r^{(r)} = 1)$ $f_r(k,\ell,t)$ grows like an i-fold iterated exponential in k. At the moment very little is known about the validity of this conjecture.

VI. Many papers have recently been written on extremal problems in graph theory. Here I only mention an old conjecture of V.T.Sós and myself ([17] p.30) and a recent conjecture of Sauer and myself.

Is it true that every $G(n;[\frac{1}{2}(k-1)n]+1)$ contains all trees having k edges ? ($G(n;t)$ is a graph of n vertices and t edges.) Our conjecture has been proved for many special trees, but no progress has been made in the general case.

Denote by $f_n(k)$ the smallest integer so that every $G(n;f_n(k))$ contains a regular subgraph of valency k. Trivially $f_n(1)=1$ and $f_n(2)=n$ but we know nothing (literally nothing) non-trivial about $f_n(k)$ for $k>2$!

For much of the recent literature on extremal problems see [18].

VII. The systematic study of random graphs was (as far as I know) started by Rényi and myself [19], [20], [21]. One of our outstanding conjectures stated that there is an absolute constant C so that almost all graphs $G(n;[C\,n\log n])$ are Hamiltonian. Pósa recently proved this conjecture. Komlós and Szemerédi later proved by his method that the result holds for $C=\frac{1}{2}+\varepsilon$.

Let $f(n)\longrightarrow\infty$ as slowly as we please. Rényi and I proved that with probability tending to 1 (as $n\longrightarrow\infty$) every vertex of a

$$G(n;[\tfrac{1}{2}\,n\log n+n\log\log n+nf(n)]) \tag{1}$$

has valency ≥ 2. Perhaps with probability tending to 1 the graphs (1) are Hamiltonian. Perhaps this is too good to be true but I have not been able to disprove it. But an even stronger conjecture is possible: Consider the graphs

$$G(n;[\tfrac{1}{2}\,n\log n+n\log n+Cn]). \tag{2}$$

With probability tending to 1 if a graph (2) has all its vertices of valency ≥ 2 then it is Hamiltonian.

I expect that for large C and $n\longrightarrow\infty$ almost all graphs $G(n;Cn)$ have a circuit of size $>(1-\varepsilon)n$. I think the strongest conjecture which could be true states as follows: There is a function $f(C)$ so that with probability tending to 1 the longest circuit of $G(n;Cn)$ has size $(1+o(1))\,f(C)n$, $f(C)\longrightarrow 1$ as $C\longrightarrow\infty$. The fact that $f(\frac{1}{2})=0$ follows from our results with Rényi, but perhaps $f(C)$ is a continuous strictly increasing function for $C\geq\frac{1}{2}$. $f(C)<1$ again follows from our results with Rényi.

VIII. Edwards [22] and I proved that every graph of m edges contains a bipartite subgraph of $\frac{m}{2} + C_1 m^{\frac{1}{2}}$ edges but that in general it does not contain a bipartite graph with $\frac{m}{2} + C_2 m^{\frac{1}{2}}$ edges. Edwards in fact proved a sharper result.

Is it true that every graph of m edges which contains no triangle contains a bipartite subgraph of $\frac{m}{2} + \left[m^{\frac{1}{2}+\alpha} \right]$ edges for a certain absolute constant $\alpha > 0$? I proved by probabilistic methods that the result certainly fails if α is close enough to $\frac{1}{2}$. On the other hand I can not even prove that our graph contains a bipartite graph of $\frac{m}{2} + \left[f(m) m^{\frac{1}{2}} \right]$ edges where $f(m)$ tends to infinity as slowly as we please.

IX. E.Koch asked me the following question: Let $G(n)$ be a graph of n vertices $x_1, \ldots, x_n$. A subsystem $x_{i_1}, \ldots, x_{i_r}$ is called dominating if every other vertex x is joined to at least one of the x_{i_j}, $1 \leq j \leq r$. Determine or estimate

$$\max_{G(n)} \min \left(\sum_{j=1}^{r} v(x_{i_j}) \right) = f(n),$$

where the minimum is to be taken over all dominating systems and the maximum over all graphs $G(n)$ of n vertices ($v(x)$ is the valency or degree of x).

