EXTREMAL RAMSEY THEORY FOR GRAPHS

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1. *Introduction.*

If G and H are graphs, define the *Ramsey number* $r(G,H)$ to be the least number p such that if the lines of the complete graph K_p are colored red and blue (say), either the red subgraph contains a copy of G or the blue subgraph contains H. Also set $r(G) = r(G,G)$; these are called the *diagonal Ramsey numbers*. These definitions are taken from Chvátal and Harary [1]; other terminology will follow Harary [2]. For a survey of known results concerning these generalized Ramsey numbers, see [3].

A natural question about Ramsey numbers is how small, or large, they can be. We make some definitions. If $\mathcal{G}$ is a set of graphs, define $\mathrm{exr}(\mathcal{G})$ by

$$\mathrm{exr}(\mathcal{G}) = \min_{G\in\mathcal{G}} r(G) ;$$

also if $\mathcal{G}$ and $\mathcal{H}$ are sets of graphs, set

$$\mathrm{exr}(\mathcal{G},\mathcal{H}) = \min_{\substack{G\in\mathcal{G}\\ H\in\mathcal{H}}} r(G,H).$$

We also define $\mathrm{Exr}(\mathcal{G})$ and $\mathrm{Exr}(\mathcal{G},\mathcal{H})$ similarly, with min replaced by max. Note that $\mathrm{exr}(\mathcal{G},\mathcal{G}) \le \mathrm{exr}\,(\mathcal{G})$. This inequality can be strict. In fact, theorem 4.1 of [4] shows that $\mathrm{exr}(\mathcal{G},\mathcal{G})/\mathrm{exr}(\mathcal{G})$ can be made arbitrarily small, even for sets $\mathcal{G}$ containing only two graphs. Likewise, theorem 2.5 of [4] shows that $\mathrm{Exr}(\mathcal{G},\mathcal{G})/\mathrm{Exr}(\mathcal{G})$ can be made arbitrarily large.

2. *Two Off-Diagonal Results.*

Define $\mathcal{C}_n$ to be the set of connected graphs on n points, $\mathcal{G}_n$ to be the set of graphs on n points with no isolates, and $\mathcal{K}_n$ to

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be the set of graphs with chromatic number $\chi = n$.

THEOREM 2.1. $\operatorname{exr}(\mathcal{C}_m, \mathcal{K}_n) = (m-1)(n-1) + 1$.

Proof. By lemma 4 of [5], $r(G,H) \geq (m-1)(n-1) + 1$, where $G \in \mathcal{C}_m$ and $H \in \mathcal{K}_n$. On the other hand, in [6] it is shown that $r(T, K_n) = (m-1)(n-1) + 1$, where T is any tree on m points; hence the result follows.

THEOREM 2.2.

$$\operatorname{exr}(\mathcal{G}_m, \mathcal{K}_n) = \begin{cases} m+n-2 & \textit{if } m \textit{ is even,} \\ \max(m+n-2,\ 2n-1) & \textit{if } m \textit{ is odd.} \end{cases}$$

Proof. Consider first a two-colored K_{m+n-3} in which the red graph consists of just a K_{m-1}, so that the blue graph is $K_{n-2} + (m-1)K_1$. Clearly the red graph cannot contain a member of $\mathcal{G}_m$. Furthermore the blue graph has chromatic number $n-1$ and so cannot contain a member of $\mathcal{K}_n$. Hence, $\operatorname{exr}(\mathcal{G}_m, \mathcal{K}_n) \geq m + n - 2$. Now suppose m to be odd. Then any member G of $\mathcal{G}_m$ has a component which has an odd number of points, and so at least three points. Hence, by theorem 2.1, $r(G,H) \geq 2n - 1$, where H is any member of $\mathcal{K}_n$. From these facts, the right-hand side of the statement of the theorem is a lower bound for $\operatorname{exr}(\mathcal{G}_m, \mathcal{K}_n)$.

