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Families of sets whose pairwise intersections have prescribed cardinals or order types

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1. *Introduction.* For a given index set I, let us consider a family $(A_\nu: \nu \in I)$ of subsets of a set E. In this note we deal with some aspects of the following question: to what extent is it possible to prescribe the cardinalities, or the order types in case E is ordered, of the sets A_ν and of their pairwise intersections? In (1) the authors have shown that, given any regular cardinal a, there is a family of a^+ sets of cardinal a whose pairwise intersections are arbitrarily prescribed to be either less than or equal to a. In Theorem 1 below we prove a stronger result which states that if a is regular, say $a = \aleph_\alpha$, and if E is well-ordered and of order type ω_α^3, then one can find a^+ subsets A_ν of E, each of type ω_α^2, whose pairwise intersections are arbitrarily prescribed to be either of type ω_α or of a type less than ω_α. By way of contrast, Theorem 2 below implies – this is its special case $m = \aleph_\omega$; $n = \aleph_2$; $p = \aleph_0$ – that, assuming the Generalized Continuum Hypothesis (GCH), there do not exist $\aleph_{\omega+1}$ sets A_μ, each of cardinal at most $\aleph_\omega$, such that $\aleph_2$ of them have pairwise finite intersections, whereas all other pairs of sets A_ν have a denumerable intersection. Theorem 3 gives another case in which some type of prescription of the sizes of the intersections cannot be satisfied. Finally, Theorem 4 asserts that in Theorem 3 the condition $cfp \neq cfm$ cannot be omitted. The paper concludes with some remarks on open questions.

2. *Notation.* We use the *obliterator* $\hat{}$, an operator which removes from a well-ordered sequence the term above which it is placed. Roman capital letters denote sets. If A is ordered then $\operatorname{tp} A$ denotes the order type of A. If A_ν is a set, for $\nu \in I$, where $I \neq \varnothing$, then we put† $A_{[I]} = \bigcap_\nu (\nu \in I) A_\nu$. The relation $A \subset B$ denotes inclusion in the wide sense, and symbols such as $\{\mu, \nu\}_{\neq}$ have their obvious meaning. For every cardinal a, we put $\underline{a} = \{\gamma: \gamma = \text{ordinal}; |\gamma| < a\}$, and if $a \geq \aleph_0$ then cfa denotes the least cardinal b such that there is a representation $a = \sum_\nu (\nu \in \underline{b}) x_\nu$, where $x_\nu < a$ for $\nu \in \underline{b}$. Thus a is *regular* if and only if $cfa = a$.

3. *Results.* THEOREM 1. *Let a be a regular cardinal, $a = \aleph_\alpha$, and $f(\mu, \nu) \in \{0, 1\}$ for $\mu < \nu < \omega_{\alpha+1}$. Then there are subsets $A(0), A(1), \ldots, \hat{A}(\omega_{\alpha+1})$ of $\{0, 1, \ldots, \hat{\omega}_\alpha^3\}$ each of type ω_α^2, such that, for $\mu < \nu < \omega_{\alpha+1}$,*

$$\left.\begin{aligned} \operatorname{tp}(A(\mu) \cap A(\nu)) &< \omega_\alpha \quad \text{if} \quad f(\mu, \nu) = 0, \\ &= \omega_\alpha \quad \text{if} \quad f(\mu, \nu) = 1. \end{aligned}\right\} \tag{1}$$

† For typographical convenience we place the conditions relating to operations Σ, $\cup$, $\cap$ next to the operational symbol.

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THEOREM 2. *Assume GCH. Let* $m, n, p \geqslant \aleph_0$; $m > n$; $m > p^+$;

$$cfm \neq p^+; \quad |I| = m^+; \quad J \subset I; \quad |J| = n$$

Then there is no family $(A_\nu: \nu \in I)$ *such that* $|A_\nu| \leqslant m$ *for* $\nu \in I$;

$$\begin{aligned} |A_\mu \cap A_\nu| &< p \quad \text{if} \quad \{\mu, \nu\}_{\neq} \subset J, \\ &= p \quad \text{if} \quad \mu \neq \nu; \quad \mu \in I - J; \quad \nu \in I. \end{aligned}$$

THEOREM 3. *Assume GCH. Let* $\aleph_0 \leqslant p < m$; $cfp \neq cfm$;

$$|I| = m^+; \quad |A| = |B_\nu| = m; \quad |A \cap B_\nu| = p \quad \text{for} \quad \nu \in I.$$

Then there is $M \subset I$ *such that* $|M| = m^+$ *and* $|A \cap B_{[M]}| = p$ *and hence* $|B_\mu \cap B_\nu| \geqslant p$ *for* $\mu, \nu \in M$.

