RATIONAL APPROXIMATION ON THE POSITIVE REAL AXIS

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Introduction

Rational Chebyshev approximation to reciprocals of certain entire functions by reciprocals of polynomials on the positive real axis has recently attracted the attention of many mathematicians. By developing certain new methods of approach we successfully attacked ([**3**]–[**6**]) some of the related problems. This paper is a continuation of our earlier papers ([**3**]–[**6**]). The results of this paper improve and extend some of the earlier results with simplified proofs (cf. Theorem 3). For a reader interested in this topic, this paper may serve as a guide by illustrating some of the techniques (old ones with refinements, as well as new) which we used to solve some of the very interesting and difficult problems of the field (cf. examples 1, 2, 3 of Theorem 5).

Notation and definitions

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be any entire function. As usual, let

$$M(r) = \max_{|z|=r} |f(z)|, \quad m(r) = \max_{n \geq 1} |a_n| r^n = |a_\nu| r^\nu,$$

where $\nu = \nu(r)$ is an increasing function of r. $M(r)$, $m(r)$, and $\nu(r)$ are known as the *maximum modulus*, *maximum term*, and the *rank of the maximum term*, respectively. If there exists more than one term which is equal to the maximum term, then we take the one with the largest index. $S_n(z)$ denotes the nth partial sum of $f(z)$. π_n denotes the class of ordinary polynomials of degree at most n, $\pi_{m,n}$ denotes the class of all rational functions of the form $r_{m,n} = p_m/q_n$, where $p_m \in \pi_m$, $q_n \in \pi_n$. Throughout our work we denote ($k \geqslant 1$):

$$l_k(x) = l_{k-1}[\log x], \quad l_0(x) = x;$$

$$e_k(x) = e_{k-1}[\exp x], \quad e_0(x) = x;$$

$$n(l_1 n)(l_2 n)(l_3 n)\ldots(l_{k+1} n)^2 = A(n);$$

$$(l_1 n)(l_2 n)(l_3 n)\ldots(l_k n)^{1+\varepsilon} = B(n);$$

$$(l_1 n)(l_2 n)(l_3 n)\ldots(l_k n) = D(n).$$

As usual we write

$$\lambda_{0,n}(f^{-1}) \equiv \lambda_{0,n} \equiv \inf_{p \in \pi_n} \left\| \frac{1}{f(x)} - \frac{1}{p(x)} \right\|_{[0,\infty)}, \tag{1}$$

where $\|\cdot\|$ is the uniform norm on $[0,\infty)$. As usual we define the order ρ of $f(z)$ as follows ([**2**], p. 8). The entire function $f(z)$ is of order ρ if

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \rho \quad (0 \leqslant \rho \leqslant \infty).$$

If ρ is positive and finite, then we define the type τ and the lower type ω, corresponding to the order ρ, as follows:

$$\limsup_{r\to\infty} r^{-\rho} \log M(r) = \tau, \quad \liminf_{r\to\infty} r^{-\rho} \log M(r) = \omega \tag{2}$$

$$(0 < \rho < \infty,\ 0 \leqslant \omega \leqslant \tau \leqslant \infty).$$

It is known ([**2**], p. 13) that for functions of finite order we can replace $\log M(r)$ by $\log m(r)$ in the above formulae. That is,

$$\limsup_{r\to\infty} \frac{\log\log m(r)}{\log r} = \rho \quad (0 \leqslant \rho \leqslant \infty),$$

$$\limsup_{r\to\infty} r^{-\rho} \log m(r) = \tau, \quad \liminf_{r\to\infty} r^{-\rho} \log m(r) = \omega \tag{2'}$$

$$(0 < \rho < \infty,\ 0 \leqslant \omega \leqslant \tau \leqslant \infty).$$

If $f(z)$ is of order zero, then we define as in [**11**], p. 145, the logarithmic order $\rho_l = \Lambda + 1$, and if Λ is strictly positive and finite then the corresponding logarithmic types are defined as follows:

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \limsup_{r\to\infty} \frac{\log\log m(r)}{\log\log r} = \Lambda + 1 \quad (0 \leqslant \Lambda \leqslant \infty),$$

$$\limsup_{r\to\infty} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = \limsup_{r\to\infty} \frac{\log m(r)}{(\log r)^{\Lambda+1}} = \tau_l,$$

$$\liminf_{r\to\infty} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = \liminf_{r\to\infty} \frac{\log m(r)}{(\log r)^{\Lambda+1}} = \omega_l \quad (0 < \Lambda < \infty,\ 0 \leqslant \omega_l \leqslant \tau_l \leqslant \infty).$$

It is also known ([**20**], p. 45) that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is of order ρ $(0 < \rho < \infty)$, type τ, and lower type ω $(0 < \omega \leqslant \tau < \infty)$, then

$$\limsup_{n\to\infty} (n/\rho e) |a_n|^{\rho/n} = \tau,$$

and

$$\liminf_{p\to\infty} (n_p/\rho e) |a_{n_p}|^{\rho/n_p} \geqslant \omega, \tag{3}$$

for a sequence of numbers n_p satisfying the condition

$$\limsup_{p\to\infty} (n_{p+1}/n_p) \leqslant x_1/x_2, \tag{4}$$

where x_1 is the greatest and x_2 the smallest root of the equation

$$x \log(x/e) + (\omega/\tau) = 0. \tag{5}$$

LEMMA 1 ([7], pp. 534–35). *Let $p(x)$ be any algebraic polynomial of degree at most n. If this polynomial is bounded by M on an interval of total length l contained in $[-1, 1]$, then in $[-1, 1]$,*

$$|p(x)| \leqslant M\,|T_n(4l^{-1}-1)| \tag{6}$$

where $2T_n(x) = (x+\sqrt{(x^2-1)})^n + (x-\sqrt{(x^2-1)})^n$.

