COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
10. INFINITE AND FINITE SETS, KESZTHELY (HUNGARY), 1973.

STRONG EMBEDDINGS OF GRAPHS INTO COLORED GRAPHS

P. ERDŐS – A. HAJNAL – L. PÓSA

§1. NOTATION

Results. By a graph $\mathscr{G}$ we mean an ordered pair $\mathscr{G} = \langle g, G \rangle$ where $G \subset [g]^2$. Let $\gamma > 0$. The sequence $\{G_\nu : \nu < \gamma\}$ is said to be an *edge coloring* of $\mathscr{G}$ by γ colors if the G_ν are disjoint and their union is G. For $h \subset g$ we denote by $\mathscr{G}(h)$ the subgraph of $\mathscr{G}$ spanned by the set h i.e. $\mathscr{G}(h) = \langle h, G \cap [h]^2 \rangle$. We say that the graph $\mathscr{H} = \langle h, H \rangle$ can be *embedded* in the ν-th color of the edge coloring $\{G_\nu : \nu < \gamma\}$ of G if $\mathscr{H}$ is isomorphic to a spanned subgraph $\mathscr{G}_\nu(G')$ of $\mathscr{G}_\nu$.

We say that $\mathscr{H}$ can be *strongly embedded* into the ν-th color if it is isomorphic to some spanned subgraph $\mathscr{G}_\nu(g')$ such that $\mathscr{G}_\nu(g') = \mathscr{G}(g')$.

In other words in both cases there exist one-to-one mappings f of h into g such that $\{x, y\} \in H$ *implies* $\{f(x), f(y)\} \in G_\nu$ for $x, y \in h$ and $\{x, y\} \notin H \wedge x, y \in h$ implies $\{f(x), f(y)\} \notin G_\nu$ is case of embeddings and $\{f(x), f(y)\} \notin G$ in case of strong embeddings respectively.

The following two partition relations were introduced in Erdős –

Hajnal [1] and Henson [2] to study embeddings and strong embeddings respectively.

Let $\mathcal{G}$, $\mathcal{H}_\nu$: $\nu < \gamma$ be graphs. The relations $\mathcal{G} \to (\mathcal{H}_\nu)_{\nu<\gamma}$. $\mathcal{G} \rightarrowtail (\mathcal{H}_\nu)_{\nu<\gamma}$ are said to hold if for every edge coloring $\{G_\nu:\ \nu < \gamma\}$ of $\mathcal{G}$ by γ colors there is a $\nu < \gamma$ such that $\mathcal{H}_\nu$ can be embedded or strongly embedded into the ν-th color respectively.

As usual $\mathcal{G} \not\to (\mathcal{H}_\nu)_{\nu<\gamma}$, $\mathcal{G} \not\rightarrowtail (\mathcal{H}_\nu)_{\nu<\gamma}$ denote that the corresponding statements are false. If all $\mathcal{H}_\nu$ are equal to $\mathcal{H}$ we write $\mathcal{G} \longrightarrow (\mathcal{H})_\gamma$ and $\mathcal{G} \rightarrowtail (\mathcal{H})_\gamma$. Both relations are generalizations of the well-known ordinary partition symbol since if $\mathcal{G}$, $\mathcal{H}_\nu$: $\nu < \gamma$ are complete graphs of cardinality α, β_ν: $\nu < \gamma$ then $\mathcal{G} \longrightarrow (\mathcal{H}_\nu)_{\nu<\gamma}$ and $\mathcal{G} \rightarrowtail (\mathcal{H}_\nu)_{\nu<\gamma}$ are both equivalent to $\alpha \to (\beta_\nu)^2_{\nu<\gamma}$.

We will mostly deal with strong embeddings, Henson asked in his paper [2] if the following generalization of Ramsey's theorem was true. For every finite sequence $\{\mathcal{H}_\nu:\ \nu < k\}$, $k < \omega$ of finite graphs there is a finite $\mathcal{G}$ such that $\mathcal{G} \rightarrowtail (\mathcal{H}_\nu)_{\nu<\gamma}$ holds.

