P. ERDÖS, D. J. KLEITMAN and B. L. ROTHSCHILD

ASYMPTOTIC ENUMERATION OF K_n-FREE GRAPHS

RIASSUNTO. — In questo lavoro si calcola il numero di grafi con n vertici che non contengono alcun K_m (grafo completo con m vertici). Fissato m, si ottiene un valore asintotico, per n che tende all'infinito, del logaritmo del numero di tali grafi. Una vasta classe di tali grafi può ottenersi spartendo i vertici in $m-1$ classi approssimativamente eguali e collegando poi due vertici se essi sono in classi diverse. Il logaritmo del numero di grafi di questo tipo tende asintoticamente al logaritmo del numero di tutti i grafi privi di K_m.

Nel caso in cui $m = 3$ (grafi privi di triangoli) si ottiene un risultato più forte: si ha infatti che il numero di grafi con n vertici e privi di triangoli è asintoticamente eguale al numero dei grafi bipartiti con n vertici.

We investigate in this paper the question of how many graphs there are containing no complete m-gon (K_m) as a subgraph; that is, no subset of m vertices with every pair joined by an edge of the graph. All graphs considered here are undirected, without loops or multiple edges, and with labelled vertices. We answer the question asymptotically for the logarithm (Corollary to Theorem 1) for $m \geq 3$, and asymptotically (Corollary to Theorem 2) for $m = 3$.

For the $m = 3$ case, a slight modification of the method used in [2] for counting asymptotically the number of partially ordered sets on n elements can be used. The method used here is basically the same, but divides the graphs into cases in different ways. The idea is to divide the graphs into several subclasses, all but one of which are asymptotically negligible. That one is the class of bipartite graphs. Thus these graphs are "almost all" bipartite.

THEOREM 1. *For every integer $k \geq 2$ and every real number $\varepsilon > 0$, there are numbers $0 \leq f_k(\varepsilon)$ and $n(k, \varepsilon)$ such that the number of graphs with n vertices, $n \geq n(k, \varepsilon)$ and at most εn^k subgraphs of type K_k, is at most $2^{(n^2/2)((1-1/(k-1))+f_k(\varepsilon))}$, where*

$$f_k(\varepsilon) \to 0 \qquad \text{as} \quad \varepsilon \to 0 .$$

COROLLARY. *Let $G_k(n)$ be the number of graphs with n vertices and with no subgraph of type K_k. Then*

$$\log_2 (G_k(n)) = \frac{n^2}{2}\left(1 - \frac{1}{k-1}\right) + o(n^2) .$$

Note: All logarithms are base 2 in this paper.

Proof of Corollary. That $n^2/2\,(1 - 1/(k-1)) + o(n^2)$ is an upper bound follows from Theorem 1 by letting $\varepsilon \to 0$ as $n \to \infty$. That it is a lower bound can be seen as follows: Divide the set of n vertices into $k - 1$ subsets as equally as possible ($[n/(k-1)]$ or $[n/(k-1)] + 1$ in each) consider all graphs with no edge joining two vertices of the same subset. There are at most $2^{(n/k-1)^2(k-1)((k-2)/2)}$ of these, or $2^{(n^2/2)(1-(1/(k-1)))}$. This construction completes the proof.

Proof of Theorem 1. We use induction on k, starting with $k = 2$. In this, case, given $\varepsilon > 0$, we get $\Sigma_{i=0}^{[\varepsilon n^2]} \left(\binom{\binom{n}{2}}{i} \right)$ graphs. For ε sufficiently small, say $\varepsilon \leq \varepsilon_0$, and n sufficiently large, depending on ε, say $n > n(\varepsilon)$, we get

$$\sum_{i=0}^{[\varepsilon n^2]} \left(\binom{\binom{n}{2}}{i} \right) \leq n^2 \binom{[n^2/2]}{[2\varepsilon n^2/2]}$$

$$< 2^{-(2\varepsilon \log \varepsilon + (1-2\varepsilon)\log(1-2\varepsilon))(n^2/2) + 2\log n}$$

$$< 2^{-2\varepsilon(\log \varepsilon) n^2}.$$

Thus for $\varepsilon > \varepsilon_0$ we let $n(2, \varepsilon) = 1, f_2(\varepsilon) = 1$, and for $\varepsilon \leq \varepsilon_0$ we let $n(2, \varepsilon) = n(\varepsilon)$, and $f_2(\varepsilon) = -2\varepsilon \log \varepsilon$. This completes the $k = 2$ case.

