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DISTINCT VALUES OF EULER'S ϕ-FUNCTION

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Introduction. Let $V(x)$ denote the number of distinct values not exceeding x taken by Euler's ϕ-function, so that we have $\pi(x) \leqslant V(x) \leqslant x$. It was shown by Erdős and Hall [1] that for each fixed $B > 2\sqrt{(2/\log 2)}$, the estimate

$$V(x) \ll \pi(x) \exp\{B\sqrt{(\log\log x)}\}$$

holds; moreover we stated that the ratio $V(x)/\pi(x)$ tends to infinity with x, faster than any fixed power of $\log\log x$. Our aim in the present paper is to prove the following result.

THEOREM. *There exist positive absolute constants A, C, such that*

$$V(x) \geqslant C\pi(x) \exp\{A(\log\log\log x)^2\}.$$

Remarks and notation. Here and throughout the paper, $\log x$ is to be interpreted as $\max(1, \log x)$ to ensure that the various iterated logarithms are well defined.

There is still a gap to be filled between this result and the estimate from above: it is not clear to us which estimate is nearer the truth.

The letters, p, q, r, are reserved for primes, also $C_1, C_2, \ldots$ are positive absolute constants, chosen to ensure the validity of every expression in which they occur.

LEMMA 1. *Let x, y be real numbers and $\{m_i : 1 \leqslant i \leqslant t\}$ be integers satisfying $1 = m_1 < m_2 < \ldots < m_t < y < x^{\frac{1}{4}}$ and for each n, let $s(n)$ denote the number of representations of n in the form $n = m_i(p-1)$, $1 \leqslant i \leqslant t, p > 2y$. Then*

$$\sum_{n\leqslant x} s(n) \geqslant C_1 \frac{x}{\log x} \sum_{i=1}^{t} \frac{1}{m_i}$$

and

$$\sum_{n\leqslant x} s^2(n) \leqslant C_2 \frac{x}{\log x} \left(\frac{(\log y)^4}{\log x} + \sum_{i=1}^{t} \frac{1}{m_i} \right). \tag{1}$$

Proof. The first inequality is clear, so we may confine our attention to the second. Notice that the sum on the left of (1) is the number of solutions of $m_i(p-1) = m_j(q-1) \leqslant x$, where $p, q > 2y$, and that the second term on the right of (1) arises from the solutions of this equation in the case $i = j$, $1 \leqslant i \leqslant t$. Let N_{ij} denote the number of solutions, *i.e.* choices for p and q, when i and j are fixed, $i \neq j$. Writing $u = u_{ij} = (m_i, m_j)$, we derive from Satz 4.2 of Prachar [2] p. 45 the estimate

$$N_{ij} \leqslant \frac{C_3\, x/u}{\phi(m_i/u)\phi(m_j/u)\log^2(xu/m_i m_j)} \prod_r \left(1-\frac{1}{r}\right)^{-1},$$

where r runs through the prime factors of $(m_i - m_j)/u$. In view of the well-known result that $m/\phi(m) = O(\log\log m)$ this gives

$$N_{ij} \leqslant \frac{C_4\, x(m_i, m_j)(\log\log y)^3}{m_i m_j \log^2 x}. \tag{2}$$

We have to sum N_{ij} over $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant t$, $i \neq j$. In fact it is sufficient, to obtain (1), to observe that with no restrictions on n_1, n_2, we have

$$\sum_{n_1 \leqslant y} \sum_{n_2 \leqslant y} \frac{(n_1, n_2)}{n_1 n_2} = O(\log^3 y).$$

Together with (2) this gives the result stated.

LEMMA 2. For each $x \geqslant 1$, define $y \geqslant 1$ by the relation $(\log y)^4 = \log x$. Then with the hypotheses and notation of Lemma 1, the number N of distinct $n \leqslant x$ representable in the form $n = m_i(p-1)$, $p > 2y$, satisfies

$$N \geqslant C_5 \frac{x}{\log x} \sum_{i=1}^{t} \frac{1}{m_i}. \tag{3}$$

Proof. By the Cauchy–Schwarz inequality,

$$\left(\sum_{n \leqslant x} s(n)\right)^2 \leqslant N \sum_{n \leqslant x} s^2(n).$$

The result follows from Lemma 1.

LEMMA 3. *Let $S(k)$ denote the sequence of distinct numbers of the form $(p_1 - 1)(p_2 - 1) \ldots (p_k - 1)$, $(p_i \neq p_j)$, and let $V_k(x)$ denote the counting function of $S(k)$. For $x > e$, define*

$$W_{k+1}(x) = \int_e^x t^{-2}\, V_k(t)\,dt.$$

Then for $x > e$ and $k = 1, 2, \ldots$ we have

$$V_{k+1}(x) \geqslant C_6 (x/\log x) W_{k+1}(y),$$

y being defined as in the previous lemma. (C_6 is independent of k.)

Proof. Let $\{m_i\} = S(k) \cap (0, y)$, so that

$$m_i = (p_1^{(i)} - 1)(p_2^{(i)} - 1) \ldots (p_k^{(i)} - 1) < y.$$

The integers n represented in Lemma 1 belong to $S(k+1)$: if $n = m_i(p-1)$, p is different from $p_j^{(i)}$, $1 \leqslant j \leqslant k$ since $p > 2y$. (This gives $p - 1 > y$ as by hypothesis, $y > e$.) Next,

$$\sum_i \frac{1}{m_i} \geqslant \sum_i \int_{m_i}^{y} t^{-2}\,dt \geqslant \int_e^y t^{-2}\,V_k(t)dt,$$

and now using Lemma 2, we obtain the result stated.

Proof of the theorem. Evidently $V(x) \geqslant V_k(x)$ for every k, and, starting with $V_1(x) \geqslant \pi(x)$, we set out to estimate the $V_k(x)$ from below by induction. The induction hypothesis is that

$$V_{k+1}(x) \geqslant C_7 \frac{x}{\log x} \frac{(C_6 \log\log x)^k}{k!} 2^{-k(k+1)}, \tag{4}$$

which is true for $k = 0$. By Lemma 3, then,

$$\begin{aligned}(x/\log x)^{-1}\,V_{k+2}(x) &\geqslant C_6\,W_{k+2}(y)\\ &\geqslant C_6\,C_7 \int_e^y \frac{(C_6 \log\log t)^k\,2^{-k(k+1)}}{k!\,t\log t}\,dt\\ &\geqslant C_7 \frac{(C_6\log\log y)^{k+1}}{(k+1)!} 2^{-k(k+1)}\\ &\geqslant C_7 \frac{(C_6\log\log x)^{k+1}}{(k+1)!} 2^{-(k+1)(k+2)},\end{aligned}$$

in view of the relation between x and y. This completes the induction, so that (4) holds for all k.

Next, since $k! \leqslant k^k$, we have, from (4) and $V(x) \geqslant V_{k+1}(x)$,

$$\log \frac{V(x)\log x}{x} \geqslant k \log\log\log x - k^2 \log 2 - C_8\,k \log k.$$

Now we choose $k = [(\log 4)^{-1} \log\log\log x]$, and obtain the result stated, for every $A < 1/\log 16$.

Finally, we would like to ask the following question: is it true that, for every $c > 1$, $\lim V(cx)/V(x) = c$?

References

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