It remains to exhibit $G \in \mathcal{G}_m$ and $H \in \mathcal{K}_n$ for which the lower bound is achieved. Specifically, we take $G = (m/2)K_2$ when m is even, $G = P_3 \cup ((m-3)/2)K_2$ when m is odd, and $H = K_n$ in either case. We could evaluate the desired Ramsey numbers by means of a result of Stahl [7], but we will evaluate them directly.

First, we prove that if m is even, then $r\left((m/2)K_2, K_n\right) = m + n - 2$. We use induction on m. The result is trivial when $m = 2$; now suppose it to have been proved for $m - 2$, $m \geq 4$, and consider a two-colored K_{m+n-2}. By hypothesis, $r\left(\left((m-2)/2\right) K_2, K_n\right) = m + n - 4$, so we may assume the red graph contains $((m-2)/2)K_2$. If the remaining n points induce any red line, we have a red $(m/2)K_2$. If not, we have a red K_n, and in either case the proof is complete. It remains to show that if m is odd, then $r\left(P_3 \cup (m-3)K_2, K_n\right) = \max(m+2, 2n-1)$. When $m = 3$, this fact follows from the result of [6] given in the proof of theorem 2.1. For $m > 3$, the desired result follows by induction in the same manner as the above. This completes the proof.

3. *Connected Graphs with Specified Chromatic Number.*

Erdös has conjectured that $\operatorname{exr}(K_n) = r(K_n)$. Except when $n = 2$ or 3 this conjecture is unsettled, and we will not consider it further. Rather we will evaluate $\operatorname{exr}(C_m \cap K_n)$ and similar extremal Ramsey numbers when m is large. We begin by giving some lemmas; all are sometimes sharp, as will be seen. Recall that if F is a graph, $F+K_1$ is formed by adjoining one point to F and connecting that point to each point of F by an edge.

LEMMA 3.1. *If* $F = F_1 \cup F_2$ *and* $G = G_1 \cup G_2$, *then*

$$r(F+K_1, G+K_1) \le \max\Big(r(F_1+K_1, G_1+K_1),\; p(F_1) + r(F_2, G+K_1) + r(F+K_1, G),\; p(G_1) + r(G_2, F+K_1) + r(G+K_1, F)\Big).$$

Proof. Let n be the right-hand side of the above and consider any two-colored K_n. Since $n \ge r(F_1+K_1, G_1+K_1)$, we have either a red F_1+K_1 or a blue G_1+K_1; without loss of generality we may assume the former. Consider the distinguished point of the F_1+K_1 and consider the lines emanating from it that do not meet the given F_1. There are at least $r(F_2, G+K_1) + r(F+K_1, G) - 1$ such lines, so either at least $r(F_2, G+K_1)$ are red or at least $r(F+K_1, G)$ are blue. But in either case we see that we have either a red $F+K_1$ or a blue $G+K_1$. This completes the proof.

Next we state a simple lemma, which is lemma 3.1 of [4].

LEMMA 3.2. *If* $t = r(F,G)$, *then* $r(F+K_1, G) \le r(K_{1,t}, G)$.

Combining this with a special case of lemma 3.1 yields the following.

LEMMA 3.3. *Under the conditions of lemma* 3.1, *if* F_2 *and* G_2 *consist entirely of isolated points, then*

$$r(F+K_1, G+K_1) \le \max\Big(r(F_1+K_1, G_1+K_1),\ p(F) + r(K_{1,t}, G_1),\ p(G) + r(K_{1,t}, F_1)\Big),$$

where $t = \max\Big(r(F_1, G_1),\ p(F),\ p(G)\Big)$.

Proof. By lemma 3.1, and since the desired result is symmetrical in F and G, it is clearly sufficient to show that $p(F_1) + r(F_2, G+K_1) + r(F+K_1, G) \le p(F) + r(K_{1,t}, G_1)$. But $r(F_2, G+K_1) = p(F_2)$ trivially, and by lemma 3.2, $r(F+K_1, G) \le r(K_{1,t}, G) = r(K_{1,t}, G_1)$, the final equality holding because $r(K_{1,t}, G_1) \ge t \ge p(G)$. This completes the proof.