Next we assume that the theorem holds for $k - 1$ and consider graphs on n vertices with at most εn^k subgraphs of type K_k. We consider two subclasses:

$A(n, \varepsilon)$: Graphs with fewer than cn^{k-1} subgraphs of type K_{k-1}, where c is such that $f_{k-1}(c') \leq 1/(2(k-1)(k-2))$ for all $c' \leq c$.

$B(n, \varepsilon)$: Graphs with a subgraph H of type $R^{[k-1]}$ (see definition below) such that at most αn vertices from the remaining $n - (k-1)R$ vertices are connected to some vertex in each of the $k - 1$ parts of H. Here we take $R = 2\varepsilon^{-1/k}$, $\alpha = 4kR^{k-1}c^{-1}\varepsilon$, and we assume ε is small enough so that $\alpha < 1$ say $\varepsilon < \varepsilon_0$. By a graph of type $R^{[l]}$ we mean a graph consisting of l disjoint sets of R vertices each (called the " parts " of $R^{[l]}$) and edges between every two vertices in distinct parts, and only these edges. Such a graph is called a complete l-partite graph.

Let $G(n, \varepsilon)$ be the class of all graphs with n vertices and at most εn^k subgraphs of type K_k.

LEMMA. $G(n, \varepsilon) = A(n, \varepsilon) \cup B(n, \varepsilon)$, *for n sufficiently large, say* $n \geq n(\varepsilon)$.

Proof. Consider a graph G with n vertices and at most $\varepsilon n^k K_k$, and suppose $G \notin A(n, \varepsilon) \cup B(n, \varepsilon)$. It therefore contains at least $cn^{k-1} K_{k-1}$ subgraphs. By a theorem of Erdös [1], for n large enough, depending on R and c (and thus only on ε), we can find a subgraph H in G of type $R^{[k-1]}$.

Now since G is not in $B(n, \varepsilon)$, there must be at least αn vertices connected to at least one vertex in each of the $k - 1$ parts of H. Thus some set S_1 of $k - 1$ vertices, one from each part, is common to at least $\alpha n / R^{k-1}$ subgraphs of type K_k.

Consider the family of sets of $k-1$ vertices of G forming subgraphs of type K_{k-1}, but excluding S_1. There are at least $cn^{k-1}-1$ of these. Thus again by the theorem of Erdös [1], there will be a subgraph H′ of type $R^{[k-1]}$, where not all $k-1$ parts of H′ contain vertices of S_1. Again, since G is not in $B(n, \varepsilon)$, there will be a set S_2 of $k-1$ vertices, one from each part of H′, such that S_2 is common to at least $\alpha n/R^{k-1}$ subgraphs of type K_k.

We repeat this argument $[(c/2)\,n^{k-1}]$ times, each time eliminating one set of $k-1$ vertices, and always leaving at least $(c/2)\,n^{k-1}$ other K_{k-1}. That this process can be continued is a consequence of the theorem of Erdös [1], which guarantees it for n large enough, depending on c. But repeating the argument $[(c/2)\,n^{k-1}]$ times guarantees the existence of at least $[(c/2)\,n^{k-1}]\ S_i$ and thus at least $\frac{1}{k}\,(\alpha n/R^{k-1})\,(c/2)\,n^{k-1}\ K_k$ (to account for the possibility that a given K_k may contain up to k of the S_i). However, this exceeds εn^k, a contradiction. This proves the lemma.

