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Cliques in random graphs

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1. *Introduction.* Let $0 < p < 1$ be fixed and denote by G a random graph with point set $\mathbb{N}$, the set of natural numbers, such that each edge occurs with probability p, independently of all other edges. In other words the random variables e_{ij}, $1 \leqslant i < j$, defined by

$$e_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is an edge of } G, \\ 0 & \text{if } (i,j) \text{ is not an edge of } G, \end{cases}$$

are independent r.v.'s with $P(e_{ij} = 1) = p$, $P(e_{ij} = 0) = 1-p$. Denote by G_n the subgraph of G spanned by the points $1, 2, \ldots, n$. *These random graphs* G, G_n *will be investigated throughout the note.* As in (1), denote by K_r a complete graph with r points and denote by $k_r(H)$ the number of K_r's in a graph H. A maximal complete subgraph is called a *clique*. In (1) one of us estimated the *minimum* of $k_r(H)$ provided H has n points and m edges. In this note we shall look at the random variables

$$Y_r = Y(n,r) = k_r(G_n),$$

the number of K_r's in G_n, and

$$X_n = \max\{r : k_r(G_n) > 0\},$$

the maximal size of a clique in G_n.

Random graphs of a slightly different kind were investigated in detail by Erdös and Rényi (2). In (4) Matula showed numerical evidence that X_n has a strong peak around $2\log n/\log(1/p)$. Grimmett and McDiarmid (3) proved that as $n \to \infty$

$$X_n/\log n \to 2/\log(1/p)$$

with probability one. Independently and earlier Matula (5) proved a considerably finer result about the peak of X_n. In particular he proved that, as $n \to \infty$, X_n takes one of at most two values depending on n with probability tending to 1.

The main aim of this note is to prove various results about the distribution of X_n. We shall also investigate the existence of infinite complete graphs in G. Finally we prove how many colours are likely to be used by a certain colouring algorithm.

2. *Cliques in finite graphs.* To simplify the notations we shall put $b = 1/p$. Note first that the probability that a given set of r points spans a complete subgraph of G is $p^{\binom{r}{2}}$. Consequently the expectation of $Y_r = Y(n,r)$ is

$$E_r = E(Y_r) = E(n,r) = \binom{n}{r} p^{\binom{r}{2}}. \tag{1}$$

Let $d = d(n)$ be the positive real number for which

$$\binom{n}{d} p^{\binom{d}{2}} = 1.$$

It is easily checked that

$$d(n) = 2\log_b n - 2\log_b \log_b n + 2\log_b (\tfrac{1}{2}e) + 1 + O(1)$$
$$= 2\log_b n + O(\log_b \log_b n) = \frac{2\log n}{\log b} + O(\log_b \log_b n).$$

Choose an ϵ, $0 < \epsilon < \frac{1}{2}$. Given a natural number $r \geqslant 2$ let n_r be the maximal natural number for which

$$E(n_r, r) \leqslant r^{-(1+\epsilon)}$$

and let n_r' be the minimal natural number for which

$$E(n_r', r) \geqslant r^{1+\epsilon}.$$

It is easily checked that

$$n_r' - n_r < \frac{3\log r}{r} b^{\frac{1}{2}r} \tag{2}$$

and

$$n_r = b^{\frac{1}{2}r} + o(b^{\frac{1}{2}r}). \tag{3}$$

Thus, with at most finitely many exceptions, one has

$$n_r < n_r' < n_{r+1},$$

$$(n_r' - n_r)/(n_{r+1} - n_r') < 4(b^{\frac{1}{2}} - 1)^{-1} r^{-1} \log r$$

and

$$\lim_{r\to\infty} (n_{r+2} - n_{r+1})/(n_{r+1} - n_r) = b^{\frac{1}{2}}.$$

THEOREM 1. *For a.e. graph G there is a constant $c = c(G)$ such that if*

$$n_r' \leqslant n \leqslant n_{r+1} \quad \textit{for some} \quad r > c$$

then

$$X_n(G) = r.$$

Proof. Let $0 < \eta < 1$ be an arbitrary constant. We shall consider the random variables $Y_r = Y(n, r)$ for

$$(1+\eta)\frac{\log n}{\log b} < r < 3\frac{\log n}{\log b}$$

and large values of n. Note first that the second moment of Y_r is the sum of the probabilities of ordered pairs of K_r's. The probability of two K_r's with l points in common is