This problem has been answered in the affirmative by W. Deuber [3] and J. Nesetril [4] and by us independently. The both investigated the problem in a more general setting. We are going to deal with generalizations of different types. First we are going to consider if the infinite form of Ramsey's theorem generalizes. The answer is negative. We prove

Theorem 1. *Let $\mathcal{H}$ be the infinite complete bipartite graph. i.e. $h = h_0 \cup h_1$, $h_0 \cap h_1 = \phi$, $|h_0| = |h_1| = \omega$, $H = [h_0, h_1]$. Then $\mathcal{G} \not\to (\mathcal{H})_2$ for all countable graphs $\mathcal{G}$.*

Let us say that $\mathcal{G}$ is locally finite if each vertex of $\mathcal{G}$ has finite valency either in $\mathcal{G}$ or in the complement $\overline{\mathcal{G}}$ of $\mathcal{G}$. We are going to prove the following Ramsey type theorem.

Theorem 2. *Let $\mathcal{H}$ be locally finite, $\mathcal{H}, \mathcal{K}$ countable. There is a countable $\mathcal{G}$ such*

$$\mathcal{G} \rightarrowtail (\mathcal{H}, \mathcal{K}) .$$

Obviously this theorem extends to the case when we have finitely many countable locally finite graphs $\mathscr{H}_i$; $i < k$, and one countable graph $\mathscr{K}$. This, of course, gives a proof of the results of [3] and [4] for finite graphs already mentioned.

As to the generalizations for larger graphs we will prove

Theorem 3. *Let* $\langle \mathscr{H}_i : i < k \rangle$ *be a finite sequence of countable graphs. Then there is a graph* $\mathscr{G}$, *with* $|\mathscr{G}| \leqslant 2^\omega$ *such that*

$$\mathscr{G} \rightarrowtail (\mathscr{H}_i)_{i<k} .$$

Surprisingly enough we do not know if this result extends for graphs with larger cardinalities. We state the simplest unsolved case.

Problem. Is it true that for all graphs $\mathscr{H}, \mathscr{K}$ of cardinality ω_1 there is a $\mathscr{G}$ (of reasonable size) such that $\mathscr{G} \rightarrowtail (\mathscr{H}, \mathscr{K})$?

§2. UNIVERSAL GRAPHS. PROOF OF THEOREM 1.

Let $\kappa \geqslant \omega$ be an infinite cardinal. The graph $\mathscr{G} = \langle g, G \rangle$ is said to be *κ-good* if for all $X \subset g$, $|X| < \kappa$ and for all $f\colon X \longrightarrow 2$ the set $\mathscr{G}(X, f) = \{y \in g\colon \forall z \in X(\{y, z\} \in G \text{ iff } f(z) = 1)\}$ has cardinality $\geqslant \kappa$. The graph $\mathscr{G}$ is said to be *κ-universal* if every graph $\mathscr{H}$ of cardinality at most κ is isomorphic to a spanned subgraph of $\mathscr{G}$. The following well-known facts are all we need about this concepts.

2.1. (i) Each κ-good graph is κ-universal.

(ii) Any two κ-good graphs of cardinality κ are isomorphic.

(iii) There exists an ω-good graph $\mathscr{U}$ of cardinality ω and there exists an ω_1-good graph $\mathscr{V}$ of cardinality 2^ω.

For the proof of Theorem 1 it will be convenient to have the following representation of $\mathscr{U}$.

2.2. Let $r_m : m < \omega$ be a one-to-one enumeration of all diadically rational numbers $r \in (0, 1)$; $r_m = 0, r_{m,0}, \ldots, r_{m,n}, \ldots$ and let $U = \{\{n, m\}\colon n > m \wedge r_{m,n} = 1\}$, $u = \omega$. Clearly $\mathscr{U}$ is the countable

ω-good graph.