LEMMA 2 ([20], p. 34). *Let $f(z) = \sum_0^\infty a_k z^k$ be any entire function of finite order ρ. Then for any $\varepsilon > 0$, and all sufficiently large $r \geqslant r_0(\varepsilon)$, we have*

$$M(r) \leqslant m(r) r^{\rho+\varepsilon}.$$

LEMMA 3 ([10], Problem [1], part I). *Let*

$$f(x) = 1 + \sum_{j=1}^{\infty} \frac{x^j}{d_1 d_2 d_3 \ldots d_j} \quad (d_{j+1} > d_j > 0,\ j \geqslant 1). \tag{7}$$

Then for $x = d_n$, the nth term of the series (7) becomes the maximum term.

For the detailed discussion of our results, we need the following known results.

THEOREM I ([8], Theorem 6). *Let $f(z) = \sum_{k=0}^\infty a_k z^k$ be an entire function of order ρ, type τ, and lower type ω $(0 < \omega \leqslant \tau < \infty)$, with $a_0 > 0$ and $a_k \geqslant 0$ for all $k \geqslant 1$. Then*

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} < 1.$$

THEOREM II ([12], Theorem 7′). *Let $f(z) = \sum_{k=0}^\infty a_k z^k$ $(a_0 > 0,\ a_k \geqslant 0,\ k \geqslant 1)$ be an entire function satisfying the assumptions that $0 < \Lambda < \infty$ and $0 < \omega_l \leqslant \tau_l < \infty$. Then*

$$\limsup_{n\to\infty} (\lambda_{0,n})^{n^{-1-(1/\Lambda)}} < 1.$$

THEOREM III ([13], Theorem D). *Let $f(z) = \sum_{k=0}^\infty a_k z^k$ $(a_0 > 0,\ a_k \geqslant 0,\ k \geqslant 1)$ be any entire function of order ρ $(0 < \rho < \infty)$, type τ, and lower type ω, with the assumption that $\tau < \theta\omega$ for a $\theta < 2$ and $0 < \omega \leqslant \tau < \infty$. Then*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \geqslant (\omega/\tau 2^{2\rho+1})^{x_1/\rho x_2}.$$

THEOREM IV ([3], Theorem 1). *Let $f(z) = \sum_{k=0}^\infty a_k z^k$ $(a_0 > 0,\ a_k \geqslant 0,\ k \geqslant 1)$ be any entire function. Then for each $\varepsilon > 0$ there exist infinitely many n such that*

$$\lambda_{0,n} \leqslant \exp(-n(\log n)^{-1-\varepsilon}).$$

THEOREM V ([3], Theorem 2). *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be any entire function of infinite order with non-negative coefficients. Then for each $\varepsilon > 0$ there exist infinitely many n for which*

$$\lambda_{0,n} \geqslant \exp(-\varepsilon n).$$

THEOREM VI ([18], Theorem). *Let $f(x) = e^x$. Then*

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n} = \tfrac{1}{3}.$$

Careful observation of the above theorems naturally leads to the following questions.

QUESTION 1. Can one obtain under the assumptions of Theorem I the fact that

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} > 0\,?$$

QUESTION 2. Is it possible to improve the upper bound and provide a simple proof to Theorem I?

QUESTION 3. What conclusion do we get by dropping the assumptions on the logarithmic types in Theorem II?

QUESTION 4. Is it possible to prove Theorem III without the assumption that $\tau < \theta\omega$?

QUESTION 5. Is it possible to replace $(\log n)^{1+\varepsilon}$ by $(l_1 n)(l_2 n)\ldots(l_k n)^{1+\varepsilon}$ for any $k \geqslant 1$ in Theorem IV?

QUESTION 6. Given an $\varepsilon_n \geqslant (\log\log n)^{-1}$ can we replace ε in Theorem V by ε_n?

QUESTION 7. Are there any other functions besides e^x for which we get, for a $\psi(n)$ which tends to infinity,

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/\psi(n)} = \delta \quad (0 < \delta < 1)\,?$$

These questions motivated the work of this paper and in it we answer all of them.

New results

THEOREM 1. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_0 > 0,\ a_k \geqslant 0,\ k \geqslant 1)$ be any entire function. Then for each $\varepsilon > 0$ and any $k \geqslant 1$, there exist infinitely many n such that*

$$\lambda_{0,n} \leqslant \exp(-n/(l_1 n)(l_2 n)\ldots(l_k n)^{1+\varepsilon}). \tag{8}$$

Proof. If $f(z) = \sum_0^{\infty} a_k z^k$ is entire, then $\lim_{n\to\infty} |a_n|^{1/n} = 0$. Let $u_n = a_n^{-1/n}$. Then $u_n \to \infty$. Now it is easy to observe from the convergence of

$$\prod_{j=e_{k+1}(2)}^{\infty} (1+[A(j)]^{-1}),$$

that there exist arbitrarily large values of n for which for each $l > 0$,

$$u_{n+l} > u_n \prod_{t=1}^{l} (1+[A(n+t)]^{-1}). \tag{9}$$

From (9) it follows, with $l = n$, that

$$u_{2n} > u_n(1+n[2A(n)]^{-1}). \tag{10}$$

Given any $\varepsilon > 0$, we can show now that there exist infinitely many n such that

$$\left\| \frac{1}{f(x)} - \frac{1}{S_{2n}(x)} \right\|_{[0,\infty)} < \exp(-2n[B(2n)]^{-1}). \tag{11}$$

By the definition of $\lambda_{0,n}$ (8) follows from (11). To prove (11), observe that, on the one hand, we have for all $x \geqslant 0$,

$$0 \leqslant \frac{1}{S_{2n}(x)} - \frac{1}{f(x)} \leqslant \frac{1}{S_{2n}(x)} \leqslant \frac{1}{a_n x^n}.$$