THEOREM 4. *Assume GCH. Let* $\aleph_0 \leqslant p \leqslant m$; $cfp = cfm$; $|I| = m^+$; $|A| = m$. *Then there is a family* $(B_\nu: \nu \in I)$ *such that* $|B_\nu| = m$ *and* $|A \cap B_\nu| = p$ *for* $\nu \in I$, *whereas* $|B_\mu \cap B_\nu| < p$ *for* $\{\mu, \nu\}_{\neq} \subset I$.

4. *Proof of Theorem* 1. Put, for $\xi, \eta < \omega_\alpha$,

$$S(\xi, \eta) = \{\omega_\alpha^2 \xi + \omega_\alpha \eta + \theta : \theta < \omega_\alpha\}.$$

We shall construct $A(\nu)$ inductively. Let $\nu_0 < \omega_{\alpha+1}$;

$$\begin{gathered} A(0), \ldots, \hat{A}(\nu_0) \subset \{0, \ldots, \hat{\omega}_\alpha^3\}, \\ \operatorname{tp} A(\nu) = \omega_\alpha^2 \quad \text{for} \quad \nu < \nu_0, \\ |A(\nu) \cap S(\xi, \eta)| = 1 \quad \text{if} \quad \nu < \nu_0 \quad \text{and} \quad \xi, \eta < \omega_\alpha. \end{gathered}$$

Suppose that (1) holds for $\mu < \nu < \nu_0$. We shall define $A(\nu_0)$, and in such a way that (1) holds for $\mu < \nu = \nu_0$.

In what follows dependence on ν_0 will often not be shown in our notation. It is clearly possible to choose sets $B(0), \ldots, \hat{B}(t)$ in such a way that

$$t \leqslant \omega_\alpha; \quad \{B(\tau): \tau < t\} = \{A(\nu): \nu < \nu_0\}$$

and, for $\mu < \nu_0$,

$$\left.\begin{aligned} |\{\tau < t: B(\tau) = A(\mu)\}| &= 1 \quad \text{if} \quad f(\mu, \nu_0) = 0 \\ &= \aleph_\alpha \quad \text{if} \quad f(\mu, \nu_0) = 1. \end{aligned}\right\} \tag{2}$$

We shall define $x(\xi, \eta) \in S(\xi, \eta)$ for $\xi, \eta < \omega_\alpha$, and we shall put

$$A(\nu_0) = \{x(\xi, \eta): \xi, \eta < \omega_\alpha\}. \tag{3}$$

Case 1. $t < \omega_\alpha$. Then, by (2), $f(\mu, \nu_0) = 0$ for $\mu < \nu_0$, and we have $\nu_0 < \omega_\alpha$. Hence we can choose, for all $\xi, \eta < \omega_\alpha$, $x(\xi, \eta) \in S(\xi, \eta) - \bigcup_\nu (\nu < \nu_0) A(\nu)$. Then, by (3), $\operatorname{tp} A(\nu_0) = \omega_\alpha^2$. Moreover, if $\mu < \nu_0$ then $f(\mu, \nu_0) = 0$ and, as required,

$$\operatorname{tp}(A(\mu) \cap A(\nu_0)) = 0 < \omega_\alpha.$$

Case 2. $t = \omega_\alpha$. We shall define $\xi(\theta)$, $\eta(\theta)$ for $\theta < \omega_\alpha$ in such a way that, for all $\theta < \omega_\alpha$,

$$\xi(\theta) < \eta(\theta) < \omega_\alpha, \tag{4}$$

$$\eta(\theta') < \xi(\theta) \quad \text{for} \quad \theta' < \theta. \tag{5}$$

Let $\theta_0 < \omega_\alpha$, and assume that $\xi(\theta)$ and $\eta(\theta)$ have been defined for $\theta < \theta_0$ in such a way that (4) and (5) hold for $\theta < \theta_0$. We shall define $\xi(\theta_0)$ and $\eta(\theta_0)$. Put

$$\bar{\eta}(\theta_0) = \sup\{\eta(\phi) \colon \phi < \theta_0\} \quad \text{if} \quad \theta_0 > 0,$$
$$= 0 \qquad\qquad\qquad \text{if} \quad \theta_0 = 0.$$