The above lemma can be applied in a variety of situations, but we will be content with the case in which F_1 and G_1 are complete.

THEOREM 3.1. *Let* $m \ge n \ge \ell \ge k$, $F_1 = K_k$, $F_2 = (m-k)K_1$, $G_1 = K_\ell$, $G_2 = (n-\ell)K_1$, $F = F_1 \cup F_2$, *and* $G = G_1 \cup G_2$. *If also* $n \ge r(K_k, K_\ell)$ *and* $m\ell + 1 \ge r(K_{k+1}, K_{\ell+1})$, *then*

$$r(F+K_1, G+K_1) = m\ell+1 .$$

Proof. That $r(F+K_1, G+K_1) \ge m\ell+1$ follows from theorem 2.1, since $p(F+K_1) = m+1$ and $\chi(G+K_1) = \ell+1$. Now we show the inequality in the other direction by applying lemma 3.3. We have $t = \max(r(K_k, K_\ell), m, n) = m$. Moreover, $p(F) + r(K_{1,t}, G_1) = m + (\ell-1)m+1 = m\ell+1$, and $p(G) + r(K_{1,t}, F_1) = n + (k-1)m + 1 \le m\ell+1$. Therefore, $r(F+K_1, G+K_1) \le m\ell+1$ and the proof is complete.

From this the next theorem follows immediately.

THEOREM 3.2. *If* $m \geq n \geq \ell \geq k$, $n-1 \geq r(K_{k-1},K_{\ell-1})$, *and* $(m-1)(\ell-1) + 1 \geq r(K_k,K_\ell)$, *then*

$$\mathrm{exr}(C_m \cap K_k, C_n \cap K_\ell) = (m-1)(\ell-1) + 1 .$$

Of course, in the case $m = n$, $k = \ell$, we have under the above conditions that $\mathrm{exr}(C_m \cap K_k) = \mathrm{exr}(C_m \cap K_k, C_m \cap K_k) = (m-1)(k-1) + 1$. Using the facts that $r(K_2,K_n) = n$, $r(K_3) = 6$, $r(K_4) = 18$, and $r(K_3,K_4) = 9$, we have the following three results:

THEOREM 3.3. *If* $m \geq n \geq 3$ *and* $m \geq 4$, *then* $\mathrm{exr}(C_m \cap K_3, C_n \cap K_3) = 2m-1$.

THEOREM 3.4. *If* $m \geq n \geq 7$, *then* $\mathrm{exr}(C_m \cap K_4, C_n \cap K_4) = 3m-2$.

THEOREM 3.5. *If* $m \geq n \geq 4$, *then* $\mathrm{exr}(C_m \cap K_3, C_n \cap K_4) = 3m-2$.

The extremal graphs in the three above results can, of course, be taken to be of the form of those in theorem 3.1, namely, a complete graph with a sufficiently large star emanating from some point. When the star is small the situation is different. It is interesting to consider the case in which the star consists of a single line, so that the graphs have the form $K_n \cdot K_2$. We conjecture that $r(K_n \cdot K_2) = r(K_n)$ when $n \geq 4$. It is not hard to see that this would follow if $r(K_m,K_n) \geq r(K_m,K_{n-1}) + m$ for all $m \geq n \geq 3$; this question in classical Ramsey theory does not seem to have been investigated. Tantalizingly, it is easy to prove that $r(K_m,K_n) \geq r(K_m,K_{n-1}) + m-1$ if $m \geq n \geq 3$, but the stronger result has resisted our efforts.