$$p^{2\binom{r}{2} - \binom{l}{2}}$$

since G must contain $2\binom{r}{2} - \binom{l}{2}$ given edges. As one can choose

$$\binom{n}{r}\binom{r}{l}\binom{n-r}{r-l}$$

ordered pairs of sets of r points with l points in common (since $n \geqslant 2r$), the second moment of Y_r is

$$E(Y_r^2) = \sum_{l=0}^{r} \binom{n}{r}\binom{r}{l}\binom{n-r}{r-l} p^{2\binom{r}{2}-\binom{l}{2}} \tag{4}$$

(cf. Matula (4)). As

$$E_r^2 = \sum_{l=0}^{r} \binom{n}{r}\binom{r}{l}\binom{n-r}{r-l} p^{2\binom{r}{2}},$$

the variance of Y_r is

$$\sigma_r^2 = \sigma^2(Y_r) = \sum_{l=2}^{r} \binom{n}{r}\binom{r}{l}\binom{n-r}{r-l} p^{2\binom{r}{2}} (b^{\binom{l}{2}} - 1)$$

and so

$$\sigma_r^2/E_r^2 = \sum_{l=2}^{r} \frac{\binom{r}{l}\binom{n-r}{r-l}}{\binom{n}{r}} (b^{\binom{l}{2}} - 1) = \sum_{l=2}^{r} F_l. \tag{5}$$

Routine calculations show that, if n is sufficiently large and $3 \leqslant l \leqslant r-1$, then

$$F_l < F_3 + F_{r-1}.$$

Consequently

$$\begin{aligned}\sigma_r^r/E_r^2 &< F_2 + F_r + r(F_3 + F_{r-1})\\ &< \frac{r^4}{2n^2}(b-1) + \frac{1}{E_r} + r\left(\frac{r^6}{6n^3}(b^3-1) + \frac{rnp^{r-1}}{E_r}\right).\end{aligned}$$

Since

$$b^r > n^{1+\eta} \quad \text{and} \quad r < \frac{3}{\log b}\log n,$$

this gives

$$\begin{aligned}\sigma_r^2/E_r^2 &< (\tfrac{1}{2}(b-1)\, r^4 n^{-2} + E_r^{-1})(1+\eta)\\ &< (b-1)\, r^4 n^{-2} + 2E_r^{-1}.\end{aligned} \tag{6}$$

Inequality (6) gives, in fact, the right order of magnitude of σ_r^2/E_r^2 since (5) implies immediately

$$\sigma_r^2/E_r^2 > F_2 + F_3 = \tfrac{1}{2}(b-1)\, r^4 n^{-2} + E_r^{-1}(1 - p^{\binom{r}{2}}).$$

We shall use inequality (6) only to conclude

$$P(Y_r = 0) < \sigma_r^2/E_r^2 < br^4 n^{-2} + 2E_r^{-1}. \tag{7}$$

In particular, by the choice of n_r',

$$\begin{aligned}P(Y(n_r', r) = 0) &< br^4(n_r')^{-2} + 2E(n_r', r)^{-1}\\ &< 3r^{-(1+\epsilon)}.\end{aligned}$$

On the other hand,

$$P(Y(n_{r+1}, r+1) > 0) < E(n_{r+1}, r+1) < r^{-(1+\epsilon)}.$$

Consequently, for a fixed r

$$P(\exists n,\ n_r' \leqslant n \leqslant n_{r+1},\quad X_n \neq r) < 4r^{-(1+\epsilon)}. \tag{8}$$

As $\sum_1^{\infty} r^{-(1+\epsilon)} < \infty$, the Borel–Cantelli lemma implies that for a.e. graph G with the exception of finitely many r's one has

$$X_n(G) = r \quad \text{for all } n, \quad n_r' \leqslant n \leqslant n_{r+1}.$$

This completes the proof of the theorem.

Let $\epsilon = \frac{1}{3}$. Then the choice of the numbers n_r, n_r' and the definition of $d(n)$ imply easily (cf. inequalities (2) and (3)) that

$$d(n_r) < r < d(n_r')$$

and if r is sufficiently large then

$$\max\{r - d(n_r), d(n_r') - r\} < \frac{7}{2 \log b} \frac{\log r}{r} < 2 \frac{\log \log n_{r+1}}{\log b \log n_{r+1}}.$$

Thus inequality (8) implies the following extension of the result of Matula (5).