Now for any given $\varepsilon > 0$,† let $x \geqslant u_n(1+[B(n)]^{-1})$. Then

$$a_n x^n \geqslant (1+[B(n)]^{-1})^n \geqslant \exp(2n[B(2n)]^{-1}). \tag{12}$$

On the other hand, let $x < u_n(1+[B(n)]^{-1})$. Then for all $n \geqslant n_1$,

$$0 \leqslant \frac{1}{S_{2n}(x)} - \frac{1}{f(x)} = \frac{f(x) - S_{2n}(x)}{f(x) S_{2n}(x)} \leqslant a_0^{-2} \sum_{k=2n+1}^{\infty} a_k x^k. \tag{13}$$

By (9) and (10), we have for all $k > 2n$,

$$a_k < u_n^{-k}(1+n[2A(n)]^{-1})^{-k}. \tag{14}$$

Thus, from (13) and (14), for $x < u_n(1+[B(n)]^{-1})$, we obtain

$$a_0^{-2} \sum_{k=2n+1}^{\infty} a_k x^k \leqslant a_0^{-2} \sum_{k=2n+1}^{\infty} \left(\frac{1+[B(n)]^{-1}}{1+n[2A(n)]^{-1}} \right)^k. \tag{15}$$

A simple calculation based on (15) gives us

$$a_0^{-2} \sum_{k=2n+1}^{\infty} a_k x^k \leqslant \exp(-2n[B(2n)]^{-1}). \tag{16}$$

Inequality (11) now follows from (12) and (16).

THEOREM 2. *For every large $c > 0$ and $k \geqslant 1$ there is an entire function of infinite order with non-negative coefficients for which there exist infinitely*

† ε may not be the same at each occurrence.

many n such that

$$\lambda_{0,n} \geqslant \exp(-cn/(l_1 n)(l_2 n)\ldots(l_k n)).$$

Proof. Let $f(x) = e_{k+1}(x)$ $(k \geqslant 1)$, and let us suppose that for all large n,

$$\left\| \frac{1}{f} - \frac{1}{p} \right\|_{[0,\infty)} < \exp(-cn[D(n)]^{-1}). \tag{17}$$

Let $x = l_k(n[D(n)]^{-1}) = r_1(k)$, say $(k \geqslant 1)$. At this point

$$f(x) = e_{k+1}(x) = \exp(n[D(n)]^{-1}).$$

Then, by (17),

$$|p| < \exp(2n[D(n)]^{-1}). \tag{18}$$

But at $x = l_k(ne^2[D(n)]^{-1}) = r_2(k)$, say,

$$f(x) = e_{k+1}(x) = \exp(ne^2[D(n)]^{-1}). \tag{19}$$

By applying Lemma 1 to (18) we get for the interval $[0, r_2]$

$$|p| \leqslant \exp\left(\frac{2n}{(l_1 n)(l_2 n)\ldots(l_k n)}\right) T_n\left(\frac{2l_k(ne^2[D(n)]^{-1})}{l_k(n[D(n)]^{-1})} - 1\right) \tag{20}$$
$$\times \exp(2n[D(n)]^{-1})\exp(4n[D(n)]^{-1}).$$

From (19) and (20), it is easy to see that

$$\left\| \frac{1}{f(x)} - \frac{1}{p(x)} \right\|_{[0,r_2]} \geqslant \exp(-cn[D(n)]^{-1}),$$

for some constant c, which contradicts our earlier assumption (17). Hence the required result is proved.

THEOREM 3. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_0 > 0,\ a_k \geqslant 0,\ k \geqslant 1)$ be any entire function of order ρ $(0 < \rho < \infty)$, type τ, and lower type ω $(0 < \omega \leqslant \tau < \infty)$. Then*

$$0 < (e\omega^2/e^{2\omega/\tau(e+1)}\tau^2(e+1)4^{\rho})^{x_1/x_2} \leqslant \liminf_{n\to\infty}(\lambda_{0,n})^{\rho/n}$$
$$\leqslant \limsup_{n\to\infty}(\lambda_{0,n})^{\rho/n} \leqslant \exp(-\omega/(e+1)\tau) < 1,$$

where x_1 is the greatest and x_2 the smallest root of equation (5).

Proof. If $f(z)$ is an entire function of order ρ $(0 < \rho < \infty)$ and type τ $(0 < \tau < \infty)$, then, for each $\varepsilon > 0$, there is an $n_2 = n_2(\varepsilon)$ such that, for all $n \geqslant n_2(\varepsilon)$, we have ([**2**], p. 11)

$$|a_n| \leqslant (\rho e\tau(1+\varepsilon)/n)^{n/\rho}. \tag{21}$$

As earlier, we have

$$(22) \qquad 0 \leqslant \frac{1}{S_{n_2}(x)} - \frac{1}{f(x)} \leqslant a_0^{-2}(f(x) - S_{n_2}(x)) \leqslant a_0^{-2} \sum_{k=n_2+1}^{\infty} a_k x^k$$

$$\leqslant a_0^{-2} \sum_{k=n_2+1}^{\infty} a_k r^k \quad (0 \leqslant x \leqslant r).$$

On the other hand, for each $r > 0$, we can find ([**2**], p. 12) an $n_3 = n_3(r)$ such that for all $n \geqslant n_3$† and $x \geqslant r$,

$$(23) \qquad 0 \leqslant \frac{1}{S_{n_3}(x)} - \frac{1}{f(x)} \leqslant \frac{1}{S_{n_3}(x)} \leqslant \frac{1}{m(r)}.$$

Now two possibilities occur in (23): (i) $n_3 > n_2$, or (ii) $n_3 \leqslant n_2$. If (i) is true, then in (22) we replace $S_{n_2}(x)$ by $S_{n_3}(x)$, that is,

$$(22') \qquad 0 \leqslant \frac{1}{S_{n_3}(x)} - \frac{1}{f(x)} \leqslant a_0^{-2} \sum_{k=n+1}^{\infty} a_k r^k.$$