Proof of Theorem 1. Let $U_0 = \{\{n, m\} \in U: n < m \wedge r_n < r_m\}$ $U_1 = \{\{n, m\}: n < m \wedge r_m < r_n\}$. This is an edge coloring of $\mathscr{U}$. It is sufficient to prove that the $\mathscr{H}$ of Theorem 1 is not isomorphic to a spanned subgraph of $\mathscr{U}_i = \langle \omega, U_i \rangle$ for $i < 2$. By symmetry it is sufficient to prove this for $i = 0$. Assume the statement is false. We may assume that then there are $|A| = |B| = \omega$, $A, B \subset \omega$, $A \cap B = \phi$ such that $[A]^2 \cap U_0 = [B]^2 \cap U_0 = \phi$ and $[A, B] \subset U_0$. We may assume that $n_0 = \min A < \min B$. There is $A' \subset A$ such that $|A'| = \omega$ and $r_{m,n} = r_{m',n}$ for all $n \leqslant n_0$ provided $m, m' \in A'$. Let now $n_0 < n_1 < n_2 < n_3 < n_4$ be such that $n_2, n_4 \in A'$ and $n_1, n_3 \in B$. By $[A, B] \subset U_0$ it follows that $r_{n_i} < r_{n_{i+1}}$ for $i < 4$. By $[A]^2 \cap U_0 = \phi$, $[A, B] \subset U_0$ it now follows that $r_{n_2, n_0} = r_{n_4, n_0} = 0$ and $r_{n_3, n_0} = 1$. Let $m_0 = \min\{m: r_{n_2, m} \neq r_{n_3, m}\}$. Then $m_0 \leqslant n_0$ and by the definition of A', $r_{n_2, m} = r_{n_4, m}$ for $m \leqslant m_0$. Hence either $r_{n_2}, r_{n_4} < r_{n_3}$ or $r_{n_2}, r_{n_4} > r_{n_3}$, a contradiction in both cases.

Note that there is still a gap between the results of Theorems 1, 2. The infinite complete bipartite graph is not the smallest one for which the above argument works, but we do not know a necessary and sufficient condition for $\mathscr{U} \succ\!\!\!\not\to (\mathscr{H})^2_2$ in the countable case.

§3. κ-GOOD GRAPHS

Let $\kappa \geqslant \omega$ be regular and let $\mathscr{G} = \langle g, G \rangle$ be a κ-good graph. We are going to associate with $\mathscr{G}$ a κ-complete proper ideal $I_{\mathscr{G}} = I$ as follows.

3.1. Let $A \subset g$. $A \in I$ iff there exists a sequence $X_\xi \in [g]^{<\kappa}$ of pairwise disjoint subsets of g and a sequence $f_\xi: X_\xi \to 2$: $\xi < \kappa$ of functions such that

$$|A \cap \mathscr{G}(X_\xi, f_\xi)| < \kappa \qquad \text{for} \qquad \xi < \kappa .$$

Obviously $B \subset A \in I$ implies $B \in I$. By the definition of a κ-good graph, $g \notin I$ and thus I is proper. We now prove that I is κ-complete.

3.2. Assume $A_\nu \in I$ for $\nu < \varphi < \kappa$. Let $A = \bigcup_{\nu<\varphi} A_\nu$. Then $A \in I$.

Proof. Let X^ν_ξ, f^ν_ξ: $\xi < \kappa$ be the sequences satisfying 3.1 for $\nu < \varphi$. By an easy transfinite induction one can pick sets $X^\nu_{\xi(\rho,\nu)}$ for $\rho < \kappa$, $\nu < \varphi$ in such a way that they are pairwise disjoint. Put $X_\rho = \bigcup_{\nu<\varphi} X^\nu_{\xi(\rho,\nu)}$, $f_\rho = \bigcup_{\nu<\varphi} f^\nu_{\xi(\rho,\nu)}$ for $\rho < \kappa$. Then $A \cap \mathscr{G}(X_\rho, f_\rho) \subset \bigcup_{\nu<\varphi} A_\nu \cap \mathscr{G}(X^\nu_{\xi(\rho,\nu)}, f^\nu_{\xi(\rho,\nu)})$ hence, by the regularity of κ 3.1 holds for X_ρ, f_ρ: $\rho < \kappa, A$.

Finally we need the following strengthening of 3.1.

3.3. Let $A \subset g$, and assume that there exist sequences $X_\xi \in [g]^{<\kappa}$ (disjoint), $f_\xi: X_\xi \to 2$: $\xi < \kappa$ such that

$$A \cap \mathscr{G}(X_\xi, f_\xi) \in I \qquad \text{for} \qquad \xi < \kappa .$$

Then $A \in I$.

Proof. Let $Y(\xi, \rho) \in [g]^{<\kappa}$ (disjoint), $f(\xi, \rho) \in {}^{Y(\xi,\rho)}2$; $\rho < \kappa$ be such that

$$|A \cap \mathscr{G}(X_\xi, f_\xi) \cap \mathscr{G}(Y(\xi, \rho), f(\xi, \rho))| < \kappa$$

for $\rho < \kappa$, $\xi < \kappa$.