Since $\aleph_\alpha$ is regular, we have $\bar{\eta}(\theta_0) < \omega_\alpha$. There is $\mu(\theta_0) < \nu_0$ such that $B(\theta_0) = A(\mu(\theta_0))$. Put†

$$C(\theta_0) = B(\theta_0) - \bigcup_\phi (\phi < \theta_0; B(\phi) \neq B(\theta_0)) B(\phi).$$

If $\phi < \theta_0$ and $B(\phi) \neq B(\theta_0)$, then $\operatorname{tp}(B(\phi) \cap B(\theta_0)) \leqslant \omega_\alpha$. Hence $\operatorname{tp} C(\theta_0) = \operatorname{tp} B(\theta_0) = \omega_\alpha^2$. It now follows that there are numbers $\xi(\theta_0)$, $\eta(\theta_0)$ such that

$$\bar{\eta}(\theta_0) < \xi(\theta_0) < \eta(\theta_0) < \omega_\alpha,$$
$$C(\theta_0) \cap S(\xi(\theta_0), \eta(\theta_0)) \neq \varnothing. \tag{6}$$

This completes the definition of $\xi(\theta)$ and $\eta(\theta)$ for $\theta < \omega_\alpha$ so that (4), (5), (6) hold for $\theta, \theta_0 < \omega_\alpha$. We now define $x(\xi, \eta)$ for $\xi, \eta < \omega_\alpha$. Let $\xi_1, \eta_1 < \omega_\alpha$. By (4) and (5) there is $\theta_0(\xi_1, \eta_1) < \omega_\alpha$ such that

$$\eta(\phi) < \max\{\xi_1, \eta_1\} \leqslant \eta(\theta_0(\xi_1, \eta_1)) \tag{7}$$

for $\phi < \theta_0(\xi_1, \eta_1)$. For, this only means that $\theta_0(\xi_1, \eta_1)$ is the least ordinal $\lambda < \omega_\alpha$ satisfying $\eta(\lambda) \geqslant \max\{\xi_1, \eta_1\}$, and such an ordinal λ exists by (4) and (5).

Case 2a. Either (i)

$$(\xi_1, \eta_1) \neq (\xi(\theta_0(\xi_1, \eta_1)), \eta(\theta_0(\xi_1, \eta_1))),$$

or (ii)
$$(\xi_1, \eta_1) = (\xi(\theta_0(\xi_1, \eta_1)), \eta(\theta_0(\xi_1, \eta_1)))$$

and
$$f(\mu(\theta_0(\xi_1, \eta_1)), \nu_0) = 0.$$

In this case we can choose

$$x(\xi_1, \eta_1) \in S(\xi_1, \eta_1) - \bigcup_\phi (\phi < \theta_0(\xi_1, \eta_1)) B(\phi).$$

Case 2b.
$$(\xi_1, \eta_1) = (\xi(\theta_0(\xi_1, \eta_1)), \eta(\theta_0(\xi_1, \eta_1)))$$

and
$$f(\mu(\theta_0(\xi_1, \eta_1)), \nu_0) = 1.$$

Then, by (6), we can choose

$$x(\xi_1, \eta_1) \in C(\theta_0(\xi_1, \eta_1)) \cap S(\xi_1, \eta_1).$$

This completes the definition of $x(\xi, \eta)$ for $\xi, \eta < \omega_\alpha$, and we can define $A(\theta_0)$ by (3). Since $x(\xi, \eta) \in S(\xi, \eta)$, we have $\operatorname{tp} A(\nu_0) = \omega_\alpha^2$. Let $\mu_0 < \nu_0$. We now show that (1) holds for $(\mu, \nu) = (\mu_0, \nu_0)$. There is a least number $\phi_0 < \omega_\alpha$ such that $B(\phi_0) = A(\mu_0)$.