If (ii) is true, then in (23) we replace $S_{n_3}(x)$ by $S_{n_2}(x)$, that is, for all $x \geqslant r$,

$$(23') \qquad 0 \leqslant \frac{1}{S_{n_2}(x)} - \frac{1}{f(x)} \leqslant \frac{1}{m(r)},$$

where

$$(24) \qquad r = (n/\rho\tau(e+1)(1+\varepsilon))^{1/\rho}.$$

In either case we choose $n = \max(n_2, n_3)$.‡ A simple calculation based on (21) and (22′) gives us, for $0 \leqslant x \leqslant r$,

$$(25) \qquad 0 \leqslant \frac{1}{S_n(x)} - \frac{1}{f(x)} \leqslant a_0^{-2} \sum_{k=n+1}^{\infty} a_k r^k$$

$$\leqslant a_0^{-2} \sum_{k=n+1}^{\infty} \left(\frac{\rho e\tau(1+\varepsilon)}{\rho\tau(e+1)(1+\varepsilon)}\right)^{k/\rho}$$

$$\leqslant a_0^{-2} \sum_{k=n+1}^{\infty} \left(\frac{e}{e+1}\right)^{k/\rho}$$

$$\leqslant a_0^{-2} \left(\frac{e}{e+1}\right)^{(n+1)/\rho} \left(\frac{(e+1)^{1/\rho}}{(e+1)^{1/\rho} - e^{1/\rho}}\right).$$

On the other hand, from (2′) and (23′), for all $r \geqslant r_3(\varepsilon)$,

$$(26) \quad m(r) \geqslant \exp[r^\rho \omega(1-\varepsilon)] = \exp\left(\frac{\omega n(1-\varepsilon)}{(e+1)\rho\tau(1+\varepsilon)}\right) = G\left(\frac{n(1-\varepsilon)}{(1+\varepsilon)}, \rho, \omega, \tau\right),$$

† n_3 denotes the rank of the maximum term.

‡ It is easy to verify that $n_2(r) \leqslant n(r)$ for the value of r given in (24).

where ε is arbitrary. We easily obtain from (25) and (26) that

$$\limsup_{n\to\infty}(\lambda_{0,n})^{1/n} \leq \exp(-\omega/\rho\tau(e+1)) < 1.$$

Now we shall prove the other inequality of Theorem 3. As the coefficients of $f(x)$ are non-negative, we have from (2), for all large $r \geq r_4(\varepsilon)$,

(27) $\quad 0 \leq f(x) \leq f(r) = M(r) \leq \exp(r^\rho\tau(1+\varepsilon)) \quad (0 \leq x \leq r,\ r \geq r_4(\varepsilon)).$

Now from (27) we have, for

$$r = \{\omega n(1+2\varepsilon)^{-1}\rho^{-1}\tau^{-2}(e+1)^{-1}\}^{1/\rho} = H(n,\rho,\omega,\tau),$$

that

$$\begin{aligned} 0 \leq f(x) &\leq f\left[\left(\frac{\omega n}{\tau^2\rho(e+1)(1+2\varepsilon)}\right)^{1/\rho}\right] \\ &\leq \exp\left(\frac{n\omega(1+\varepsilon)}{(1+2\varepsilon)\tau\rho(e+1)}\right) \\ &< G(n,\rho,\omega,\tau) = \exp\left(\frac{n\omega}{\tau(e+1)\rho}\right) \leq \frac{1}{\lambda_{0,n}} \end{aligned}$$

for all $n \geq n_4$. Next, we take the rational function $r^*_{0,n} = 1/p^*_n$ ($r^*_{0,n} \in \pi_{0,n}$) which gives the best approximation in the sense of (1); that is, for all $n \geq n_5$,

(28) $$\lambda_{0,n} \equiv \left\|\frac{1}{f(x)} - \frac{1}{p^*(x)}\right\|_{[0,\infty)}.$$

A simple manipulation based on (28) gives us

(29) $$\begin{aligned} -f^2(x)/\left(f(x)+\frac{1}{\lambda_{0,n}}\right) &\leq p^*_n - f(x) \\ &\leq f^2(x)/\left(\frac{1}{\lambda_{0,n}} - f(x)\right) \quad (0 \leq x \leq H(n,\rho,\omega,\tau)). \end{aligned}$$

Clearly the right-hand side of inequality (29) is monotonic increasing with x. Hence we write

(30) $$\|p^*_n - f(x)\| \leq G(2n,\rho,\omega,\tau)/\left(\frac{1}{\lambda_{0,n}} - G(n,\rho,\omega,\tau)\right) \quad (0 \leq x \leq H(n,\rho,\omega,\tau)).$$

Next, let

(31) $$E_n(f) \equiv \inf_{p_n \in \pi_n} \|p_n(x) - f(x)\|_{[0,H(n,\rho,\omega,\tau)]}.$$

From (30) and (31) we get, for all $n \geq n_6$,

(32) $$E_n(f) \leq G(2n,\rho,\omega,\tau)/\left(\frac{1}{\lambda_{0,n}} - G(n,\rho,\omega,\tau)\right).$$

To obtain a lower bound for E_n we use a result of Bernstein ([**1**], p. 10) which gives us, in the interval $[0, H(n, \rho, \omega, \tau)]$,

$$E_n \geqslant \left(\frac{n\omega}{\tau^2(e+1)\rho(1+2\varepsilon)}\right)^{(n+1)/\rho} \frac{a_{n+1}}{2^{2n+1}}. \tag{33}$$

From (32) and (33) we get

$$\frac{a_{n+1}[H(n, \rho, \omega, \tau)]^{n+1}}{2^{2n+1}} \leqslant G(2n, \rho, \omega, \tau) \Big/ \left(\frac{1}{\lambda_{0,n}} - G(n, \rho, \omega, \tau)\right) \tag{34}$$

for all $n \geqslant n_7$.