By an easy transfinite induction one can pick Y_ξ, g_ξ: $\xi \in K$ for some $|K| = \kappa$ such that

$$Y_\xi = Y(\xi, \rho), \qquad g_\xi = f(\xi, \rho)$$

for some ρ and X_ξ, Y_ξ: $\xi \in \kappa$ are pairwise disjoint.

Let $Z_\xi = X_\xi \cup Y_\xi$, $h_\xi = f_\xi \cup g_\xi$ for $\xi \in K$.

Then $|A \cap \mathscr{G}(Z_\xi, h_\xi)| < \kappa$ for $\xi \in K$.

§4. PROOF OF THE POSITIVE RELATIONS

Let now $\mathscr{G}$ be a graph and let $G = G_0 \cup G_1$ be an edge coloring of it. Let further $X \in [g]^{<\kappa}$, $f: X \to 2$.

We denote by $\mathscr{G}^i(X, f)$ the set

$$\{y \in g\colon \forall z \in X(f(z) = 1 \Rightarrow \{y, z\} \in G_i \wedge f(z) = 0 \Rightarrow \{y, z\} \notin G)\}$$

for $i < 2$.

Note that this is not the same as $\mathscr{G}_i(X, f)$ for the subgraph $\mathscr{G}_i = \langle g, G_i \rangle$. We will use the following notation for $u \in g$

$$\mathscr{G}(u, i) = \{v \in g\colon \{u, v\} \in G_i\} \qquad \text{for} \qquad i < 2$$

$$\mathscr{G}(u) = \{v \in g\colon \{u, v\} \in G\}$$

$$\bar{\mathscr{G}}(u) = \{v \in g\colon \{u, v\} \notin G\}\ .$$

Obviously $\mathscr{G}(u, 0) \cup \mathscr{G}(u, 1) = \mathscr{G}(u)$. We prove the following

Lemma. *Let $\kappa \geqslant \omega$ be regular. Assume $\mathscr{G} = \langle g, G \rangle$ is κ-good, $G = \bigcup_{i<2} G_i$ is an edge coloring of it, $A \subset g$, $A \notin I$. Let further $\mathscr{K} = \langle \omega, K \rangle$ be a countable graph. Then one of the following conditions holds*

(i) *There are $B, C \subset A$; $B \cap C = \phi$; $B, C \notin I$ such that $\mathscr{G}(u, 1) \cap C \in I$ for all $u \in B$.*

(ii) *There is $D \subset A$ such that $\mathscr{G}(D) = \mathscr{G}_1(D)$ is isomorphic to $\mathscr{K}$.*

Proof. We assume (i) is false. We define a one-to-one sequence $d_n\colon n < \omega$ of elements of A, by induction on n. Put first f_n for the function satisfying $D(f_n) = n$, $R(f_n) \subset 2$ and $f_n(m) = 1$ iff $\{m, n\} \in K$ for $m < n$. Assume $d_m \in A$ is defined for $m < n$ in such a way that for $D_n = \{d_m : m < n\}$, $\mathscr{G}(D_n) = \mathscr{G}_1(D_n)$ the mapping $m \longrightarrow d_m$ is an isomorphism of $\mathscr{K}(n)$ onto $\mathscr{G}_1(D_n)$ and for all $f \in {}^n2$

$$\mathscr{G}^1(D_n, f) \cap A \notin I\ .$$

We claim that there is an element u of $\mathscr{G}^1(D_n, f_n) \cap A$ satisfying the following condition

(1) For all $f \in {}^n2$

$$\mathscr{G}(u, 1) \cap \mathscr{G}^1(D_n, f) \cap A \notin I \qquad \text{and}$$

$$\bar{\mathscr{G}}(u) \cap \mathscr{G}^1(D_n, f) \cap A \notin I .$$

In fact the second requirement holds for all but fewer than κ elements of $\mathscr{G}_1(D_n, f_n)$ hence it holds for $u \in B'$ for some $B' \subset \subset \mathscr{G}^1(D_n, f_n) \cap A$, $B' \notin I$. If the first requirement fails for all $u \in B'$ for some $f \in {}^n 2$, then there are a subset $B \subset B'$, $B \notin I$ and an $f \in {}^n 2$ such that for $C = \mathscr{G}^1(D_n, f) \cap A$, $C \notin I$ and $\mathscr{G}(u, 1) \cap C \in I$ for all $u \in B$. This contradicts our assumption that (i) is false. It follows that (1) holds. Let $d_n \in \mathscr{G}^1(D_n, f_n) \cap A$ satisfy (1).