Case A. $f(\mu_0, \nu_0) = 0$. We shall show that

$$A(\mu_0) \cap A(\nu_0) \subset \bigcup_{\xi, \eta} (\xi, \eta \leqslant \eta(\phi_0)) S(\xi, \eta), \tag{8}$$

which would imply $\operatorname{tp}(A(\mu_0) \cap A(\nu_0)) \leqslant (\eta(\phi_0) + 1)^2 < \omega_\alpha$. Assume that ξ_2, η_2 are such that $\eta(\phi_0) < \max\{\xi_2, \eta_2\} < \omega_\alpha$. Then, by (7),

$$\eta(\phi_0) < \max\{\xi_2, \eta_2\} \leqslant \eta(\theta_0(\xi_2, \eta_2))$$

† See footnote in section 2.

and hence, by (4) and (5), $\phi_0 < \theta_0(\xi_2, \eta_2)$. If, in the definition of $x(\xi_2, \eta_2)$, Case $2a$ applies, then we conclude that

$$x(\xi_2, \eta_2) \notin B(\phi_0) = A(\mu_0). \tag{9}$$

If, on the other hand, Case $2b$ applies in the definition of $x(\xi_2, \eta_2)$, then

$$B(\phi_0) = A(\mu_0) \neq B(\theta_0(\xi_2, \eta_2)),$$

in view of $f(\mu_0, \nu_0) = 0$ and $f(\mu(\theta_0(\xi_2, \eta_2)), \nu_0) = 1$. By the definition of $C(\theta_0(\xi_2, \eta_2))$, we again deduce that (9) holds. This proves (8).

Case B. $f(\mu_0, \nu_0) = 1$. Then we can write

$$\{\phi < \omega_\alpha : B(\phi) = A(\mu_0)\} = \{\phi(0), \ldots, \hat{\phi}(\omega_\alpha)\}_<.$$

We shall show that

$$\left.\begin{aligned} A(\mu_0) \cap A(\nu_0) \subset \bigcup_{\xi, \eta} (\xi, \eta \leqslant \eta(\phi(0))) \, S(\xi, \eta) \\ \cup \{x(\xi(\phi(\beta)), \eta(\phi(\beta))) : 0 < \beta < \omega_\alpha\}. \end{aligned}\right\} \tag{10}$$

Let $\eta(\phi(0)) < \max\{\xi_2, \eta_2\} < \omega_\alpha$. Then, by (4), (5) and (7), $\phi(0) < \theta_0(\xi_2, \eta_2)$.

Case B1. $\theta_0(\xi_2, \eta_2) \neq \phi(\beta)$ for $\beta < \omega_\alpha$. Then it follows from the procedure in the Cases $2a$ and $2b$ that (9) holds.

Case B2. $\theta_0(\xi_2, \eta_2) = \phi(\beta_0)$ for some $\beta_0 < \omega_\alpha$. Then $\beta_0 > 0$. Let

$$(\xi_2, \eta_2) \neq (\xi(\phi(\beta_0)), \eta(\phi(\beta_0))).$$

Then, again, (9) follows. This completes the proof of (10). The relations (4) and (5) imply that

$$\operatorname{tp}(A(\mu_0) \cap A(\nu_0)) \leqslant \omega_\alpha. \tag{11}$$

On the other hand, we shall now show that

$$x(\xi(\phi(\beta)), \eta(\phi(\beta))) \in A(\mu_0) \cap A(\nu_0) \quad \text{for} \quad \beta < \omega_\alpha. \tag{12}$$

Let $\beta < \omega_\alpha$ and $(\xi_3, \eta_3) = (\xi(\phi(\beta)), \eta(\phi(\beta)))$. Then

$$B(\phi(\beta)) = A(\mu_0); \quad f(\mu_0, \nu_0) = 1; \quad \xi_3 < \eta_3 < \omega_\alpha,$$

