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A NOTE ON REGULAR METHODS OF SUMMABILITY AND THE BANACH-SAKS PROPERTY

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ABSTRACT. Using the Galvin-Prikry partition theorem from set theory it is proved that every bounded sequence in a Banach space has a subsequence such that either every subsequence of which is summable *or* no subsequence of which is summable.

The infinite matrix $\{a_{ij}\}_{i\in\omega,j\in\omega}$ (ω is the set of natural numbers) is called a regular method of summability if given a sequence $\langle e_i\rangle_{i\in\omega}$ of elements of a Banach space B, converging in norm to e, then the sequence $e'_i = \sum_{j=0}^{\infty} a_{ij}e_j$ converges also to e. The sequence $\langle e_i\rangle_{i\in\omega}$ is called summable with respect to $\langle a_{ij}\rangle_{i\in\omega,j\in\omega}$ if $e'_i = \sum_{j=0}^{\infty} a_{ij}e_j$ converges in norm. (See [2, p. 75] for reference.) It is well known [2] that $\langle a_{ij}\rangle_{i\in\omega,j\in\omega}$ is a regular method of summability if and only if

(a) l.u.b. $\sum_{j=0}^{\infty} |a_{ij}| < M < \infty$,
(b) $\lim_{i\to\infty} a_{ij} = 0$ for every j,
(c) $\lim_{i\to\infty} \sum_{j=0}^{\infty} a_{ij} = 1$.

In this note we prove:

THEOREM. *Let $\langle e_i\rangle_{i\in\omega}$ be a bounded sequence of elements in a Banach space B, and $\langle a_{ij}\rangle_{i\in\omega,j\in\omega}$ a regular method of summability; then there exists a subsequence of $\langle e_i\rangle_{i\in\omega}$, $\langle e_{i_k}\rangle_{k\in\omega}$ such that*:

(a) *every subsequence of $\langle e_{i_k}\rangle_{k\in\omega}$ is summable with respect to $\langle a_{ij}\rangle_{i\in\omega,j\in\omega}$, each being summed to the same limit*; *or*

(b) *no subsequence of $\langle e_{i_k}\rangle_{k\in\omega}$ is summable with respect to $\langle a_{ij}\rangle_{i\in\omega,j\in\omega}$.*

PROOF. Let $P(\omega)$ be the set of all infinite subsets of ω. There exists a natural topology on $P(\omega)$ generated by the subbasis $\{A_n\}_{n\in\omega} \cup \{B_n\}_{n\in\omega}$ where

$$A_n = \{p \mid p \in P(\omega), n \in p\},\quad B_n = \{p \mid p \in P(\omega), n \notin p\}.$$

Define a partition of $P(\omega)$ into two Borel sets:

$$A = \{p \mid \langle e_i\rangle_{i\in p} \text{ is summable w.r.t. } \langle a_{ij}\rangle_{i\in\omega,j\in\omega}\},$$

$$B = P(\omega) - A$$

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($\langle e_i \rangle_{i \in p}$ is the subsequence of $\langle e_i \rangle_{i \in \omega}$ obtained by enumerating e_i for $i \in p$ in the natural order of the i's).

We prove that A is a Borel subset of $P(\omega)$. Let

$$B_{\varepsilon,m,n} = \left\{ p \middle| \left\| \sum_{j=0}^{\infty} a_{nj} \cdot e_{k_j} - \sum_{j=0}^{\infty} a_{mj} e_{k_j} \right\| < \varepsilon \right.$$

$$\left. \text{where } k_j \text{ is a monotone enumeration of } p \right\}.$$

$B_{\varepsilon,m,n}$ is open in our topology on $P(\omega)$, because if $p \in B_{\varepsilon,m,n}$, pick ε' such that

$$\left\| \sum a_{mj} e_{k_j} - \sum_{j=0}^{\infty} a_{nj} e_{k_j} \right\| < \varepsilon' < \varepsilon.$$

Let J be large enough such that

$$T\left(\sum_{j=J}^{\infty} |a_{mj}| + \sum_{j=J}^{\infty} |a_{nj}| \right) < \frac{\varepsilon - \varepsilon'}{2}$$

where T is a bound for $\|e_i\|$. (J exists because $\langle a_{ij} \rangle_{i \in \omega, j \in \omega}$ is a regular method of summability.)

The set $C = \{q | q \in P(\omega), q \cap \{l | l < J\} = p \cap \{l | l < J\}\}$ is an open subset of $P(\omega)$. $p \in C$ and $C \subseteq B_{\varepsilon,m,n}$. This last inclusion is true since if $q \in C$ and l_j is a monotone enumeration of q, then $l_j = k_j$ for $j < J$. Hence,