We now prove

$$C_1 n^{\frac{3}{2}} < f(n) < n^{\frac{3}{2}}. \tag{1}$$

Let x_{i_1} be one of the vertices of maximal valency. Suppose we already have found $x_{i_1}, \ldots, x_{i_r}$. Let $y_1, \ldots, y_m$ be the vertices not joined to any of the x_{i_j}'s. If there is a vertex $x \neq x_{i_j}$ $(1 \leq j \leq r)$ which is joined to at least $\left[\frac{1}{2} \sqrt{n} \right]$ of the y's we put $x_{i_{r+1}} = x$. If no such vertex exists our dominating set simply is $x_{i_1}, \ldots, x_{i_r}, y_1, \ldots, y_m$. We now show

$$\sum_{j=1}^{r} v(x_{i_j}) \leq \tfrac{1}{2} n^{\frac{3}{2}}, \qquad \sum_{i=1}^{m} v(y_i) < \tfrac{1}{2} n^{\frac{3}{2}}, \tag{2}$$

which of course will imply the upper bound in (1).

By our assumption each vertex of G is joined to at most $\left[\frac{1}{2}\sqrt{n}\right] - 1$ y's. Thus by a simple argument

$$\sum_{i=1}^{m} v(y_i) < n\left(\left[\tfrac{1}{2}\sqrt{n}\right] - 1\right). \tag{3}$$

A simple argument shows that $r \le \frac{n^{\frac{3}{2}}}{v(x_1)}$ (this follows since x_1 eliminates $1 + v(x_1)$ vertices and every further vertex x_{i_j} eliminates at least $1 + \left[\frac{1}{2}\sqrt{n}\right]$ vertices). Thus we obtain

$$\sum_{j=1}^{r} v(x_{i_j}) \le \tfrac{1}{2}n^{\frac{3}{2}}. \tag{4}$$

(3) and (4) proves (2) and thus the upper bound in (1) is proved.

Spencer and I obtained the lower bound in (1) as follows: For simplicity assume $n = 4m$. Our graph $G(n)$ will be bipartite, $x_1, \dots, x_n$, $y_1, \dots, y_n$ are the white, $z_1, \dots, z_{2n}$ the black vertices. Every x is joined to every z. Every z is joined to $[\sqrt{n}]$ y's and every y is joined to $2[\sqrt{n}]$ z's. A dominating set must contain at least $[\sqrt{n}]$ z's thus since every z is joined to every x we obtain $f(4n) > n^{\frac{3}{2}}$. The lower bound in (1) could easily be improved but I do not see at the moment a proof for

$$\lim_{n\to\infty} \frac{f(n)}{n^{\frac{3}{2}}} = C.$$

It would be interesting to determine $f(n)$ explicitly or if this is too difficult at least one should prove

$$f(n) = (C + 1)\, n^{\frac{3}{2}}$$

for a certain C. I have not been able to prove this.

E.Koch also wanted to determine $f(n;m)$ where

$$f(n;m) = \max_{G(n;m)} \left(\min \Sigma\, v(x_i)\right).$$

The same proof gives

$$C_1 n\, m^{\frac{1}{2}} < f(n;m) < C_2 n\, m^{\frac{1}{2}}.$$

X. Finally I state a very attractive conjecture of Faber, Lovász and myself: Let $1 \le k \le n$, be n sets satisfying $|A_k| = n$, $|A_i \cap A_j| \le 1$, $1 \leqq i < j \le n$. Is is true that elements of $\bigcup_{i=1}^{n} A_i$ can be colored by n colors so that every set A_k gets all the n colors ?

It is easy to see that the theorem fails if the number of sets can be $n+1$. Greenwell and Lovász proved the conjecture if the number of sets is at most $\left[\frac{n+1}{2}\right]$.

I thought of this generalisation: Let $|A_k| = n$, $1 \le k \le m$, $|A_i \cap A_j| \le 1$, $1 \le i < j \le m$. Determine (or estimate) the smallest $f(n,m)$ so that we can color the elements of $\bigcup_{k=1}^{m} A_k$ by $f(n,m)$ colors so that no A_k contains two elements of the same color. Many further generalisations are possible. I only state one which I formulated during our excursion and which was nearly completely solved on the spot by R.C.Bose and L.Lovász. Denote by A_i, $|A_i| = n+1$, $1 \le i \le n^2+n+1$ the lines of a finite geometry. I want to color the n^2+n+1 elements by $n+1$ colors in such a way that each of the sets get as many colors as possible.

First of all Bose observed that if $n = m^2 = p^{2\alpha}$ then it follows from the results of J.Freeman [23] that one can color the elements by $n+1$ colors so that every line gets $m^2 - m$ colors and then Lovász proved that one of the lines always gets at most $m^2 - m + o(m)$ colors. New problems arise if we insist that every line has at most two points of the same color.

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