From (3) we have, for a sequence of values of $n = n_p + 1$ satisfying the assumption (4),

$$a_{n_p+1} \geqslant (\rho e\omega(1-\varepsilon)/(n_p+1))^{(n_p+1)/\rho}. \tag{35}$$

From (34) and (35) we get

$$\left(\frac{\rho e\omega(1-\varepsilon)}{n_p+1}\right)^{(n_p+1)/\rho} \frac{[H(n_p, \rho, \omega, \tau)]^{n_p+1}}{2^{2n_p+1}} \leqslant G(2n, \rho, \omega, \tau) \Big/ \left(\frac{1}{\lambda_{0,n}} - G(n, \rho, \omega, \tau)\right). \tag{36}$$

It is easy to obtain from (36) that

$$\begin{aligned} \frac{1}{\lambda_{0,n_p}} &\leqslant G(n_p, \rho, \omega, \tau) + \frac{G(2n_p, \rho, \omega, \tau)(n_p+1)^{(n_p+1)/\rho} 2^{2n_p+1}}{[\{\rho e\omega(1-\varepsilon)\}^{1/\rho} H(n_p, \rho, \omega, \tau)]^{n_p+1}} \\ &\leqslant 2\left[\exp\left(\frac{2n_p\omega}{\tau(e+1)\rho}\right)\right] \left(\frac{4^\rho \tau^2 \rho(e+1)(1+2\varepsilon)}{\omega^2(1-\varepsilon)e}\right)^{(n_p+1)/\rho} \left(\frac{n_p+1}{n_p}\right)^{(n_p+1)/\rho}. \end{aligned} \tag{37}$$

Now by adopting the technique used on p. 373 of [**19**] we can easily obtain from (37) and (4) the fact that

$$\liminf_{n\to\infty} (\lambda_{0,n})^{\rho/n} \geqslant \left(\frac{e\omega^2}{e^{2\omega/\tau(e+1)}\tau^2(e+1)4^\rho}\right)^{x_1/x_2}. \tag{38}$$

REMARKS. (1) Under the assumptions of Theorem 3, Reddy ([**16**]) has recently obtained the following sharper result

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \geqslant (2^{2+1/\rho}\tau^{1/\rho}\omega^{-1/\rho} - 1)^{-2}.$$

(2) There exist entire functions which fail to satisfy the assumptions of Theorem 3, but for which we can still find two constants c_1 and c_2 $(0 < c_1 \leqslant c_2 < 1)$ such that

$$0 < c_1 \leqslant \liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant c_2 < 1.$$

EXAMPLE 1. Let

$$f(x) = 1 + \sum_{k=2}^{\infty} \left(\frac{\log n}{n}\right)^n x^n.$$

This is an entire function of order $\rho = 1$ and type $\tau = \infty$. For this function the maximum module $M(r)$ is given by

$$f(r) = M(r) \sim \exp\left(\frac{r}{e} \log \frac{r}{e}\right). \tag{39}$$

This function clearly satisfies the growth condition (3.1) of [8]; hence there exists a constant $q > 1$ for which

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} = \frac{1}{q} < 1. \tag{40}$$

From (4) it is easy to see that, for all large $n \geqslant n_8(\varepsilon)$,

$$q_1^n = [q(1-\varepsilon)]^n \leqslant 1/\lambda_{0,n}. \tag{41}$$

From (39) we have

$$0 \leqslant f(x) \leqslant f(r) < \exp(r \log(r/e)).$$

Let

$$r \log(r/e) = \tfrac{1}{2} n \log q_1. \tag{42}$$

That is,

$$r \sim \frac{n \log q_1}{2 \log(\frac{1}{2} n \log q_1)}.$$

From (41) and (42) we get, for all $n \geqslant n_9$,

$$0 \leqslant f(x) \leqslant f(r) \leqslant \exp(r \log(r/e)) \leqslant q_1^{n/2} < 1/\lambda_{0,n}.$$

Now proceeding exactly as in the proof of the second part of Theorem 3 we get, for the value of r given in (42) and for all $n \geqslant n_{10}$,

$$\left(\frac{n}{\log(\frac{1}{2} n \log q_1)}\right)^{n+1} \frac{a_{n+1}}{2^{2n+1}} \leqslant q_1^n \Big/ \left(\frac{1}{\lambda_{0,n}} - q_1^{n/2}\right). \tag{43}$$

From (43) it is easy to see that

$$\begin{aligned} \frac{1}{\lambda_{0,n}} &\leqslant \frac{2^{2n+1} q_1^n [\log(\frac{1}{2} n \log q_1)]^{n+1}}{n^{n+1} a_{n+1}} + q_1^{n/2} \\ &\leqslant 2^{2n+1} q_1^n \left(\frac{\log(\frac{1}{2} n \log q_1)}{n \log(n+1)}\right)^{n+1} (n+1)^{n+1} + q_1^{n/2}. \end{aligned}$$

In other words,

$$\frac{1}{\lambda_{0,n}} \leqslant 2^{2n+2} q_1^n \left(\frac{n+1}{n}\right)^{n+1} \left(\frac{\log(\frac{1}{2} n) + \log\log q_1}{\log(n+1)}\right)^{n+1}. \tag{44}$$

A simple manipulation based on (44) gives us, if we take q_1 to be very close to q,

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \geq 1/4q.$$

EXAMPLE 2. Let

$$f(x) = 1 + \sum_{n=2}^{\infty} \left(\frac{x}{n \log n}\right)^n.$$

This is an entire function of order $\rho = 1$ and type $\tau = 0$. For this function

$$f(r) \sim \exp\left(\frac{r}{e \log(r/e)}\right). \tag{45}$$

$f(r)$ satisfies the growth condition (3.1) given in [8]; hence there exists a $q > 1$ such that

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} = \frac{1}{q} < 1. \tag{46}$$

As before from (46) we can find, for all $n \geq n_{11}(\varepsilon)$, a $q_1 < q$ such that

$$\lambda_{0,n} \leqslant q_1^{-n}. \tag{47}$$

Let

$$\frac{r}{2 \log(r/e)} = \tfrac{1}{2} n \log q_1. \tag{48}$$

Then for some $c > 0$, $r \sim n \log(cn \log q_1)$. From (45), (47), and (48) we obtain, for all $n \geq n_{12}$,

$$0 \leqslant f(x) \leqslant f(r) \leqslant \exp\left(\frac{(1+\varepsilon)r}{e \log(r/e)}\right) \leqslant q_1^{n/2} < \frac{1}{\lambda_{0,n}}$$

and

$$\frac{n^{(n+1)}[\log(cn \log q_1)]^{n+1} a_{n+1}}{2^{2n+1}} \leqslant q_1^n \Big/ \left(\frac{1}{\lambda_{0,n}} - q_1^{n/2}\right). \tag{49}$$