We first show that $\theta_0(\xi_3, \eta_3) = \phi(\beta)$. This means that $\eta(\phi(\beta)) \geqslant \eta_3$ and $\eta(\phi) < \eta_3$ for $\phi < \phi(\beta)$. But these two statements are true because of the equation $\eta_3 = \eta(\phi(\beta))$ and the fact that, by (4) and (5), $\eta(\phi)$ increases with ϕ. This proves that $\theta_0(\xi_3, \eta_3) = \phi(\beta)$. We conclude that

$$\xi(\theta_0(\xi_3, \eta_3)) = \xi(\phi(\beta)) = \xi_3,$$
$$\eta(\theta_0(\xi_3, \eta_3)) = \eta(\phi(\beta)) = \eta_3,$$

and that $\mu(\theta_0(\xi_3, \eta_3)) = \mu(\phi(\beta)) = \mu_0$, by the definitions of $\mu(\theta)$ and $\phi(\beta)$. Finally, we have

$$f(\mu(\theta_0(\xi_3, \eta_3)), \nu_0) = f(\mu_0, \nu_0) = 1.$$

Hence, by Case $2b$,

$$x(\xi_3, \eta_3) \in C(\theta_0(\xi_3, \eta_3)) = C(\phi(\beta)) \subset B(\phi(\beta)) = A(\mu_0),$$

and this implies (12). However, (12) yields $\operatorname{tp}(A(\mu_0) \cap A(\nu_0)) \geqslant \omega_\alpha$ which, together with (11), gives $\operatorname{tp}(A(\mu_0) \cap A(\nu_0)) = \omega_\alpha$. This completes the proof of Theorem 1.

5. *Proof of Theorem* 2. Let the family $(A_\nu: \nu \in I)$ satisfy the hypothesis of the theorem. Put

$$m = \aleph_\alpha; \quad n = \aleph_\beta; \quad p = \aleph_\gamma; \quad cfm = \aleph_\delta.$$

Then $\alpha > \beta$; $\alpha > \gamma+1$; $\delta \neq \gamma+1$. By enlarging the sets A_ν suitably, we can achieve that, in addition, $|A_\nu| = m$ for $\nu \in I$. Also, without loss of generality, we assume that $I = \underline{m}^+$ and $J = \underline{n}$. Let μ, ν, ρ, σ always denote ordinals such that

$$\mu, \nu < \omega_\beta \leqslant \rho, \sigma < \omega_{\alpha+1}.$$

Put $S = \bigcup_{\mu,\nu} (\mu < \nu) A_\mu \cap A_\nu$. Then $|S| \leqslant np < m$. Put $A^*_\mu = A_\mu - S$ for all μ. Then $|A^*_\mu| = m$ and $A^*_\mu \cap A^*_\nu = \varnothing$ for $\mu < \nu$. Put

$$N(\rho) = \{\mu: A^*_\mu \cap A_\rho \neq \varnothing\}; \quad W = \{\rho: |N(\rho)| \leqslant p\}.$$

Case 1. $|W| = m^+$. Since $|\{A_\rho \cap S: \rho \in W\}| \leqslant 2^{|S|} \leqslant m$, there are sets W' and S_0 such that $W' \subset W$; $|W'| = |W|$ and $A_\rho \cap S = S_0$ for $\rho \in W'$.

Let $\{\rho, \sigma\}_+ \subset W'$. Then

$$|S_0| = |(A_\rho \cap S) \cap (A_\sigma \cap S)| \leqslant |A_\rho \cap A_\sigma| = p.$$

Since $|\{N(\rho): \rho \in W'\}| \leqslant 2^n \leqslant m$, there are sets W'', N_0 such that

$$W'' \subset W'; \quad |W''| = |W'|; \quad |N_0| \leqslant p; \quad N(\rho) = N_0 \quad \text{for} \quad \rho \in W''.$$

Let $\rho_0 \in W''$ and $\mu \notin N_0$. Then

$$\mu \notin N_0 = N(\rho_0); \quad A^*_\mu \cap A_{\rho_0} = \varnothing; \quad A_\mu \cap A_{\rho_0} \subset S; \quad A_\mu \cap A_{\rho_0} \subset A_{\rho_0} \cap S = S_0.$$

Since $|\{A_\mu \cap A_{\rho_0}: \mu \notin N_0\}| \leqslant 2^{|S_0|} \leqslant m$, there are numbers $\mu_1, \mu_2 \notin N_0$ such that $\mu_1 \neq \mu_2$; $A_{\mu_1} \cap A_{\rho_0} = A_{\mu_2} \cap A_{\rho_0}$. Then

$$p = |A_{\mu_1} \cap A_{\rho_0}| = |(A_{\mu_1} \cap A_{\rho_0}) \cap (A_{\mu_2} \cap A_{\rho_0})| \leqslant |A_{\mu_1} \cap A_{\mu_2}| < p,$$

which is the required contradiction.