COROLLARY 1. (i) *For a.e. graph G there is a constant $\tilde{c} = \tilde{c}(G)$ such that if $n \geqslant \tilde{c}(G)$ then*

$$\left[d(n) - 2\frac{\log \log n}{\log n \log b}\right] \leqslant X_n \leqslant \left[d(n) + 2\frac{\log \log n}{\log n \log b}\right].$$

(ii) *If r is sufficiently large and $n_r \leqslant n \leqslant n_{r+1}$ then*

$$P\left\{\left[d(n) - 2\frac{\log \log n}{\log n \log b}\right] \leqslant X_n \leqslant \left[d(n) + 2\frac{\log \log n}{\log n \log b}\right], \forall\, n, n_r \leqslant n \leqslant n_{r+1}\right\} \geqslant 1 - 10 r^{-\frac{1}{3}}.$$

Remark. Note that the upper and lower bounds on X_n in Corollary 1 differ by at most 1 if n is large and for most values of n they simply coincide.

Let us estimate now how steep a peak X_n has got near $d(n)$. More precisely, we shall estimate

$$P(X_n \leqslant r(n)) \quad \text{and} \quad P(X_n \geqslant r'(n))$$

for certain functions $r(n)$, $r'(n)$ with $r(n) < d(n) < r'(n)$. The expectation gives a trivial but fairly good bound for the second probability. As for $r > d(n)$ one has $E(n,r) < n^{d(n)-r}$,

$$P(X_n \geqslant r'(n)) < n^{d(n)-r'(n)}$$

whenever $r'(n) > d(n)$. Furthermore, if $0 < d(n) - r(n)$ is bounded, $K > 0$ is a constant and n is sufficiently large, then $E(n,r) > K n^{d(n)-r(n)}$. Consequently it follows from (7) that if $0 < \delta < 2$, $0 < c$ and n is sufficiently large (depending on p, δ and c) then

$$P(X_n \leqslant d(n) - \delta) < c n^{-\delta}. \tag{9}$$

Our next result extends this inequality.

THEOREM 2. (i) *Let $0 < \epsilon$, $0 < r(n) < d(n)$, $r(n) \to \infty$ and put*

$$t = t(n) = [d(n) - \epsilon - r(n)] - 1.$$

Then

$$P(X_n \leqslant r(n)) < n^{-[b^{\frac{1}{2}t}]}$$

if n is sufficiently large.

(ii) *Let $0 < \epsilon < \delta < 1$. Then*

$$P(X_n \leqslant (1-\delta)d(n)) < n^{-n^{2\epsilon}}$$

if n is sufficiently large.

Proof. (i) Put $s = [b^{\frac{1}{2}t}]$ and choose subsets $V_1, \ldots, V_s$ of $\{1, \ldots, n\}$ such that $|V_i \cap V_j| \leqslant 1$ and $|V_i| \geqslant n/s$ $(1 \leqslant i, j \leqslant n, i \neq j)$. Then

$$d(|V_i|) > d(n) - \tfrac{1}{2}\epsilon - \frac{2\log s}{\log b} > d(n) - \tfrac{1}{2}\epsilon - t \geqslant r(n) + 1 - \tfrac{1}{2}\epsilon.$$

Thus for large n the probability that the subgraph spanned by V_i does not contain a K_l with $l > r(n)$ is less than n^{-1}. As these subgraphs are independent of each other,

$$P(X_n \leqslant r(n)) < n^{-s}$$

if n is sufficiently large.