Considering that for $f \in {}^{n+1} 2$

$$\mathscr{G}^1(D_{n+1}, f) = \mathscr{G}^1(D_n, f \mid n) \cap \mathscr{G}(d_n, 1) \qquad \text{if} \qquad f(n) = 1$$

and

$$\mathscr{G}^1(D_{n+1}, f) = \mathscr{G}^1(D_n, f \mid n) \cap \bar{\mathscr{G}}(d_n) \qquad \text{if} \qquad f(n) = 0$$

the new d_n satisfies all the necessary requirements. Put $D = \{d_n : n < \omega\}$. Then $\mathscr{G}(D) = \mathscr{G}_1(D)$ and $n \longrightarrow d_n$ is an isomorphism of $\mathscr{K}$ onto $\mathscr{G}_1(D)$. This proves the Lemma.

Proof of Theorem 2. Let $\mathscr{H}$, $\mathscr{K}$ be countable graphs. Assume $\mathscr{H}$ is locally finite. Let $\mathscr{G}$ be the countable ω-good graph and let $G = G_0 \cup G_1$ be an arbitrary edge coloring of it. If there exists $A \subset g$ such that $\mathscr{G}(A) = \mathscr{G}_1(A)$ is isomorphic to $\mathscr{K}$ we are done. Hence by the Lemma, we may assume that for all $A \notin I$ (i) of the Lemma holds.

Let $\mathscr{H} = \langle \omega, H \rangle$. Since $\mathscr{H}$ is locally finite there exist functions $\varphi \in {}^\omega \omega$, $\psi \in {}^\omega 2$ such that φ is strictly increasing $\varphi(0) > 1$ and

$$\forall m \geqslant \varphi(n)(\{n, m\} \in H \qquad \text{iff} \qquad \psi(n) = 1) .$$

We are to define a strong embedding of $\mathscr{H}$ into the color 0. We define a function ψ_k on the interval $[k+1, \varphi(k)]$ by the stipulation $\psi_k(j) = 0$ for $\{k, j\} \notin H$ and $\psi_k(j) = 1$ for $\{k, j\} \in H$. Note that $\psi_k(\varphi(k)) = \psi(k)$.

We now define sequences $A_n, B_{n,m}$ of subsets of g by induction on n

as follows. Put $A_0 = g \notin I$. Assume that $A_n, B_{i,j}$ has already been defined for some $n < \omega$, $i < n$, $j < n - 1$, in such a way that A_n, $B_{i,j} \notin I$; the sets $B_{i,0}, A_n$ are disjoint, $B_{i,0} \supset B_{i,j}$; $\mathcal{G}(u, 1) \cap B_{l,0} \in I$ for $u \in B_{i,0}$, $i < l < n$ and $\mathcal{G}(u, 1) \cap A_n \in I$ for $u \in B_{i,0}$, $i < n$. We will define $B_{i,n-i}$ for $i \leqslant n$ and A_{n+1}. First we choose two sets $B', C' \subset A_n$; $B', C' \notin I$, $B' \cap C' = \phi$ in such a way that $\mathcal{G}(u, 1) \cap C' \in I$ for all $u \in B'$. We now distinguish two cases (i) $n + 1 \notin R(\varphi)$, (ii) $n + 1 \in R(\varphi)$.

Case (i). Put $B_{n,0} = B'$, $A_{n+1} = C'$, $B_{j,n-j} = B_{j,n-j-1}$ for $j < n$.

Case (ii). Let $k = \varphi^{-1}(n + 1)$. Put $B_{j,n-j} = B_{j,n-1}$ for $j \leqslant k < n$ (since $\varphi(k) \geqslant k + 2$ by the assumption). We now claim that there is a $u \in B_{k,n-k}$ such that

$$(2) \quad \mathcal{G}(u, 0) \cap B_{j,n-j-1} \notin I, \ \bar{\mathcal{G}}(u) \cap B_{j,n-j-1} \notin I \ \text{ for } \ k < j < n$$

$$\mathcal{G}(u, 0) \cap B' \notin I\,, \qquad \bar{\mathcal{G}}(u) \cap B' \notin I\,,$$

$$\mathcal{G}(u, 0) \cap C' \notin I\,, \qquad \bar{\mathcal{G}}(u) \cap C' \notin I\,.$$