We next obtain bounds for $A_n = |A(n, \varepsilon)|$ and $B_n = |B(n, \varepsilon)|$. Let $G_n = |G(n, \varepsilon)|$. We already saw that $\log G_n \geq (1 - 1/(k-1))\,\frac{n^2}{2}$ (in the proof of the Corollary). By induction on k we have, for n large enough,

$$\log A_n \leq \left((1 - 1/(k-2)) + f_{k-1}(c)\right)\frac{n^2}{2}$$
$$\leq \left((1 - 1/(k-1)) - \frac{1}{2\,(k-1)\,(k-2)}\right)\frac{n^2}{2}.$$

Thus

$$\log\left(\frac{A_n}{G_n}\right) \leq -\left(\frac{1}{2}\,\frac{1}{(k-1)\,(k-2)}\right)\frac{n^2}{2}. \tag{1}$$

Next we consider $B(n, \varepsilon)$, and we estimate B_n by (over-) counting the numbers of ways to form graphs in $B(n, \varepsilon)$ by starting with one on $n-(k-1)\,R$ vertices and at most $\varepsilon n^k\,K_k$ subgraphs. By definition of $B(n, \varepsilon)$, we can obtain all graphs in $B(n, \varepsilon)$ by adding one of type $R^{[k-1]}$ to one on $n-(k-1)\,R$ vertices.

First we choose $(k-1)$ R-sets $\left(\text{at most } \binom{n}{R}^{k-1} \text{ choices}\right)$, and then we choose a graph on the remaining vertices with at most $\varepsilon n^k\,K_k$ (at most $G_{n-(k-1)R}$ choices). Next we choose at most αn vertices to be connected to all $k-1$ of the R-sets $\left(\text{at most } n\binom{n}{[\alpha n]} \text{ choices}\right)$, and then we connect them (at most $2^{(k-1)R n\alpha}$ ways). We then connect the rest of the $n-(k-1)\,R$ vertices by choosing for each of these vertices one of the R-sets to which it will not be connected (at most $(k-1)^{n-(k-1)R}$ choices), and then connecting them (at most $2^{(n-R(k-1))(k-2)R}$ ways. This completes the construction of all graphs in $B(n, \varepsilon)$ and gives

$$\begin{aligned}\log\left(\frac{B_n}{G_{n-(k-1)R}}\right) &\leq kR\log n + \log n\\ &\quad - n\,(\alpha\log\alpha + (1-\alpha)\log(1-\alpha)) + (k-1)\,R\alpha n\\ &\quad + (n-(k-1)\,R)\log(k-1) + (n-(k-1)\,R)\,(k-2)\,R\,.\end{aligned}$$

For α sufficiently small (and thus ε sufficiently small) and n sufficiently large, depending on R, α (and thus on ε), we get

$$\log\left(\frac{B_n}{G_{n-(k-1)R)}}\right) \leq (k-2)\,Rn + dn$$

where

$$d = 100\,kR\alpha = 100\,k^2\,2^{k+1}\,c^{-1}\,.$$

d is thus a constant independent of ε. We get

$$(2) \qquad \log\left(\frac{B_n}{G_{n-(k-1)R}}\right) \leq \left(\left(1-\frac{1}{k-1}\right)+\frac{d}{(k-1)R}\right)(k-1)\,Rn\,.$$

Let ε_1 be sufficiently small so that for each fixed $\varepsilon \leq \varepsilon_1$ there is a number $n'(\varepsilon)$ such that (1), (2) and the lemma hold for all $n \geq n'(\varepsilon)$.

