SOME PROBLEMS ON ELEMENTARY GEOMETRY

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Elementary geometry occupied mathematicians for thousands of years. Nevertheless it is rather easy to find new and, in my opinion, interesting and probably difficult unsolved problems if one asks problems where metrical and combinatorial questions are considered.

Let x_1, x_2, x_3 be three points in the plane in general position, i.e. they are not on a line. Mrs. E. Szekeres observed that there always is a unique point x_4 so that x_1, x_2, x_3, x_4 are not on a circle and so that the radii of the four circles determined by the four triples x_i, x_j, x_ℓ $1 \le i < j < \ell \le 4$ are the same. It suffices to choose x_4 as the orthocentre of the triangle x_1, x_2, x_3.

This observation of E. Szekeres led me to the following problems which seem to me of some interest and which perhaps are quite difficult.

Let $x_1, \ldots, x_n$ be n distinct points in the plane. The triples x_i, x_j, x_ℓ, $1 \le i < j < \ell \le n$ determine $\binom{n}{3}$ circles. These circles do not have to be all different since we do not assume that the points are in general position. Denote now by $f(n)$ the largest integer so that there are $f(n)$ distinct circles of radius one determined by the $\binom{n}{3}$ triples. Obviously

$$\frac{3n}{2} < f(n) \le n(n-1) \tag{1}$$

The lower bound is given by the triangular lattice, the upper bound follows from the fact that to two points x_i and x_j there are at most two circles of radius one passing through them. I am sure that

$$\frac{f(n)}{n^2} \to 0 \text{ and } \frac{f(n)}{n} \to \infty \tag{2}$$

I was not able to prove (2) but perhaps I overlook a simple idea. On the other hand I am fairly sure that it will be very difficult to give an asymptotic formula for $f(n)$ and the exact determination of $f(n)$ may not be possible, i.e. there may not be a simple exact expression for $f(n)$.

Let me state a few modifications of the above problem. First of all assume that our points are in general position, i.e. no four are on a circle and no three on a line. Determine or estimate $f(n)$ under these conditions.

Assume that not all of our points are on a unit circle. Denote by $g(n)$ the maximum number of triples x_i, x_j, x_ℓ $1 \le i < j < \ell \le n$ so that the circumscribed circle has unit radius. Estimate or determine $g(n)$. This perhaps is not too difficult. It seems reasonable to assume that the maximum is achieved if $n - 1$ of our n points are on a unit circle.

It might be more interesting but also more difficult to determine $g(n)$ if the points are assumed to be in general position.

Let there be given n points in the plane in general position. Denote by $h(n)$ the largest integer so that there are at least $h(n)$ circles of different radii passing through three of our points. Estimate or if possible determine $h(n)$. How does $h(n)$ get modified if we only assume that not all our points are on a circle?

Finally I state a problem of a slightly different character. Is it true that to every k there is an n_k so that if there are given n_k points in the plane in general position one can always find k of them so that all the $\binom{k}{3}$ triples determine circles of different radii? At present I cannot even prove that n_k exists.

I was led to this problem by an old question of E. Klein (= Mrs. E. Szekeres) formulated in the distant days of our youth: Is it true that to every k there is an m_k so that if m_k points are given in the plane no three of them on a line then one can select always k of them which form the vertices of a convex k-gon. She proved $m_4 = 5$, Turán and Makai proved $m_5 = 9$ and G. Szekeres conjectured $m_k = 2^{k-2} + 1$.

REFERENCES

Related problems were considered by Purdy and myself: P. Erdös and G. Purdy, Some extremal problems in geometry, J. Combinatorial Theory 10 (1971) 246–252, see also P. Erdös, On sets of distances of n points, Amer. Math. Monthly 53 (1946) 248–250.

On the problem of E. Klein see P. Erdös and G. Szekeres, A combinatorial problem in geometry, Compositio Math. 2 (1935) 463–470, see also P. Erdös and G. Szekeres, On some extremal problems in elementary geometry, Ann. Univ. Sci. Budapest 3–4 (1961) 313–320.