From (49) we get as before, for $a_n = (n \log n)^{-n}$,

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \geq 1/4q.$$

THEOREM 4. *Let $f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$, $a_k = (d_1 d_2 \ldots d_k)^{-1}$ with $d_{k+1} > d_k > 0$ $(k \geq 1)$, be an entire function of finite order ρ. Then for any $\varepsilon > 0$ and all large $n \geq n_{13}(\varepsilon)$, we have*

$$\frac{d_1 d_2 d_3 \ldots d_n}{2^{4n} d_n^{2(\rho+\varepsilon)} d_{n+1} d_{n+2} \ldots d_{2n}} \leqslant \lambda_{0,2n-1} \leqslant \frac{d_1 d_2 d_3 \ldots d_n d_{2n+1}}{d_{n+1} d_{n+2} \ldots d_{2n} (d_{2n+1} - d_{2n})}. \tag{50}$$

Proof. The second half of (50) follows from the proof of Theorem 5 of [**4**].

To prove the first half of (50) observe that, for $0 \leqslant x \leqslant r = d_n$, Lemmas 2 and 3 hold and that, for all $n \geqslant \hat{n}(\varepsilon)$,

$$\begin{aligned}(51) \qquad 0 \leqslant f(x) &\leqslant M(r) \leqslant m(r) r^{\rho+\varepsilon} \\ &= m(d_n) d_n^{\rho+\varepsilon} \\ &= \frac{d_n^{\,n} d_n^{\,\rho+\varepsilon}}{d_1 d_2 \ldots d_n} < \frac{d_{n+1} d_{n+2} \ldots d_{2n}(d_{2n+1} - d_{2n})}{2 d_{2n+1}} = \frac{(d_{2n+1} - d_{2n})}{2J(d_n) d_{2n+1}}.\end{aligned}$$

This follows because when $x = d_n$ we know from Lemma 3 that the nth term becomes the maximum term, that is,

$$m(d_n) = d_n^{\,n}/d_1 d_2 \ldots d_n.$$

As before we choose $p^*(x) \in \pi_{2n-1}$ such that $p^*(x)$ gives the best approximation to $f(x)$ in the sense of (1), that is,

$$(52) \qquad \lambda_{0,2n-1} \geqslant \left\| \frac{1}{f(x)} - \frac{1}{p^*(x)} \right\|_{[0,\infty)} \quad (p^* \in \pi_{2n-1}).$$

A simple calculation based on (52) gives us (as in the proof of Theorem 3)

$$(53) \qquad \|f - p^*_{2n-1}\| < \{f(x)\}^2 \Big/ \left(\frac{1}{\lambda_{0,2n-1}} - f(x) \right) \quad (0 \leqslant x \leqslant r = d_n).$$

From (51) and (53) we obtain

$$\begin{aligned}(54) \qquad &\|f - p^*_{2n-1}\| \\ &\leqslant d_n^{\,2n} d_n^{\,2(\rho+\varepsilon)} / (d_1 d_2 \ldots d_n)^2 \left(\frac{1}{\lambda_{0,2n-1}} - \frac{d_n^{\,n} d_n^{\,\rho+\varepsilon}}{d_1 d_2 \ldots d_n} \right) \quad (0 \leqslant x \leqslant d_n).\end{aligned}$$

Let

$$(55) \qquad E_{2n-1}(f) = \min_{g_{2n-1} \in \pi_{2n-1}} \|f - g_{2n-1}\|_{[0,d_n]}.$$

From (54) and (55) we get

$$\begin{aligned}(56) \qquad &E_{2n-1}(f) \\ &\leqslant d_n^{\,2n} d_n^{\,2(\rho+\varepsilon)} / (d_1 d_2 \ldots d_n)^2 \left(\frac{1}{\lambda_{0,2n-1}} - \frac{d_n^{\,n} d_n^{\,\rho+\varepsilon}}{d_1 d_2 \ldots d_n} \right) \quad (0 \leqslant x \leqslant d_n).\end{aligned}$$

To obtain a lower bound for $E_{2n-1}(f)$ we use (as before) a result of Bernstein ([1], p. 10), which gives us for $x = d_n$,

$$(57) \qquad E_{2n-1}(f) \geqslant a_{2n} d_n^{\,2n} / 2^{2n} 2^{2n-1}.$$

From (56) and (57) we get, for all $n \geqslant n_{14}$,

$$(58) \qquad J(d_n) \leqslant 2^{4n-1} d_n^{\,2(\rho+\varepsilon)} \Big/ \left(\frac{1}{\lambda_{0,2n-1}} - \frac{d_n^{\,n} d_n^{\,\rho+\varepsilon}}{d_1 d_2 \ldots d_n} \right).$$

From (58) it is easy to calculate that

$$\frac{1}{\lambda_{0,2n-1}} \leqslant \frac{2^{4n} d_n^{2(\rho+\varepsilon)}}{J(d_n)}.$$

Therefore for all large n, we have

$$\frac{d_1 d_2 d_3 \ldots d_n}{2^{4n} d_n^{2(\rho+\varepsilon)} d_{n+1} d_{n+2} \ldots d_{2n}} \leqslant \lambda_{0,2n-1} \leqslant \frac{d_1 d_2 \ldots d_n d_{2n+1}}{d_{n+1} d_{n+2} \ldots d_{2n}(d_{2n+1} - d_{2n})}.$$

EXAMPLES. (1) Let

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{2^{\log 2} 3^{\log 3} \ldots n^{\log n}}.$$

For this function we get from (50)

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/(n \log n)} = \tfrac{1}{2}.$$

It is interesting to note that this function fails to satisfy the assumptions of Theorem 7 of [**8**] and Theorem 7′ of [**12**], because $\rho_l = \Lambda + 1 = \infty$. But our present method which is much simpler than the methods used in [**8**] and [**12**] gives us more precise information.