4. *Connected Graphs.*

In this section we consider $\mathrm{ex\acute{r}}(C_n,C_n)$ which, as will be seen, also equals $\exr(C_n)$. Since every connected graph has a spanning subtree, the extremal graphs may be taken to be trees, and hence bipartite graphs. With this in mind, define $\mathcal{B}_{k,\ell}$ to be the set of connected bipartite graphs with maximal independent sets of k and ℓ points. For the following lemma, for each $k,\ell \geq 2$ define $S_{k,\ell}$ to be the following tree: take a copy of P_4 (a path of length three) and append a star $K_{1,k-2}$ to one end and a star $K_{1,\ell-2}$ to the other end. Note that $S_{k,\ell} \in \mathcal{B}_{k,\ell}$.

LEMMA 4.1. *If* $k \geq \ell \geq 2$, *then*

$$r(S_{k,\ell}) = \max(2k-1,\ k+2\ell-1)\ .$$

Proof. That $r(S_{k,\ell}) = \max(2k-1,\ k+2\ell-1)$ follows from lemma 1 of [3]. To prove the reverse inequality consider a two-colored complete graph on $\max(2k-1,\ k+2\ell-1)$ points. By a result of Rosta (personal communication; see [3]), $r(K_{1,k-1} \cup K_{1,\ell}) = \max(2k-1,\ k+2\ell-1)$, so that without loss of generality we may assume that we have a red $K_{1,k-1} \cup K_{1,\ell}$. Let U denote the set of endpoints of the $K_{1,k-1}$ in question and let u denote the center of this star. Similarly, let V and v denote respectively the set of endpoints and the center of the $K_{1,\ell}$ in question. Let W denote all points not in the $K_{1,k-1} \cup K_{1,\ell}$, and note that W has at least $\ell-2$ points.

If any line joining U and V is red, we have a red $S_{k,\ell}$, so we may assume all such lines are blue. Now if any line joining V and $W \cup \{u\}$ is blue, we have a blue $S_{k,\ell}$, so we may assume that all lines joining V and $W \cup \{u\} \cup \{v\}$ are red. But now the red graph contains a copy of $S_{k,\ell}$ and the proof is complete.

THEOREM 4.1. *Let* $k \geq \ell \geq 1$. *If* $\ell = 1$ *and* k *is odd, then* $\exr(\mathcal{B}_{k,\ell}) = 2k$; *otherwise* $\exr(\mathcal{B}_{k,\ell}) = \max(2k-1,\ k+2\ell-1)$. *In all cases,* $\exr(\mathcal{B}_{k,\ell}, \mathcal{B}_{k,\ell}) = \exr(\mathcal{B}_{k,\ell})$.

Proof. If $\ell = 1$, $\mathcal{B}_{k,\ell} = \{K_{1,k}\}$ and the theorem follows in this case from the evaluation of $r(K_{1,k})$ in [8]. If $\ell \geq 2$, that $\max(2k-1,\ k+2\ell-1)$ is a lower bound for $\mathrm{exr}(\mathcal{B}_{k,\ell})$ and $\mathrm{exr}(\mathcal{B}_{k,\ell},\ \mathcal{B}_{k,\ell})$ follows again from lemma 1 of [3]. That it is an upper bound for $\mathrm{exr}(\mathcal{B}_{k,\ell})$, and hence for $\mathrm{exr}(\mathcal{B}_{k,\ell},\ \mathcal{B}_{k,\ell})$, follows from lemma 4.1.

We may now apply this to C_n.

THEOREM 4.2. *If* $n \geq 3$, *then*

$$\mathrm{exr}(C_n) = \mathrm{exr}(C_n, C_n) = \left\lceil \frac{4n-1}{3} \right\rceil.$$

Proof. In view of theorem 4.1, it suffices (except in the trivial case $n = 2$) to show that for integral k,

$$\min_{1 \leq k \leq n-1} \max(2k-1,\ 2n-k-1) = \left\lceil \frac{4n-1}{3} \right\rceil.$$