We have from the lemma and (1)

$$G_n \leq A_n + B_n \leq B_n + G_n\,2^{-(n/2k)^2},$$

or

$$G_n \leq B_n(1-2^{-(n/2k)^2})^{-1}.$$

Then from (2) we get

$$\begin{aligned}\log\left(\frac{G_n}{G_{n-(k-1)R}}\right) &\leq \left(1-\frac{1}{k-1}+\frac{d}{(k-1)R}\right)(k-1)\,Rn\\ &\quad -\log(1-2^{-(n/2k)^2})\\ &\leq \left(1-\frac{1}{k-1}+\frac{2d}{(k-1)R}\right)(k-1)\,Rn\\ &\quad -\left(1-\frac{1}{k-1}+\frac{2d}{(k-1)R}\right)\frac{(k-1)^2R^2}{2}\end{aligned}$$

for n sufficiently large, say $n \geq n''(\varepsilon) \geq n'(\varepsilon)$. Thus if

$$\log(G_{n-(k-1)R}) \leq \left(1-\frac{1}{k-1}+\frac{2d}{(k-1)R}\right)\frac{(n-(k-1)R)^2}{2}+K\,,$$

we get

$$\log(G_n) \leq \left(1-\frac{1}{k-1}+\frac{2d}{(k-1)R}\right)\frac{n^2}{2}+K\,.$$

Let K be large enough so that

$$\log(G_n) \leq \left(1-\frac{1}{k-1}+\frac{2d}{(k-1)R}\right)\frac{n^2}{2}+K$$

for all $n \leq n''(\varepsilon)$. Then by the last remark, this inequality holds for all n. Now let $n(\varepsilon) \geq \max(n''(\varepsilon), ((2K(k-1)R)/d)^{\frac{1}{2}})$. Then

$$\log G_n \leq (1-(1/(k-1))+3d/(k-1)\,R)\,n^2/2 \quad \text{for all } n \geq n(\varepsilon).$$

The proof of Theorem 1 is complete if we let $f_k(\varepsilon)=1$ for $\varepsilon > \varepsilon_1$, and $f_k(\varepsilon) = 3d/(k-1)\,R$ for $\varepsilon \leq \varepsilon_1$, and if we let $n(k,\varepsilon)=1$ for $\varepsilon > \varepsilon_1$, and $n(k,\varepsilon)=n(\varepsilon)$ for $\varepsilon \leq \varepsilon_1$.

We now turn to the case $k = 3$, or triangle-free graphs. We show that " almost all " such graphs are bipartite. That is, if T_n is the number of triangle-free graphs on a set of n vertices, and if S_n is the number of bipartite graphs on the set n vertices, then

THEOREM 2. $T_n = S_n\left(1 + o\left(\frac{1}{n}\right)\right)$.

To prove Theorem 2 we use some lemmas, each concerning a special subclass of triangle-free graphs. We consider a set V of $m + 1$ vertices.

LEMMA 1. *Let* $A(V)$ *be those graphs on* V *with a vertex* v *connected to at most* $m/64$ *others. Then*

$$\log\left(\frac{|A(V)|}{T_m}\right) \leq \frac{m}{4}.$$

Proof. All graphs in $A(V)$ are obtained as follows: a vertex v is chosen ($m + 1$ ways); a graph on $V - \{v\}$ is chosen (T_m ways); and the connections to v are chosen $\left(\text{at most } m\binom{m}{[m/64]} \text{ ways}\right)$. This gives

$$\log\left(\frac{|A(V)|}{T_m}\right) \leq \log m + \log(m+1) + \log\binom{m}{[m/64]} \leq m/4,$$

for m sufficiently large.

LEMMA 2. *Let* $B(V)$ *be the graphs on* V *with a vertex* v *connected to a set* Q *of* $[m^{\frac{1}{2}}]$ *vertices, where the set* R *of all vertices connected to any vertex of* Q *satisfies* $|R| \geq m/2 + \frac{1}{2}m^{5/8}$. *Then* $\log(|B(V)|/T_m) \leq m/2 - m^{5/8}/4$ *for* m *sufficiently large.*

Proof. Graphs in $B(V)$ are all obtained as follows: v is chosen ($m + 1$ ways); a graph on $V - \{v\}$ is chosen (T_m ways); a set Q is chosen to satisfy the conditions for R $\left(\text{at most }\binom{m}{[m^{\frac{1}{2}}]} \text{ ways}\right)$; and v is connected to $V - (\{v\} \cup R)$ (at most $2^{m/2 - m^{5/8}/2}$ ways). This gives

$$\log\frac{|B(V)|}{T_m} \leq 2m^{\frac{1}{2}}\log m + m/2 - \frac{1}{2}m^{5/8} \leq \frac{m}{2} - \frac{m^{5/8}}{4}$$

for m large enough.