$$\begin{aligned}
&\left\| \sum_{j=0}^{\infty} a_{mj} e_{l_j} - \sum_{j=0}^{\infty} a_{nj} e_{l_j} \right\| \\
&\quad \leqslant \left\| \sum_{j=0}^{J-1} a_{mj} e_{l_j} - \sum_{j=0}^{J-1} a_{nj} e_{l_j} \right\| + \left\| \sum_{j=J}^{\infty} a_{mj} e_{l_j} - \sum_{j=J}^{\infty} a_{nj} e_{l_j} \right\| \\
&\quad \leqslant \left\| \sum_{j=0}^{J-1} a_{mj} e_{k_j} - \sum_{j=0}^{J-1} a_{nj} e_{k_j} \right\| + T\left(\sum_{j=J}^{\infty} |a_{mj}| + \sum_{j=J}^{\infty} |a_{nj}| \right) \\
&\quad < \left\| \sum_{j=0}^{\infty} a_{mj} e_{k_j} - \sum_{j=0}^{\infty} a_{nj} e_{k_j} \right\| + \left\| \sum_{j=J}^{\infty} a_{mj} e_{k_j} - \sum_{j=J}^{\infty} a_{nj} e_{k_j} \right\| + \frac{\varepsilon - \varepsilon'}{2} \\
&\quad < \varepsilon' + T \cdot \left(\sum_{j=J}^{\infty} |a_{mj}| + \sum_{j=J}^{\infty} |a_{nj}| \right) + \frac{\varepsilon - \varepsilon'}{2} \\
&\quad < \varepsilon' + (\varepsilon - \varepsilon')/2 + (\varepsilon - \varepsilon')/2 = \varepsilon.
\end{aligned}$$

Thus every element of $B_{\varepsilon,m,n}$ has an open neighborhood included in $B_{\varepsilon,m,n}$. Hence $B_{\varepsilon,m,n}$ is open.

The set A is $\bigcap_k \bigcup_N \bigcap_{m,n \geqslant N} B_{1/k,m,n}$. ($A$ is the set of those p such that $\sum_{j=0}^{\infty} a_{ij} e_{k_j}$ is a Cauchy sequence if k_j is a monotone enumeration of p.) By a theorem of F. Galvin and K. Prikry [3] there is $q \in P(\omega)$ such that either

(I) for every $t \subseteq q$, $t \in P(\omega) \Rightarrow t \in A$, or

(II) for every $t \subseteq q$, $t \in P(\omega) \Rightarrow t \in B$.

For the sequence $\langle e_i \rangle_{i \in q}$ either (b) holds (in case (II)) or in case (I) we shall indicate how to pick a subsequence of it for which (a) holds. If we assume that (I) holds, then every subsequence of $\langle e_i \rangle_{i \in q}$ is summable to a limit which lies in the subspace spanned by $\langle e_i \rangle_{i \in q}$. Call it B', which is of course separable. For every $n \in \omega$, $n \neq 0$, let $\{A^n_m | m \in \omega\}$ be a family of open balls of radius $1/n$ covering B'. By induction we get a sequence $\cdots \subseteq q^1_3 \subseteq q^1_2 \subseteq q^1_1 \subseteq q$ such that either (A) every subsequence of $\langle e_i \rangle_{i \in q^1_k}$ is summable to a limit in A^1_k or (B) every subsequence of $\langle e_i \rangle_{i \in q^1_k}$ is summable to a limit which is outside A^1_k. (We can get the q^1_{k+1} from q^1_k by again using the Galvin-Prikry result, noting as before that the partition of $P_\omega(q_k)$ is Borel.) Clearly for some k_1 we get (A) to hold. Let q^1_∞ be elements of the diagonal sequence of the natural enumerations of q^1_k. Now get $\cdots \subseteq q^2_2 \subseteq q^2_1 \subseteq q^1_\infty$ such that either (A): every subsequence of $\langle e_i \rangle_{i \in q^2_k}$ is summable to a limit in A^2_k or (B): every subsequence of $\langle e_i \rangle_{i \in q^2_k}$ is summable to a limit outside A^2_k. Again we get k_2 for which (A) holds. q^2_∞, q^3_∞, etc., and $k_1, k_2, k_3, \ldots$ are defined as before. Let t be the set of elements of the diagonal sequence of the sequence generated by the q^n_∞. Every subsequence of $\langle e_i \rangle_{i \in t}$ is summable to a limit which is in $A^n_{k_n}$, for every n hence to a limit in $A^n_{k_n}$ which contains at most one point. Hence the sequence $\langle e_i \rangle_{i \in t}$ satisfies (a).

REMARKS. (1) By using the theorem countably many times (using the fact that finitely many changes in a sequence do not influence its summability), we can get the conclusion to hold simultaneously for a countable sequence of regular summability methods such that the limit for those of them for which (I) holds is the same.

(2) A Banach space is said to have the Banach-Saks property with respect to the regular method of summability $\langle a_{ij} \rangle_{i \in \omega}$ if every bounded sequence has a summable subsequence. (See [1]. The problem solved by this note is due to Louis Sucheston.) As a corollary to the theorem we get: If B has the Banach-Saks property with respect to the regular method of summability $\langle a_{ij} \rangle_{i \in \omega, j \in \omega}$, then every bounded sequence has a subsequence such that each of its subsequences is summable with respect to $\langle a_{ij} \rangle_{i \in \omega, j \in \omega}$.

REFERENCES

1. Antoine Brumel et Louis Sucheston, *Sur quelques conditions equivalentes à la super-reflexivité dans les espaces de Banach*, C.R. Acad. Sci. Paris **275** (1972), 993–994.

2. Nelson Dunford and Jacob T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

3. Fred Galvin and Karel Prikry, *Borel sets and Ramsey's theorem*, J. Symbolic Logic **38** (1973), 193–198. MR **49** #2399.

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