Case 2. $|W| \leqslant m$. Put $W^* = \{\rho: \omega_\beta \leqslant \rho < \omega_{\alpha+1}\} - W$. Then $|W^*| = m^+$; $N(\rho) > p$ for $\rho \in W^*$. Since

$$\{N(\rho): \rho \in W^*\} = \bigcup_M (M \subset \underline{n}; |M| = p^+) \{N(\rho): \rho \in W^*; N(\rho) \supset M\}$$

and $$|\{M \subset \underline{n}: |M| = p^+\}| = n^{p^+} \leqslant 2^{np^+} \leqslant m,$$

there are sets W^{**}, N_1 such that $W^{**} \subset W^*$; $|W^{**}| = |W^*|$; $|N_1| = p^+$; $N(\rho) \supset N_1$ for $\rho \in W^{**}$. If $\rho \in W^{**}$ and $\mu \in N_1$, then $A_\rho \cap A^*_\mu \neq \varnothing$, and we can choose $x_{\rho\mu} \in A_\rho \cap A^*_\mu$. Put $X_\rho = \{x_{\rho\mu}: \mu \in N_1\}$ for $\rho \in W^{**}$. Then $x_{\rho\mu} \neq x_{\rho\nu}$ if $\rho \in W^{**}$ and $\{\mu, \nu\}_+ \subset N_1$. If $\{\rho, \sigma\}_+ \subset W^{**}$, then

$$|X_\rho \cap X_\sigma| \leqslant |A_\rho \cap A_\sigma| = p < |N_1| = |X_\rho|.$$

Hence $(X_\rho: \rho \in W^{**})$ is a family of m^+ almost disjoint transversals of the family $(A^*_\mu: \mu \in N_1)$ of p^+ disjoint sets of cardinal m.

On the other hand, by (2), for $r, s \geqslant \aleph_0$, no family of r disjoint sets of cardinal s has s^+ almost disjoint transversals, provided $cfr \neq cfs$ and $cfr \neq s^+$. When applying this result with $r = p^+$ and $s = m$ we obtain a contradiction, and this establishes Theorem 2.

6. *Proof of Theorem* 3. *Case* 1. $p < cfm$. Then, by GCH, $m^p < m^+$, and there are sets X, M such that $|X| = p$; $M \subset I$; $|M| = m^+$; $A \cap B_\nu = X$ for $\nu \in M$. Then $A \cap B_{[M]} = X$.

Case 2: $cfm < cfp$. Then we can write $A = \bigcup_\beta (\beta \in \underline{cfm}) A_\beta$, where $|A_\beta| < m$ for $\beta \in \underline{cfm}$. Let $\alpha \in I$. Then $A \cap B_\alpha = \bigcup_\beta (\beta \in \underline{cfm}) A_\beta \cap B_\alpha$. Because of $cfm < cfp$, there is $\beta(\alpha) \in \underline{cfm}$ such that $|A_{\beta(\alpha)} \cap B_\alpha| = p$ for $\alpha \in I$. Then there is a number $\beta' \in \underline{cfm}$ and a set $M' \subset I$ with $|M'| = m^+$, such that $\beta(\alpha) = \beta'$ for $\alpha \in M'$. Then $|A_{\beta'} \cap B_\alpha| = p$ for $\alpha \in M'$. Since $|A_{\beta'}|^p \leqslant 2^{|A_{\beta'}|p} < m^+$, there are sets X, M satisfying $|X| = p$; $M \subset M'$; $|M| = m^+$; $A_{\beta'} \cap B_\alpha = X$ for $\alpha \in M$. But now we have

$$A \cap B_{[M]} \supset A_{\beta'} \cap B_{[M]} = X.$$

Case 3. $cfp \leqslant cfm \leqslant p$. If $cfm = p$, then $cfp = p = cfm$ which is false. Hence $cfp < cfm < p$. We can write $A = \bigcup_\beta (\beta \in \underline{cfm}) A_\beta$, where $|A_\beta| < m$ for $\beta \in \underline{cfm}$. There is a representation $p = \sum_\delta (\delta \in \underline{cfp}) p_\delta$, where $p_\delta < p$ for $\delta \in \underline{cfp}$. Then $\sup\{p_\delta : \delta \in \underline{cfp}\} = p$. Let $\alpha \in I$ and $\delta \in \underline{cfp}$. Then there is a number $\gamma_\alpha(\delta) \in \underline{cfm}$ such that

$$|\bigcup_\beta (\beta < \gamma_\alpha(\delta)) A_\beta \cap B_\alpha| > p_\delta. \tag{13}$$