LEMMA 3. *Let* $C(V)$ *be the graphs on* V *with a vertex* v *connected to a set* Q *of* $[m^{\frac{1}{2}}]$ *vertices where the set* R *of vertices connected to any vertex of* Q *satisfies* $|R| \leq m/2 - \frac{1}{2}m^{5/8}$. *Then*

$$\log\left(\frac{|C(V)|}{T_{m-[m^{\frac{1}{2}}]}}\right) \leq \frac{1}{2}m^{3/2} - \frac{1}{4}m^{9/8}$$

for m sufficiently large.

Proof. All graphs in $C(V)$ are obtained as follows: v is chosen (($m + 1$) ways); Q is chosen $\left(\binom{m}{[m^{\frac{1}{2}}]} \text{ ways}\right)$; a graph on $V - (\{v\} \cup Q)$ is chosen ($T_{m-[m^{\frac{1}{2}}]}$ ways); R is chosen $\left(\text{at most } \Sigma_{j=0}^{[m/2 - \frac{1}{2}m^{5/8}]}\binom{m - [m^{\frac{1}{2}}]}{j} \leq m/2\binom{m}{[m/2]} \text{ ways}\right)$;

the connections from Q to R are chosen (at most $2^{[m^{\frac{1}{2}}](m/2-\frac{1}{2}m^{5/8})}$ ways); and finally the remaining connections of v are chosen (at most 2^m ways). This gives

$$\log\left(\frac{|\mathrm{C(V)}|}{\mathrm{T}_{m-[m^{\frac{1}{2}}]}}\right) \leq 3m + \frac{m^{3/2}}{2} - \frac{m^{9/8}}{2} \leq \frac{m^{3/2}}{2} - \frac{m^{9/8}}{4}$$

for m large enough.

LEMMA 4. *Let* $\mathrm{D(V)}$ *be the graphs on* V *with two adjoint vertices* x *and* y, *with their corresponding* Q_x, Q_y *and* R_x, R_y *as above, with* $\left| |\mathrm{R}_x| - \frac{m}{2} \right| \leq \frac{m^{5/8}}{2}$ *and similarly for* R_y, *and with* $|(\mathrm{V}-\mathrm{R}_x) \cap (\mathrm{V}-\mathrm{R}_y)| \geq \frac{m}{40}$. *Then*

$$\log\left(\frac{|\mathrm{D(V)}|}{\mathrm{T}_{m-1}}\right) \leq m - \frac{m}{160},$$

for m *large enough.*

Proof. All graphs in $\mathrm{D(V)}$ are obtained as follows: x, y are chosen (at most $(m+1)^2$ ways); a graph on $\mathrm{V}-\{x, y\}$ is chosen (T_{m-1} ways); Q_x and Q_y are chosen so that R_x and R_y satisfy the conditions above $\left(\text{at most } \binom{m}{[m^{\frac{1}{2}}]}^2 \text{ ways}\right)$; and finally x and y are connected as follows: Let $\mathrm{S} = (\mathrm{V}-\mathrm{R}_x) \cap (\mathrm{V}-\mathrm{R}_y)$. Then x can be connected to $(\mathrm{V}-\mathrm{R}_x)-\mathrm{S}$ in at most $2^{(m/2)+(m^{5/8}/2)-|\mathrm{S}|}$ ways, and similarly for y and $(\mathrm{V}-\mathrm{R}_y)-\mathrm{S}$. S can then be connected to x and y in $3^{|\mathrm{S}|}$ ways, since x and y are to be adjacent. This gives at most $2^{m+m^{5/8}+|\mathrm{S}|\log 3 - 2|\mathrm{S}|}$ ways. Then

$$\log\left(\frac{\mathrm{D(V)}|}{\mathrm{T}_{m-1}}\right) \leq 3m^{\frac{1}{2}}\log m + m + m^{5/8} - \frac{m}{40}(2-\log 3)$$
$$\leq m - \frac{m}{160}$$

for m large enough.