If k is permitted to assume rational values, the minimum occurs at $k = 2n/3$. Hence to find the desired minimum one need only consider $k = [2n/3]$ and $k = [(2n+2)/3]$. If n is of the form $3m$, $k = 2m$ in either case, and $\max(2k-1,\ 2n-k-1) = 4m-1 = \left\lceil \frac{4n-1}{3} \right\rceil$. If $n = 3m+1$, $k = 2m$ or $2m+1$ and in either case $(2k-1,\ 2n-k-1) = 4m+1 = \left\lceil \frac{4n-1}{3} \right\rceil$. Finally, if $n = 3m+2$, $k = 2m+1$ or $2m+2$. In the former case $\max(2k-1,\ 2n-k-1) = 4m+2$; in the latter, $\max(2k-1,\ 2n-k-1) = 4m+3$. Thus the desired minimum is $4m+2 = \left\lceil \frac{4n-1}{3} \right\rceil$ again. This completes the proof.

5. *Arbitrary Graphs Without Isolates.*

In this section we consider $\mathrm{exr}(G_n, G_n)$; as will be seen, it has not been possible to obtain an exact result. It is clear that we may restrict our attention to forests of stars, since every member of G_n has a spanning forest of nontrivial stars. We begin with results leading to an upper bound.

LEMMA 5.1. *If* $k, n \geq 1$, *then* $r(kK_{1,n}) \leq kn+2k+2n$.

Proof. We use induction on k. As was mentioned in the proof of theorem 4.1, $r(K_{1,n}) \le 2n$, so the theorem holds for $k = 1$. Now assume the theorem to have been proved for $k-1$, and consider a two-colored complete graph on $kn+2k+2n$ points. Suppose the graph contains a red and a blue $K_{1,m}$ with all their endpoints in common. (In the terminology of [9], this would be called a "bowtie".) Then if these $n+2$ points are removed, $(k-1)n+2(k-1)+2n$ points remain, and by the induction hypothesis these points induce a monochromatic $(k-1)K_{1,n}$. Combining this with the $K_{1,n}$ of the appropriate color from the bowtie, we have the desired $kK_{1,n}$. Hence the theorem holds for k if the graph contains a bowtie.

We now show that if the graph does not contain a bowtie it contains a monochromatic $kK_{1,n}$. Consider any point p of the graph. From this point emanate red lines leading to a set X of points and blue lines leading to a set Y. If both X and Y have at least $2n$ points, note that at least half of the lines between X and Y are one color, say red. Hence, some point of X has at least n points leading to Y, yielding a bowtie; so we may assume that Y (say) has $\le 2n-1$ points, leaving $\ge kn+2k$ points in X. If X induces a blue $K_{1,n}$ we again have a bowtie. But it is easily seen, either directly or by lemma 1 of [9], that $r\left((k-1)K_{1,n}, K_{1,n}\right) \le kn+k$. Hence, if X does not induce a blue $K_{1,n}$, it induces a red $(k-1)K_{1,n}$. Add to this one more red $K_{1,n}$ induced by n of the remaining points of X and the point p, yielding a red $kK_{1,n}$. This completes the proof.

THEOREM 5.1. *For some constant* c,

$$\operatorname{exr}(G_n, G_n) \le \operatorname{exr}(G_n) \le n+c\sqrt{n} .$$

Proof. By lemma 5.1, $r(kK_{1,k}) \le k^2 + 4k$. Hence, when $n = k^2 + k$ the theorem holds (with $c = 4$, say) and since $\operatorname{exr}(G_n)$ is a monotone function of n, it is clear that the theorem holds in general with some larger value of c.

Now we consider lower bounds.