(2) Let

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{2^2 3^3 4^4 \ldots n^n}.$$

This function also fails to satisfy the assumptions of Theorem 7 of [**8**] and Theorem 7′ of [**12**], because in this case $\Lambda = 1$, that is, $\rho_l = 2$ and $\tau_l = 0$. For this function we get by (50)

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n^2 \log n} = e^{-1/4}.$$

(3) Let

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{\delta^{2^n}} \quad (1 < \delta < \infty).$$

For this function $\Lambda = 0$, whence it also fails to satisfy the assumptions of Theorem 7 of [**8**] and Theorem 7′ of [**12**]; but we obtain by (50)

$$\lim_{n\to\infty} (\lambda_{0,n})^{2^{-n-1}} = 1/\delta.$$

THEOREM 5. *Let* $f(x) = 1 + \sum_{k=1}^{\infty} (d_1 d_2 \ldots d_k)^{-1} x^k$ $(d_{k+1} > d_k > 0, k \geqslant 1)$ *be an entire function of infinite order. Then for each* $\varepsilon > 0$, *there exist infinitely many* n *for which*

$$\lambda_{0,2n-1} \geqslant [K(d_n)]^{1+\varepsilon}.$$

Proof. As before when $x = d_n$, $d_n^n/d_1d_2d_3\ldots d_n$ becomes the maximum term of f. Let $t_n = x^n/d_1d_2\ldots d_n$. Then

$$f(x) = t_n + t_{n-1}\left(1 + \frac{t_{n-2}}{t_{n-1}} + \frac{t_{n-3}}{t_{n-2}} + \ldots + t_0\right) \tag{59}$$

$$+ t_{n+1}\left(1 + \frac{t_{n+2}}{t_{n+1}} + \frac{t_{n+3}}{t_{n+1}} + \ldots\right).$$

But

$$\frac{t_{n+2}}{t_{n+1}} = \frac{x}{d_{n+2}} < \frac{d_{n+1}}{d_{n+2}},$$

$$\frac{t_{n+3}}{t_{n+1}} = \frac{x^2}{d_{n+2}d_{n+3}} < \left(\frac{d_{n+1}}{d_{n+2}}\right)^2,$$

and so on. Hence

$$\left|\frac{t_{n+2}}{t_{n+1}} + \frac{t_{n+2}}{t_{n+1}} + \ldots\right| < \frac{d_{n+1}}{d_{n+2} - d_{n+1}}. \tag{60}$$

Similarly

$$\frac{t_{n-2}}{t_{n-1}} \leqslant \left(\frac{d_{n-1}}{d_n}\right), \quad \frac{t_{n-3}}{t_{n-1}} \leqslant \left(\frac{d_{n-1}}{d_n}\right)^2,$$

and so on. Therefore

$$\left|\frac{t_{n-2}}{t_{n-1}} + \frac{t_{n-3}}{t_{n-1}} + \ldots\right| < \frac{d_{n-1}}{d_n - d_{n-1}}. \tag{61}$$

From (59), (60), and (61), we obtain

$$f(x) = t_n + t_{n-1}(1 + \varphi) + t_{n+1}(1 + \varphi_1), \tag{62}$$

where

$$|\varphi| < \frac{d_{n-1}}{d_n - d_{n-1}}, \quad |\varphi_1| < \frac{d_{n+1}}{d_{n+2} - d_{n+1}}.$$

From (62) we get, for $x = d_n$,

$$f(x) \leqslant \frac{d_n^{\,n}}{d_1d_2d_3\ldots d_n} + t_{n-1}(1 + \varphi) + t_{n+1}(1 + \varphi_1) = [K(d_n)]^{-1/8}.$$

For all sufficiently large n we can find an $\varepsilon > 0$ such that

$$f(d_n) \leqslant [K(d_n)]^{-(1+\varepsilon)/8}. \tag{63}$$

Now let us suppose that for all large n and all x $(0 \leqslant x < \infty)$

$$\left|\frac{1}{f(x)} - \frac{1}{p_{2n-1}(x)}\right| \leqslant [K(d_n)]^{1+\varepsilon}. \tag{64}$$

From (63) and (64) we get

$$|p| < [K(d_n)]^{-(1+\varepsilon)/4}.$$

Because of the assumption that $f(x)$ is of infinite order for every large $r > 0$ we can find sufficiently large n such that

$$f[d_n(1+r^{-1})] \geqslant [f(d_n)]^{18}. \tag{65}$$

Therefore from (62) and (65) we get

$$f[d_n(1+r^{-1})] \geqslant [K(d_n)]^{-2(1+\varepsilon)}. \tag{66}$$

On the other hand by applying Lemma 1 to $p(x)$ we get

$$p[d_n(1+r^{-1})] \leqslant [K(d_n)]^{-(1+\varepsilon)/2}. \tag{67}$$

From (66) and (67) we get

$$\begin{aligned}[K(d_n)]^{1+\varepsilon} &\leqslant [K(d_n)]^{(1+\varepsilon)/2}\{1-[K(d_n)]^{3(1+\varepsilon)/2}\} \\ &\leqslant \frac{1}{p_n(d_n(1+r^{-1}))} - \frac{1}{f(d_n(1+r^{-1}))},\end{aligned} \tag{68}$$

that is,

$$\{K(d_n)\}^{(1+\varepsilon)/2} + \{K(d_n)\}^{3(1+\varepsilon)/2} < 1,$$

which gives the required result, because the conclusion (68) contradicts our earlier assumption (64).