(ii) Put $r = [(1-\delta)d(n)+1]$ and let q be a prime between n^ϵ and $2n^\epsilon$. Put

$$Q = q^2+q+1, \quad m = [n/Q].$$

Divide $\{1, 2, \ldots, n\}$ into Q classes, $C_1, \ldots, C_Q$, each having m or $m+1$ elements. Consider the sets $C_1, \ldots, C_Q$ as the points of a finite projective geometry. If e is a line of this projective geometry, let G_e be the subgraph of G with point set $V_e = \bigcup_{C_i \in e} C_i$ and with all those edges of G that join points belonging to different classes. It is clear that almost every r-tuple of V_e is such that no two points belong to the same class, since

$$\binom{q+1}{r} m^r \sim \binom{|V_e|}{r}.$$

Furthermore,

$$r < d(|V_e|) - 2.$$

Consequently inequality (7) implies that the probability of G_e not containing a K_r is less than n^{-1}. As e runs over the set of lines of the projective geometry the subgraphs G_e are independent since they have been chosen independently of the existence of edges. Therefore

$$P(G_n \text{ does not contain a } K_r) < n^{-Q} < n^{-n^{2\epsilon}},$$

as claimed.

Remark. Up to now we have investigated the maximal order of a clique. Let us see now which natural numbers are likely to occur as orders of cliques (maximal complete subgraphs). We know that cliques of order essentially greater than $d(n)$ are unlikely to occur. It turns out that cliques of order roughly less than $\frac{1}{2}d(n)$ are also unlikely to occur but every other value is likely to be the order of a clique. The probability that r given points span a clique of G_n is clearly

$$(1-p^r)^{n-r} p^{\binom{r}{2}}.$$

Thus if $Z_r = Z_r(G_n)$ denotes the number of cliques of order r in G_n then the expectation of Z_r is

$$E(Z_r) = \binom{n}{r}(1-p^r)^{n-r} p^{\binom{r}{2}}.$$

Denote by $\tilde{d}(n)$ the minimal value of $r > 2$ for which the right hand side is 1. One can prove rather sharp results analogous to Theorem 1 stating that the orders of cliques occurring are almost exactly the numbers between $\tilde{d}(n) \sim \frac{1}{2}d(n)$ and $d(n)$, but we shall formulate only the following very weak form of the possible results.

Given $\epsilon > 0$ a.e. graph G is such that whenever n is sufficiently large and

$$(1+\epsilon)\frac{\log n}{\log b} < r < (2-\epsilon)\frac{\log n}{\log b},$$

G_n contains a clique of order r, but G_n does not contain a clique of order less than

$$(1-\epsilon)\frac{\log n}{\log b},$$

or greater than

$$(2+\epsilon)\frac{\log n}{\log b}.$$

3. *Infinite complete subgraphs.* We denote by $K(x_1, x_2, \ldots)$ (resp. $K(x_1, \ldots, x_n)$) the infinite (resp. finite) complete graph with vertex set $\{x_1, x_2, \ldots\}$ (resp. $\{x_1, \ldots, x_n\}$). We shall always suppose that $1 \leqslant x_1 < x_2 < \ldots$. We would like to determine the infimum c_0 of those positive constants c for which a.e. G contains a $K(x_1, x_2, \ldots)$ such that

$$x_n \leqslant c^n \quad \text{for every} \quad n \geqslant n(G).$$

Corollary 1 implies that $c_0 \geqslant b^{\frac{1}{2}}$. At the first sight $c_0 = \frac{1}{2}$ does not seem to be impossible since a.e. G is such that for every sufficiently large n it contains a $K(x_1, \ldots, x_n)$ satisfying $x_n < b^{(\frac{1}{2}+\epsilon)n}$. However, it turns out that a sequence cannot be continued with such a density and, in fact, $c_0 = b$. We have the following more precise result.

THEOREM 3. (i) *Given $\epsilon > 0$ a.e. graph G is such that for every $K(x_1, \ldots) \subset G$*

$$x_n > b^{n(1-\epsilon)}$$

holds for infinitely many n.

(ii) *Given $\epsilon > 0$ a.e. G contains a $K(x_1, \ldots)$ such that*

$$x_n < b^{n(1+\epsilon)}$$

for every sufficiently large n.

Proof. (i) Let $1 \leqslant x_1 < x_2 < \ldots < x_n \leqslant b^{n(1-\epsilon)}$. Then the probability that there is a point $x_{n+1} \leqslant b^{(n+1)(1-\epsilon)}$ joined to every x_i, $1 \leqslant i \leqslant n$, is less than

$$b^{(n+1)(1-\epsilon)}p^n = b^{1-(n+1)\epsilon}.$$

Thus the probability that G contains a $K(x_1, \ldots, x_N)$ satisfying

$$x_k \leqslant b^{k(1-\epsilon)} \quad (k = n, \ldots, N)$$

is less than

$$P_{n,N} = \binom{b^n}{n} \prod_{n+1}^{N} b^{1-k\epsilon}.$$

$\left(\text{There are at most } \binom{b^n}{n} \text{ sequences } 1 \leqslant x_1 < \ldots < x_n \leqslant b^{n(1-\epsilon)}.\right)$ Clearly $P_{n,N} \to 0$ as $N \to \infty$, so the assertion follows.