THEOREM 5.2. *There is a* $c_0 > 0$ *such that for all* $n \geq 3$,

$$n + \log_2 n - c_0 \ \ell n \ \ell n \ 3n < \mathrm{exr}(G_{n}, G_{n}) \leq \mathrm{exr}(G_{n}).$$

Proof. The constant c_0 will be chosen during the course of the proof. Let F and G be two members of G_n; we may assume that

$$F = K_{1,k_1} \cup \dots \cup K_{1,k_s},$$

$$G = K_{1,\ell_1} \cup \dots \cup K_{1,\ell_t}.$$

Note that $k_1 + \dots + k_s = n-s$, $\ell_1 + \dots + \ell_t = n-t$. Set $q = [\log_2 n - c_0 \ \ln \ \ln \ 3n]$ and consider the two-colored complete graph on $n+q$ points in which the blue graph is K_{n-1}. The blue graph cannot contain G. Furthermore the red graph can contain F only if $s \leq q + 1$ since at least one point of each star must be in the complement of the K_{n-1}. Hence, $r(F,G) \geq \log_2 n - c_0 \ \ln \ \ln \ 3n$ unless $s \leq q + 1$, so we may assume this inequality holds, and similarly for t.

Let X be the set of all numbers of the form

$$k_{i_1} + k_{i_2} + \dots + k_{i_u} + u ,$$

where $i_1 < i_2 < \dots < i_u$, and let Y be the set of all numbers of the form

$$\ell_{i_1} + \ell_{i_2} + \dots + \ell_{i_v} + t - v ,$$

where $i_1 < i_2 < \dots < i_v$.

The number j of elements of $X \cup Y$ is no more than

$$\begin{aligned} 2^s + 2^t &\leq 2 \cdot 2^{q+1} \\ &\leq n/(\log_2 n + 2) , \end{aligned}$$

provided c_0 has been chosen appropriately.

Consequently, if we arrange the elements of $X \cup Y$ in increasing order

$$0 = z_1 < z_2 < \ldots < z_j = n ,$$

then $z_r - z_{r-1} \geq \log_2 n + 2 \geq q + 2$ for some r. Now form a two-colored complete graph on $n+q$ points in which the red graph is

$$K_{z_r - 1} \cup K_{n+q-z_r+1} .$$

The red graph cannot contain G, since at most z_{r-1} points of the K_{z_r-1} can be used to help form G, so that the total number of points that could be used is no more than

$$z_{r-1} + n + q - z_r + 1 \leq n - 1 .$$

Likewise, the blue graph cannot contain F, since again at most z_{r-1} points of the K_{z_r-1} can be used. This completes the proof.

We conjecture that theorem 5.2 gives the true behavior of $\mathrm{exr}(G_n)$, and that the extremal graphs are roughly of the form

$$K_{1,[n/2]} \cup K_{1,[n/4]} \cup K_{1,[n/8]} \cup \cdots .$$

Therefore, it would be highly desirable to extend Rosta's result, mentioned in the proof of lemma 4.1, to forests of more than two stars.

6. *Problems and Conjectures.*

Various questions have already been raised in the course of this paper. We call particular attention to the problem of determining, or at least improving the estimates of, $\mathrm{exr}(G_n)$.

We have not given any results here concerning $\mathrm{Exr}(\mathcal{G})$ or $\mathrm{Exr}(\mathcal{G},\mathcal{H})$. One very interesting problem is that of determining $\mathrm{Exr}(\mathcal{T}_n)$ and $\mathrm{Exr}(\mathcal{T}_n,\mathcal{T}_n)$, where $\mathcal{T}_n$ is the class of trees on n points.

We conjecture that $\mathrm{Exr}(T_n) = \mathrm{Exr}(T_n, T_n) = 2n - 2$ when n is even and $2n - 3$ when n is odd, with the extremal graphs being stars. The best that is presently known is $\mathrm{Exr}(T_n) \leq 4n + 1$; see [10].

Another interesting set of graphs is L_n, the set of graphs with n lines. Presumably, when $n = \binom{k}{2}$, $k \geq 4$, $\mathrm{Exr}(L_n) = r(K_k)$, but this seems hard. Perhaps even more difficult to treat is $\mathrm{exr}(L_n)$. Here we do not even have a reasonable conjecture.

Finally, we call attention to [4] which both raises and partially solves a number of problems which may be considered extremal in nature.

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