THEOREM 6. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ ($a_0 > 0$, $a_k \geqslant 0$, $k \geqslant 1$) be any entire function satisfying the assumption that*

$$1 \leqslant \limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \Lambda + 1 < \infty.$$

Then for any $\varepsilon > 0$,

$$\liminf_{n\to\infty} (\lambda_{0,n})^{n^{-1-(1/\Lambda+\varepsilon)}} < 1.$$

Proof. We get from Theorems 1 and 3 of [**11**]

$$\limsup_{n\to\infty} \frac{\log n}{\log\{n^{-1}\log a_n^{-1}\}} = \Lambda.$$

From this we easily obtain that, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} |a_n|^{1/n} \exp(n^{1/(\Lambda+\varepsilon)}) = 0.$$

As earlier, let $u_n = a_n^{-1/n}$; and let $h = 1/(\Lambda+\varepsilon)$. Then $u_n \exp(-n^h) \to \infty$. This implies that there exist infinitely many n for which

$$\frac{u_{n+l}}{\exp[(n+l)^h]} \geqslant \frac{u_n}{\exp(n^h)} \quad (l = 0, 1, 2, \ldots). \tag{69}$$

Now let

$$x \geqslant \theta^{n^h} u_n \quad (1 < \theta < \exp(2^{h-1}).$$

Then

$$(70) \qquad S_{2n}(x) \geqslant a_n x^n \geqslant a_n u_n{}^n \theta^{n^{h+1}} = \theta^{n^{h+1}}.$$

Hence, as usual we get from (70), for $x \geqslant \theta^{n^h} u_n$,

$$(71) \qquad 0 \leqslant \frac{1}{S_{2n}(x)} - \frac{1}{f(x)} \leqslant \frac{1}{S_{2n}(x)} \leqslant \frac{1}{a_n x^n} \leqslant \theta^{-n^{h+1}}.$$

On the other hand, let

$$x < u_n(\theta)^{n^h}.$$

Then as before it is easy to note from (69) that, for any $k > 2n$,

$$|a_k| < u_n{}^{-k} \exp(k(n^h - k^h)).$$

This formula implies that

$$\begin{aligned}
\sum_{k=2n+1}^{\infty} a_k x^k &\leqslant \sum_{k=2n+1}^{\infty} \{\exp k(n^h - k^h)\} \theta^{n^h k} \\
&\leqslant \sum_{k=2n+1}^{\infty} \left(\frac{(\theta e)^{n^h}}{\exp k^h}\right)^k \\
&\leqslant \left(\frac{(\theta e)^{n^h}}{\exp(2n)^h}\right)^{2n+1} \left(\frac{\exp(2n)^h}{\exp(2n)^h - (\theta e)^{n^h}}\right).
\end{aligned}$$

From this we have

$$\begin{aligned}
(72) \qquad \lambda_{0,2n} &\leqslant \left\| \frac{1}{f(x)} - \frac{1}{p_{2n}(x)} \right\| \\
&\leqslant a_0{}^{-2} \sum_{k=2n+1}^{\infty} a_k x^k \\
&\leqslant \left(\frac{(\theta e)^{n^h}}{\exp(2n)^h}\right)^{2n+1} a_0{}^{-2} \left(\frac{\exp(2n)^h}{\exp(2n)^h - (e\theta)^{n^h}}\right).
\end{aligned}$$

From (71) and (72), with $\theta e < \exp(2^h)$, we easily obtain that

$$\liminf_{n\to\infty} (\lambda_{0,2n})^{1/[(2n)^{h+1}]} < e^{-1}(\theta e)^{1/[2^h]} < 1.$$

THEOREM 7. *Let* $f(z) = a_0 + \sum_{k=1}^{\infty} a_{n_k} z^{n_k}$, *with* $a_0 > 0$, $a_{n_k} > 0$ $(k \geqslant 1)$, *and* $\liminf_{k\to\infty}(n_{k+1}/n_k) \geqslant \beta > 1$, *be an entire function of finite order* ρ. *Then for any* α $(0 < \alpha < 1)$ *and any* $\varepsilon > 0$, *we have*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{(1-\alpha)(\rho+\varepsilon)/n} \leqslant \beta^{-1}.$$

The proof of this result is very similar to the proof of the preceding theorem. Hence the details are left to the reader.

Concluding remarks

Theorem 3 of this paper answers questions 1, 2, and 4 in the affirmative. The answer to question 5 follows from Theorem 1. Question 3 is resolved in Theorem 6. The examples given at the end of Theorem 4 answer question 7. Question 6 must be answered in the negative; this follows from Theorem 4.

It may be of some interest to know whether it is possible to obtain a lower bound for

$$\limsup_{n\to\infty}(\lambda_{0,n})^{n^{-1-(1/\Lambda)}} \quad \text{(cf. Theorem II).}$$

This has been solved in [15]. We have proved in [15], under the assumptions of Theorem II, that

$$\limsup_{n\to\infty}(\lambda_{0,n})^{n^{-1-(1/\Lambda)}} \geqslant \exp\left(\frac{-\Lambda}{(\Lambda+1)}\left(\frac{1}{(\Lambda+1)\tau_l}\right)^{1/\Lambda}\right).$$

It is natural to ask whether we can do much better by using the general rational functions of the form $p_n(x)/Q_n(x)$ than by using $1/Q_n(x)$ in the above results. We are not able to settle this question in general (see, however, [9] and [17]). But we are able to prove the following theorem.

THEOREM (cf. [14]). *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ ($a_k \geqslant 0$, $k \geqslant 0$) be any entire function of order ρ ($0 < \rho < \infty$), type τ, and lower type ω ($0 < \omega \leqslant \tau < \infty$). Then one cannot find algebraic polynomials $p(x)$ and $Q(x)$ with non-negative coefficients and of degree at most n for which*

$$\liminf_{n\to\infty}\left\{\left\|\frac{1}{f(x)}-\frac{P(x)}{Q(x)}\right\|_{[0,\infty)}\right\}^{\rho\omega/n\tau} \leqslant (2\sqrt{2})^{-1}.$$

The examples 1 and 2 of Theorem 3 fail to satisfy the assumptions of the above theorem; but the conclusion of the theorem still holds for these examples, in a slightly different form.

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