(ii) Let P_n be the probability that G contains a $K(x_1, \ldots, x_n)$ with $x_n < b^{n(1+\epsilon)}$. We know that $P_n \to 1$. Given $1 \leq x_1 < \ldots < x_n < b^{n(1+\epsilon)}$ let us estimate the probability Q_{n+1} that there is a point x_{n+1}, $x_n < x_{n+1} < b^{(n+1)(1+\epsilon)}$, which is joined to *every* x_i, $1 \leq i \leq n$. There are

$$[b^{(n+1)(1+\epsilon)} - x_n] \geq [b^{(n+1)(1+\epsilon)} - b^{n(1+\epsilon)}] > b^{(n+1)(1+\eta)} = B_{n+1}$$

independent choices for x_{n+1}, where $0 < \eta < \epsilon$ and n is sufficiently large. The probability that a point is *not* joined to *each* of $\{x_1, \ldots, x_n\}$ is $1 - p^n$. Consequently

$$1 - Q_{n+1} \leq (1-p^n)^{B_{n+1}} \leq e^{-b^{n\eta}}.$$

Therefore the probability that G contains a $K(x_1, \ldots, x_n)$, $x_n < b^{n(1+\epsilon)}$, which can be extended to a $K(x_1, x_2, \ldots)$ by choosing first $x_{n+1} < b^{(n+1)(1+\epsilon)}$, then $x_{n+2} < b^{(n+2)(1+\epsilon)}$, etc., is at least

$$R_n = P_n \prod_{n+1} Q_k.$$

As $R_n \to 1$ $(n \to \infty)$, the proof is complete.

4. *Colouring by the greedy algorithm.* Given a graph G with points $1, 2, \ldots$, the *greedy algorithm* (see (3)) colours G with colours $c_1, c_2, \ldots$ as follows. Suppose the points $1, \ldots, n$ have already been coloured. Then the algorithm colours $n+1$ with colour c_j where j is the maximal integer such that for each $i < j$ the point $n+1$ is joined to a point $x_i \leq n$ with colour c_i. In other words the algorithm colours the point $n+1$ with the colour having the minimal possible index. Denote by $\tilde{\chi}_n = \tilde{\chi}_n(G) = \tilde{\chi}(G_n)$ the number of colours used by this algorithm to colour G_n. Out next result extends a theorem of Grimmett and McDiarmid (3) stating that $\tilde{\chi}_n[(\log n)/n] \to \log 1/q$ in mean, where $q = 1-p$. An immediate corollary of our result is that $\tilde{\chi}_n[(\log n)/n] \to \log 1/q$ in any mean (with a given rate of convergence) and almost surely. As usual, $\{x\}$ denotes the least integer not less than x.

THEOREM 4. (i) *Let* $0 < \gamma < \frac{1}{2}$ *be fixed and let* $u(n) \geq \gamma^{-\frac{1}{2}}$ *be an arbitrary function. If* n *is sufficiently large then*

$$P\left\{\tilde{\chi}_n \frac{\log n}{n} < \log 1/q(1 + u(n)(\log 1/q)^{\frac{1}{2}}(\log n)^{-\frac{1}{2}})^{-1}\right\} < n^{-\gamma u^2(n)+1}.$$