LEMMA 5. *Let* $\mathrm{E(V)}$ *be the graphs on* V *with vertices* x, y *adjacent respectively to sets* Q_x, Q_y *of* $[m^{\frac{1}{2}}]$ *vertices, where* Q_x, Q_y *are connected to* R_x, R_y, *respectively, with* $\left||\mathrm{R}_x| - m/2\right| < m^{5/8}/2$, *and similarly for* R_y. *Further let no two vertices of* R_x *be adjacent, and assume* u, $v \in \mathrm{R}_x$ *have no common adjacent vertex. Then* $\log\left(\frac{|\mathrm{E(V)}|}{\mathrm{T}_{m-3}}\right) \leq \frac{15}{8}m$ *for* m *sufficiently large.*

Proof. Graphs in $\mathrm{E(V)}$ are obtained as follows: x, y, u, v are chosen (at most in m^4 ways); a graph on $\mathrm{V}-\{x, y, u, v\}$ is chosen (at most T_{m-3} ways); Q_x and Q_y are chosen so that R_x and R_y can satisfy the conditions above $\left(\text{at most } \binom{m}{[m^{\frac{1}{2}}]}^2 \text{ ways}\right)$; x and y are connected to $\mathrm{V}-\mathrm{R}_x$ and $\mathrm{V}-\mathrm{R}_y$ respectively (at most $2^{m+m^{5/8}}$ ways); finally u and v are connected to $\mathrm{V}-\mathrm{R}_x$ (at most $3^{(m/2)+m^{5/8}/2}$ ways). This gives

$$\log\left(\frac{|\mathrm{E(V)}}{\mathrm{T}_{m-3}}\right) \leq 3m^{\frac{1}{2}}\log m + 2m^{5/8} + m\left(1+\frac{\log 3}{2}\right) < \frac{15}{8}m$$

for m large enough.

LEMMA 6. *Let* F(V) *be the graphs on* V *with* v , Q *and* R *as above, with* $||R| - m/2| < m^{5/8}/2$, *and with a vertex* u *not adjacent to any vertex of* R *nor to any vertex adjacent to* v. *Then*

$$\log\left(\frac{|F(V)|}{T_{n-1}}\right) \leq \frac{7}{8}\, m\,.$$

Proof. Graphs in F(V) are obtained as follows: u , v are chosen (at most $(m+1)^2$ ways); a graph on $V - \{u, v\}$ is chosen (at most T_{n-1} ways); Q is chosen $\left(\text{at most } \binom{m}{[m^{\frac{1}{2}}]} \text{ ways}\right)$; then u , v are connected to $V - R$ (at most $3^{m/2+m^{5/8}/2}$ ways). This gives

$$\log\left(\frac{|F(V)|}{T_{m-1}}\right) \leq 2m^{\frac{1}{2}} \log m + m^{5/8} + \frac{m}{2} \log 3$$
$$< \frac{7}{8}\, m \qquad \text{for } m \text{ large enough.}$$

LEMMA 7. *For* m *sufficiently large,*

$$\frac{m^2}{4} + m - 2\log m \leq \log S_m \leq \frac{m^2}{4} + m$$

(Recall S_m is the number of bipartite graphs on m vertices).

Proof. To obtain bipartite graphs on m vertices we divide them into two subsets (at most 2^m ways), and connect the two parts (at most $2^{m^2/4}$ ways). This gives $\log S_m \leq m^2/4 + m$.

To get a lower bound we must construct a special subclass of bipartite graphs and count them without duplication. We do this as follows. First we divide the vertices into two sets of sizes $[m/2]$ and $m - [m/2]$. There are at least $\frac{1}{2}\binom{m}{[m/2]}$ ways to do this. Not counting the effect of duplication this gives a contribution of $\binom{m}{[m/2]} 2^{[m/2](m-[m/2])-1}$ ways to construct the graphs.