By 3.3 and by the assumption $B_{j,n-j-1} \notin I$, $B', C' \notin I$ these requirements hold for all but finitely many elements of u of $B_{k,n-k} \notin I$ provided $\mathcal{G}(u, 0)$ is replaced by $\mathcal{G}(u)$. Considering that by the inductive assumption $\mathcal{G}(u, 1) \cap B_{j,n-j-1} \in I$ for $k < j < n$ and $\mathcal{G}(u, 1) \cap B' \in I$, $\mathcal{G}(u, 1) \cap C' \in I$ it now follows that (2) holds for all but finitely many elements of $B_{k,n-k}$. Let a_k be an element of $B_{k,n-k}$ satisfying (2). Let now $k < j < n$. Put

$$B_{j,n-j} = B_{j,n-j-1} \cap \bar{\mathcal{G}}(a_k) \qquad \text{if} \qquad \psi_k(j) = 0\,,$$

$$B_{j,n-j} = B_{j,n-j-1} \cap \mathcal{G}(a_k, 0) \ \ \text{if} \qquad \psi_k(j) = 1\,.$$

Put

$$B_{n,0} = B' \cap \bar{\mathcal{G}}(a_k) \qquad \text{if} \qquad \psi_k(n) = 0$$

$$B_{n,0} = B' \cap \mathcal{G}(a_{k,0}) \qquad \text{if} \qquad \psi_k(n) = 1$$

$$A_{n+1} = C' \cap \bar{\mathcal{G}}(a_k) \qquad \text{if} \qquad \psi_k(n + 1) = \psi(k) = 0$$

$$A_{n+1} = C' \cap \mathcal{G}(a_{k,0}) \qquad \text{if} \qquad \psi_k(n + 1) = \psi(k) = 1\,.$$

In both cases A_{n+1}, $B_{j,n-j} \notin I$ for $j \leqslant n$, $A_{n+1}, B_{n,0}$ are disjoint and do not meet any of the $B_{j,0}$ for $j < n$. Finally by the choice of B', C' we have $\mathscr{G}(u, 1) \cap A_{n+1} \in I$ for $u \in B_{n,0}$. Thus the sets $A_n, B_{n,m}$ are defined for all n, m and a_k is defined for all $k < \omega$ since for all k there is $n + 1 \in \omega$ with $\varphi(k) = n + 1$. The elements a_k are all different. Put $A = \{a_k : k < \omega\}$. We claim that $\mathscr{G}(A) = \mathscr{G}_0(A)$ and that the mapping $k \to a_k$ is an isomorphism of $\mathscr{H}$ onto $\mathscr{G}(A)$. Let $k < s$ by given. Assume first $n + 1 = \varphi(k) \leqslant s$. Then by the construction $a_s \in B_{s,j}$ for some j, $B_{s,j} \subset B_{s,0} \subset A_{n+1}$

$$A_{n+1} \subset \bar{\mathscr{G}}(a_k) \qquad \text{if} \qquad \psi(k) = 0 \,,$$

$$A_{n+1} \subset \mathscr{G}(a_{k,0}) \qquad \text{if} \qquad \psi(k) = 1 \,.$$

Hence

$$\{a_k, a_s\} \notin G \qquad \text{if} \qquad \{k, s\} \notin H \,,$$

$$\{a_k, a_s\} \in G_0 \qquad \text{if} \qquad \{k, s\} \in H \,.$$

Assume now that $k < s < n + 1 = \varphi(k)$. Then $\varphi(k) < \varphi(s)$, $a_s \in A_{s,m-s}$ for $m = \varphi(s) - 1$, hence; $A_{s,m-s} \subset A_{s,n-s}$, $A_{s,m-s} \subset \bar{\mathscr{G}}(a_k)$ if $\psi_k(s) = 0$, $A_{s,n-s} \subset \mathscr{G}(a_{k,0})$ if $\psi_k(s) = 1$. It follows again that $\{a_k, a_s\} \notin G$ if $\{k, s\} \in H$ and $\{a_k, a_s\} \in G_0$ if $\{k, s\} \in H$. This proves Theorem 2.