For otherwise we would have

$$\begin{aligned} |A \cap B_\alpha| &= |\bigcup_\gamma (\gamma \in \underline{cfm}) \bigcup_\beta (\beta < \gamma) A_\beta \cap B_\alpha| \\ &\leqslant \sum_\gamma (\gamma \in \underline{cfm}) |\bigcup_\beta (\beta < \gamma) A_\beta \cap B_\alpha| \leqslant (cfm) p_\delta < p, \end{aligned}$$

which is a contradiction. Since $cfp < cfm = cfcfm$, we have $\sup\{\gamma_\alpha(\delta) : \delta \in \underline{cfp}\} = \bar{\gamma}_\alpha$, say, where $\bar{\gamma}_\alpha \in \underline{cfm}$. Then, by (13), $|\bigcup_\beta (\beta < \bar{\gamma}_\alpha) A_\beta \cap B_\alpha| > p_\delta$ for $\delta \in \underline{cfp}$, and hence

$$|\bigcup_\beta (\beta < \bar{\gamma}_\alpha) A_\beta \cap B_\alpha| \geqslant p = |A \cap B_\alpha| \geqslant |\bigcup_\beta (\beta < \bar{\gamma}_\alpha) A_\beta \cap B_\alpha|,$$

so that $|\bigcup_\beta (\beta < \bar{\gamma}_\alpha) A_\beta \cap B_\alpha| = p$ for $\alpha \in I$. Now there is an ordinal $\gamma' \in \underline{cfm}$ and a set $M' \subset I$ with $|M'| = m^+$, such that $\bar{\gamma}_\alpha = \gamma'$ for $\alpha \in M'$. Then $|\bigcup_\beta (\beta < \gamma') A_\beta \cap B_\alpha| = p$ for $\alpha \in M'$. We have $|\bigcup_\beta (\beta < \gamma') A_\beta| < m$ and hence $|\bigcup_\beta (\beta < \gamma') A_\beta|^p < m^+$. Therefore we can find sets X, M such that $|X| = p$; $M \subset M'$; $|M| = m^+$;

$$(\bigcup_\beta (\beta < \gamma') A_\beta) \cap B_\alpha = X \quad \text{for} \quad \alpha \in M.$$

Then $A \cap B_{[M]} \supset \bigcup_\beta (\beta < \gamma') A_\beta \cap B_{[M]} = X$, and the theorem follows.

7. *Proof of Theorem* 4. By a theorem of Tarski(3), there are almost disjoint sets $B'_\nu \subset A$ for $\nu \in I$ such that $|B'_\nu| = p$ for $\nu \in I$. Put, for $\nu \in I$, $B_\nu = B'_\nu \cup D_\nu$, where the D_ν are any sets satisfying $|D_\nu| = m$ for $\nu \in I$ and $A \cap D_\nu = B'_\mu \cap D_\nu = \varnothing$ for $\mu, \nu \in I$, and $D_\mu \cap D_\nu = \varnothing$ for $\mu \neq \nu$. Then $|B_\nu| = m$ and $|A \cap B_\nu| = |A \cap B'_\nu| = p$ for $\nu \in I$, and

$$|B_\mu \cap B_\nu| = |B'_\mu \cap B'_\nu| < p \quad \text{for} \quad \mu \neq \nu.$$

This completes the proof.

8. *Open questions.* Let A be a set, well-ordered and of order type ω_α^β. One can ask this question: how far is it possible to choose subsets A_γ of A such that, for all γ, δ, the sets $A_\gamma \cap A_\delta$ are prescribed to have either an order type less than ω_α or a type $\omega_\alpha^{g(\gamma,\delta)}$, where $g(\gamma, \delta)$ is a given ordinal less than β? In Theorem 1 we only deal with a relatively simple special case. We have some further results but do not state them as they have not yet reached a satisfactory state.

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