The only graphs counted more than once here are those with more than one connected component. Suppose the vertices are divided into two subsets K and L, with k and l vertices respectively, such that no vertex of K is connected to any in L. Using the upper bound established above, we get at most $2^{k^2/4+l^2/4+k+l}$ such graphs for fixed K and L. The upper bound we have for S_m includes all multiplicities obtained in the construction above. Thus to (more than) compensate for multiplicities we subtract $\sum_{k=1}^{[m/2]} \binom{m}{k} 2^{(m^2/4)+m-k(m-k)/2}$. This can be rewritten as

$$2^{m^2/4+m}\left(\sum_{k=1}^{[m/2]} 2^{(-k(m-k))/2}\right) \leq \frac{m}{2}\, 2^{(-m+1)/2}\, 2^{m^2/4+m}.$$

Hence we get

$$S_m \geq 2^{(m^2/4)-2}\binom{m}{[m/2]} - 2^{(m^2/4)+m}\,\frac{m}{2}\, 2^{-(m+1)/2}$$
$$\geq 2^{(m^2/4)+m}\left(2^{-2-\log m} - 2^{-m/2+\log m - \frac{1}{2}}\right)$$
$$\geq 2^{(m^2/4)+m}\left(2^{-3-\log m}\right)$$
$$\geq 2^{(m^2/4)+m-2\log m}$$

for m large enough.

COROLLARY. *For m large enough*

$$\log\left(\frac{S_m}{S_{m+1}}\right) \leq -m/2 + 3\log m .$$

LEMMA 8. *Let* $S(V)$ *be the bipartite graphs on* V. *Then* $A(V) \cup \cdots \cup F(V) \cup S(V)$ *contains all the triangle-free graphs on* V, *for m sufficiently large.*

Proof. Let G be a triangle-free graph not in $A(V)$ through $F(V)$. By $A(V)$, $B(V)$ and $C(V)$, every vertex in G is connected to some set Q which in turn is adjacent to a set R of vertices with $\left| |R| - m/2 \right| \leq \frac{1}{2} m^{5/8}$.

We claim that G contains no pentagons (5-cycles). For let a, b, c, d, e be the vertices of such a pentagon, in order. For each of these there is a corresponding Q and R, say $R_a, \cdots, R_e$. By $D(V)$,

$$|(V - R_a) \cap (V - R_b)| \leq \frac{m}{40}, \qquad |(V - R_b) \cap (V - R_c)| \leq \frac{m}{40} \cdot$$

Since $(V - R_a)$, $(V - R_b)$, $(V - R_c)$ all have $(m/2) \pm \frac{1}{2} m^{5/8}$ vertices, we get $|(V - R_a) \cap (V - R_c)| \geq (m/2) - (3 m^{5/8}/2) - m/20$. Similarly, $|(V - R_c) \cap \cap (V - R_e)| \geq (m/2) - 3 m^{5/8}/2) - m/20$. This implies that

$$|(V - R_a) \cap (V - R_e)| \geq \frac{m}{2} - \frac{7}{2} m^{5/8} - \frac{m}{10} > \frac{m}{20}$$

for m large enough, contradicting $D(V)$, since a and e are adjacent. Thus G has no pentagons.

Now consider any two adjacent vertices x and y in G. Let S_x and S_y be those vertices distance 1 from x and y respectively, and R_x, R_y those at distance 2. Since there are no pentagons, R_x and R_y are disjoint. S_x and S_y are disjoint also, as there are no triangles. Thus by $A(V)$, $B(V)$ and $C(V)$, $|R_x|$ and $|R_y|$ are both $m/2 \pm m^{5/8}/2$. Since there are no pentagons or triangles, no two vertices of R_x are adjacent, and similarly for R_y. (Also for S_x and S_y).