Proof of Theorem 3. Let $\mathscr{G}$ be a graph, $|\mathscr{G}| = 2^\omega$, which is ω_1-good. Let $\mathscr{H}_i = \langle \omega, H_i \rangle$: $i < k < \omega$ be a countable graphs and let $G = \bigcup_{i<k} G_i$ be an edge coloring of $\mathscr{G}$ by k colors. Let I be the ω_1-complete ideal corresponding to $\mathscr{G}$. Let $C, D \notin I$, $C \cap D = \phi$. We say that C, D is a good pair for the color i if for all $C' \subset C$, $D' \subset D$; $C', D' \notin I$ there is $u \in C'$ such that $\mathscr{G}(u, i) \cap D' \notin I$. First we claim

(3) $\exists i \, \exists A \notin I \, \forall B (B \subset A \wedge B \in I \Rightarrow \exists C, D (C, D \subset B$ and C, D is a good pair for the color i)).

To see this assume (3) is false. Suppose that for all $A \notin I$ and $i < k$ there is a B for which the requirement is false. It follows that there is a sequence $B_0 \supset \ldots \supset B_{k-1}$ in such a way that the requirement is false for B_i with i, and $B_0, \ldots, B_{k-1} \notin I$. It results that no pair $C, D \notin I$, $C \cap D \neq \phi$; $C, D \subset B_{k-1}$ is good for $i < k$. Let now $C, D \subset B_{k-1}$, $C \cap D = \phi$, $C, D \notin I$. By induction we can pick two sequences $C \supset C_0 \supset \ldots \supset C_{k-1}$ and $D \supset D_0 \supset \ldots \supset D_{k-1}$ such that $C_i, D_i \notin I$, $\mathscr{G}(u, i) \cap D_i \in I$ for $u \in C_i$, $i < k$. Then $C_{k-1}, D_{k-1} \notin I$ and $\mathscr{G}(u, i) \cap D_{k-1} \in I$ for all $u \in C_{k-1}$, $i < k$. Considering that $\mathscr{G}(u) = \bigcup_{i<k} \mathscr{G}(u, i)$ for all u, and that $\mathscr{G}(u) \cap D_{k-1} \notin I$ for all but fewer than ω_1 elements of g, this is a contradiction.

Let i and A satisfy (3). We define sequences A_n, B_n: $n < \omega$ by induction on n as follows. Put $A_0 = A$. Assume A_κ and $B_i \notin I$ are already defined for $i < n$ so that $A_n \subset A$. By (3), then there is a good pair for i, $B_n, A_{n+1} \notin I$, $B_n \subset A_n$, $A_{n+1} \subset A_n \subset A$, $B_n \cap A_{n+1} = \phi$. This defines the sequences A_n, B_n: $n < \omega$. Obviously the A_n are decreasing the B_n are pairwise disjoint and $\bigcup_{m \geqslant n} B_m \subset A_n$.

We now define a sequence $b_n \in B_n$ and a sequence $B_{n,m}$ for $m \leqslant n$ for $n < \omega$ by induction on n as follows. Put $B_{k,0} = B_k$ for $k < \omega$. Assume $n < \omega$; b_m, $m < n$ and $B_{k,0} \supset \ldots \supset B_{k,n}$ is already defined for all k, and $B_{k,n} \notin I$.

We claim that there is a $u \in B_{n,n}$ such that

$$(4) \qquad \begin{matrix} \mathscr{G}(u, i) \cap B_{k,n} \notin I \\ \bar{\mathscr{G}}(u) \cap B_{k,n} \notin I \end{matrix} \qquad \text{for all } n < k < \omega .$$

The second requirement holds for all but countably many elements of $B_{n,n}$ hence for all $u \in B'$ for some $B' \subset B_{n,n} \subset B_n$, $B' \notin I$. Assume $\mathscr{G}(u, i) \cap B_{n,k} \notin I$ fails for some k, $n < k < \omega$ for all $u \in B'$. Let then $B'_k = \{u \in B' : \mathscr{G}(u, i) \cap B_{n,k} \in I\}$. There is k, $n < k < \omega$ such that $B'_k \notin I$. Considering that $B_{n,k} \notin I$, $B'_k \subset B_n$, $B_{n,k} \subset A_{n+1}$ this contradicts the fact that the pair B_n, A_{n+1} is good for i. Let now b_n be an element of $B_{n,n}$ satisfying (4). Put $B_{k,n+1} = B_{k,n} \cap \mathscr{G}(b_n, i)$ if $\{n, k\} \in H_i$ and $B_{k,n+1} = B_{k,n} \cap \bar{\mathscr{G}}(b_n)$ if $\{n, k\} \notin H$. Then $B_{k,n+1} \notin I$