JOURNAL OF APPROXIMATION THEORY 17, 393–394 (1976)

Note

Müntz's Theorem and Rational Approximation

In this note we prove the following

THEOREM 1. *Let $f(x)$ be any nonvanishing continuous function defined on $[0, \infty)$ for which $f(x) \to \infty$ as $x \to \infty$. Then for every sequence of integers: $0 = n_0 < n_1 < \cdots$ satisfying $\sum_{k=1}^{\infty} 1/n_k = \infty$ there is a sequence of polynomials $\sum_{i=0}^{k} a_i^{(k)} x^{n_i}$, $k = 1, 2,\ldots,$ for which*

$$\lim_{k\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{i=0}^{k} a_i^{(k)} x^{n_i}} \right\|_{L_\infty[0,\infty)} = 0. \tag{1}$$

Proof. By the well known theorem of Müntz and Szász every continuous function defined on a finite closed interval can be uniformly approximated as close as we like by polynomials $Q_{n_k}(x) = \sum_{i=0}^{k} a_i^{(k)} x^{n_i}$, where $\{n_i\}$ satisfies the above conditions. Thus,

$$\max_{0\leqslant x\leqslant 2A_k} |f(x) - Q_{n_k}(x)| < \epsilon_k, \tag{2}$$

where $\epsilon_k \to 0$ and $n_k \to \infty$.

Let now $n_q > n_k$ be sufficiently large. We prove

$$\left| \frac{1}{f(x)} - \frac{1}{Q_{n_k}(x) + (x/A_k)^{n_q}} \right| < 2\epsilon_k \tag{3}$$

for $0 \leqslant x < \infty$. Clearly (3) is only a restatement of our theorem.

Now (3) is trivially satisfied for $0 \leqslant x \leqslant \frac{1}{2}A_k$ if n_q is sufficiently large since, by (2), $Q_{n_k}(x)$ is bounded away from 0 in $[0, 2A_k]$ and $(x/A_k)^{n_q} \leqslant (1/2)^{n_q}$ in $0 \leqslant x \leqslant \frac{1}{2}A_k$. Next we prove (3) for $x > \frac{1}{2}A_k$. Clearly $1/f(x) < \epsilon_k/2$ for $x > \frac{1}{2}A_k$ since $f(x) \to \infty$ as $x \to \infty$.

$$\frac{1}{Q_{n_k}(x) + (x/A_k)^{n_q}} < \epsilon_k \text{ in } \frac{1}{2} A_k < x \leqslant 2A_k,$$

since $Q_{n_k}(x) > 1/\epsilon_k$ there and $(x/A_k)^{n_q} > 0$.

Now finally for $x > 2A_k$,

$$Q_{n_k}(x) + (x/A_k)^{n_q} > 1/\epsilon_k \tag{4}$$

trivially holds for sufficiently large n_q (in fact we can start the proof by choosing n_q so that (4) should be satisfied for every $x \geqslant 2A_k$). This completes the proof of our theorem.

By using a well-known result of Clarkson and Erdös (*Duke Math. J.* **10** (1943), Theorem 3) we can easily prove

THEOREM 2. *Let $f(x)$ be a non vanishing continuous function defined on* $[0, \infty)$. *If there exists a sequence of polynomials $P_k(x) = \sum_{i=0}^{k} a_i^{(k)} x^{n_i}$ for which*

$$\lim_{P_k \to \infty} \left\| \frac{1}{f(x)} - \frac{1}{P_k(x)} \right\|_{L_\infty[0,\infty)} = 0,$$

where $0 = n_0 < n_1 < n_2 < \cdots < n_k$ and $\sum_{k=1}^{\infty} 1/n_k < \infty$, then $f(x)$ is the restriction to $[0, \infty)$ *of an entire function.*

RECEIVED: October 28, 1974.

PAUL ERDÖS
Hungarian Academy of Science
Budapest, Hungary

A. R. REDDY
Institute for Advanced Study
Princeton, New Jersey 08540

Communicated by Oved Shisha