(ii) *Let* $3 \leq v(n) < \log n(\log\log n)^{-1}$ *and put*

$$t(n) = 1 - v(n)\log\log n(\log n)^{-1},$$

$$c(n) = \left\{\frac{n \log 1/q}{t(n)\log n}\right\}.$$

Then for every sufficiently large n *we have*

$$P(\tilde{\chi}_n > c(n)) < e^{-(\log n)^{v(n)-2}}.$$

Proof. (i) Let M_j be the probability that G_n has at least

$$k = k(n) = \frac{\log n}{\log 1/q} + u(n)(\log n)^{\frac{1}{2}}(\log 1/q)^{-\frac{1}{2}} = k_1(n) + k_2(n) = k_1 + k_2$$

points of colour c_j. It suffices to show that if n is sufficiently large then

$$M_j < n^{-\gamma u^2},$$

since then the probability that there is a colour class with at least $k(n)$ points is at most

$$nn^{-\gamma u^2} = n^{-\gamma u^2+1}.$$

Grimmett and McDiarmid showed ((3), p. 321), that

$$M_j \leqslant \prod_{i=1}^{[k(n)]-1} (1-(1-q^i)^n).$$

Since $k_2 \to \infty$ as $n \to \infty$ we may suppose that $k_2^2 - 5k_2 + 6 > 2\gamma k_2^2$. Then taking into account that $q^{k_1} = n^{-1}$, $q^{k_2^2} = n^{-u^2}$,

$$M_j \leqslant \prod_{i=[k_1+1]}^{[k-1]} nq^i \leqslant n^{k_2-2} q^{\frac{1}{2}(k_2-2)(2k_1+k_2-3)}$$

$$= q^{\frac{1}{2}(k_2^2-5k_2+6)} < q^{\gamma k_2^2} = n^{-\gamma u^2(n)},$$

as required.

(ii) Let us estimate the probability that the greedy algorithm has to use more than $c(n)$ colours to colour the first n points. The probability that this happens when k_i points have colour i, $i \leqslant c(n)$, is exactly

$$\prod_{i=1}^{c(n)} (1-q^{k_i}) \leqslant (1-q^{n/c(n)})^{c(n)}$$

since

$$\sum_{1}^{c(n)} k_i \leqslant n-1.$$

Thus the probability that more than $c(n)$ colours have to be used to colour the points $1, \ldots, n$ is less than

$$S_n = n(1-q^{n/c(n)})^{c(n)}.$$

Now

$$(1-q^{n/c(n)})^{c(n)} < e^{-c(n)(1/q)^{-n/c(n)}}$$

and

$$\log(c(n)(1/q)^{-n/c(n)}) > (v(n)-\tfrac{3}{2}) \log\log n$$

if n is sufficiently large. Consequently

$$S_n < e^{\log n - (\log n)^{v(n)-1}} < e^{-(\log n)^{v(n)-2}}$$

for n sufficiently large, completing the proof.

5. *Final remarks.* (i) *Colouring random graphs.* It is very likely that the greedy algorithm uses twice as many colours as necessary and, in fact, $\chi(G_n)\dfrac{\log n}{n} \to \frac{1}{2}\log 1/q$ for a.e. graph G. ($\chi(G_n)$ denotes the chromatic number of G_n.) One would have a good chance of proving this if the bound $n^{-n^{2\epsilon}}$ in Theorem 2 (ii) could be replaced by $e^{-c_\epsilon n}$ for some positive constant c_ϵ, depending on ϵ.

(ii) *Hypergraphs.* Let $k > 2$ be a natural number and consider random k-hypergraphs (k-graphs) on $\mathbb{N}$ such that the probability of a given set of k points forming a k-tuple of the graph is p, independently of the existence of other k-tuples. The proofs of our edge graph results can easily be modified to give corresponding results about random k-graphs. Let us mention one or two of these results. As before, denote by X_n the maximal order of a complete subgraph of G_n.

The expectation of the number of complete graphs of order r is clearly

$$\binom{n}{r} p^{\binom{r}{k}}.$$

Let $d_k(n) > 1$ be the minimal value for which this is equal to 1. It is easily seen that

$$d_k(n) \sim \left(\frac{k!\log n}{\log 1/p}\right)^{1/(k-1)}.$$

Then, corresponding to Corollary 1, we have the following result.

For every $\epsilon > 0$

$$\lim_{n\to\infty} P([d_k(n)-\epsilon] \leqslant X_n \leqslant [d_k(n)+\epsilon]) = 1.$$

Denote by $\tilde{\chi}_n^k(G)$ the number of colours used by the greedy algorithm to colour G_n. Then

$$\tilde{\chi}_n^k \frac{(\log n)^{1/(k-1)}}{n} \to ((k-1)!\log 1/q)^{1/(k-1)}$$

in any mean. It is expected that in general the greedy algorithm uses $k^{1/(k-1)}$ times as many colours as necessary.

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