By $E(V)$, every two vertices of R_x have a common adjacent vertex, and similarly for R_y. (Strictly speaking, we first choose a set Q of $[m^{\frac{1}{4}}]$ vertices from S_x (resp. S_y) so that the two vertices under consideration are adjacent to Q, and then apply $E(V)$).

Now consider a vertex z not in $U = (\{x, y\} \cup S_x \cup S_y \cup R_x \cup R_y)$. By $A(V)$, z must be connected to some vertices in U. By definition of the S's and R's, z can be connected only to $R_x \cup R_y$, and not $\{x, y\} \cup S_x \cup S_y$. But if z is connected to $u \in R_x$ and $v \in R_y$, then by $A(V)$, v must be connected to some other vertex $w \in R_x$, and by our observation above, w and u must both be adjacent to some other vertex t. Then t, w, v, z, u form a pentagon, which is forbidden. Thus z can be connected to only one of R_x and R_y. But by $F(V)$ there can be no such z.

Thus G consists entirely of x, y, S_x, S_y, R_x, R_y. We claim G is bipartite with parts $\{x\} \cup S_y \cup R_x$ and $\{y\} \cup S_x \cup R_y$. We already know that within each of R_x, R_y, S_x, S_y there are no edges. Furthermore x cannot be connected to S_y nor R_x or a triangle would result. Similarly y is not connected to S_x nor R_y. Finally, there can be no edges between S_x and R_y nor R_x and S_y or there would be a pentagon.

Thus $G \in S(V)$ and the proof of Lemma 8 is complete.

Proof of Theorem 2. We prove the following statement by induction on n: $T_n \leq (1 + (C/n))\, S_n$ for all n, where C is large enough so that $T_n \leq (1 + (C/n))\, S_n$ for $n \leq N$, and N is large enough so that all the lemmas above are valid, and $N \geq 10^{10}$.

For $n \leq N$ statement is true by choice of C. We assume that it holds for all $n \leq m$, where $m \geq N$, and we show

$$T_{n+1} \leq \left(1 + \frac{C}{n+1}\right) S_{n+1}\,.$$

By Lemma 8, we need only show that

$$\frac{|A(V)| + \cdots + |F(V)|}{T_{n+1}} \leq \frac{C}{n+1}\,, \qquad \text{for} \quad |V| = n+1\,.$$

We use induction and the inequalities from the lemmas to show that each of $|A(V)|/S_{n+1}, \cdots, |F(V)|/S_{n+1}$ are less than $1/6\ \dfrac{C}{n+1}\cdot$ The arguments are all similar and we give only a couple here.

$$\begin{aligned}
\frac{|C(V)|}{S_{n+1}} &= \frac{|C(V)|}{T_{n-[n^{\frac12}]}}\,\frac{T_{n-[n^{\frac12}]}}{S_{n-[n^{\frac12}]}}\,\frac{S_{n-[n^{\frac12}]}}{S_{n-[n^{\frac12}]+1}} \cdots \frac{S_n}{S_{n+1}} \\
&\leq 2^{\frac12 n^{3/2} - \frac14 n^{9/8}} \left(1 + \frac{C}{n-[n^{\frac12}]}\right) \\
&\quad \cdot \left(2^{-(n-[n^{\frac12}])/2 + 3\log n}\right)^{[n^{\frac12}]+1} \\
&< 2^{-1/8\, n^{9/8}} \left(1 + \frac{C}{n-[n^{\frac12}]}\right) < \frac{1}{6}\,\frac{C}{n+1} \\
\frac{|F(V)|}{S_{n+1}} &< \frac{|F(V)|}{T_{n-1}}\,\frac{T_{n-1}}{S_{n-1}}\,\frac{S_{n-1}}{S_n}\,\frac{S_n}{S_{n+1}} \\
&< 2^{7/8\, n} \left(1 + \frac{C}{n-1}\right) 2^{-(n-1)+6\log n} \\
&< \frac{1}{6}\,\frac{C}{n+1}\,.
\end{aligned}$$

These complete the proof of Theorem 2.

References

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