ON GRAPHS OF RAMSEY TYPE

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1. *Introduction*

If F, G, and H are (finite, simple) graphs, write $F \to (G,H)$ to mean that if the edges of F are colored red and blue (say) in any fashion, then either the red subgraph of F contains a copy of G or the blue subgraph contains H. Write $F \to G$ for $F \to (G,G)$. A natural question to consider is that of characterizing those F for which $F \to (G,H)$ for a given G and H. This question is in general extremely difficult, although in a later section we will answer it in a few special cases. In general we will here discuss simpler questions of what properties such graphs can have.

One result of the type sought is already known. Folkman [1] proved the remarkable fact that there exists an F having clique number $\max(m,n)$ for which $F \to (K_m,K_n)$. Nešetřil and Rődl [2] have extended this to more than two colors. In another direction, [3] considers for certain G and H the question of how few edges F can have, given that $F \to (G,H)$. We will consider here two major questions: Given that $F \to (G,H)$, what can be said about the chromatic number $\chi(F)$, and given that $F \to (K_m,K_n)$, what can be said about the minimum and maximum degree of F?

To make certain questions meaningful, it is necessary to require that F be minimal for the given Ramsey property; that is, that F lose the property upon removal of any edge.

2. *Chromatic Number*

We first consider the question of how small $\chi(F)$ can be, given that $F \to (G,H)$. In fact, we generalize this question somewhat. Let $\mathcal{G}$ and $\mathcal{H}$ be sets of graphs. Write $F \to (\mathcal{G},\mathcal{H})$ to mean that if the edges of F are colored red and blue, there is either a red subgraph isomorphic to a member of $\mathcal{G}$ or a blue subgraph isomorphic to a member of $\mathcal{H}$. By the chromatic Ramsey number $r_c(\mathcal{G},\mathcal{H})$ we mean the least c such that there exists an F with $\chi(F) = c$ for which $F \to (\mathcal{G},\mathcal{H})$. We will evaluate $r_c(\mathcal{G},\mathcal{H})$ in a sense; we need some more definitions. By the Ramsey number $r(\mathcal{G},\mathcal{H})$ we mean the least integer n such that $K_n \to (\mathcal{G},\mathcal{H})$. Write $r_c(\mathcal{G})$ for $r_c(\mathcal{G},\mathcal{G})$. Also in any of the above definitions, if $\mathcal{G} = \{G\}$ or $\mathcal{H} = \{H\}$ we may write G or H as an argument.

We say that ϕ is a homomorphism of the graph G into the graph H if ϕ is a mapping of $V(G)$ into $V(H)$ and $(x,y) \in E(G) \Rightarrow (\phi(x), \phi(y)) \in E(H)$. The homomorphism is called *onto* if all points and edges of H arise as images of points and edges of G.

For any graph G, we denote by hom G the set of homomorphic images of G, i.e., the set of graphs having onto homomorphisms with G. If $\mathcal{G}$ is a class of graphs, we define hom $\mathcal{G}$ in the obvious way.

The direct product $G \times H$ of two graphs G, H is defined by

$$V(G \times H) = V(G) \times V(H)$$

and $(x,y) \in V(G \times H)$ is adjacent to $(x',y') \in V(G \times H)$ if and only if $(x,x') \in E(G)$ and $(y,y') \in E(H)$. Harary [4] writes this as $G \wedge H$

and calls it the conjunction. Otherwise, we generally follow the terminology of [4]. The mappings

$$\pi_1 : V(G \times H) \to V(G), \quad \pi_2 : V(G \times H) \to V(H),$$

defined by $\pi(x,y) = x$, $\pi_2(x,y) = y$ are called the projections of $G \times H$ onto G and H, respectively. These are homomorphisms onto G and H.

The following proof, and its application to corollary 1 below, are due in part to V. Chvátal (personal communication).

THEOREM 1. *For any classes* $\mathcal{G}$ *and* $\mathcal{H}$ *of graphs*

$$r_c(\mathcal{G},\mathcal{H}) = r(\text{hom } \mathcal{G}, \text{hom } \mathcal{H}).$$

Proof: Set $n = r(\text{hom } \mathcal{G}, \text{hom } \mathcal{H})$. By the definition of n, the edges of K_{n-1} can be 2-colored such that no subgraphs isomorphic to any element of $\text{hom } \mathcal{G}$ or $\text{hom } \mathcal{H}$ is monochromatic of the appropriate color. Let F be any $(n-1)$-colorable graph, then α, an $(n-1)$-coloring of F, can be regarded as a homomorphism of F into such a two-colored K_{n-1}. Let us color $(x,y) \in E(F)$ with the color of $(\alpha(x), \alpha(y))$. Then no red subgraph G is isomorphic to any member of $\mathcal{G}$; for otherwise $\alpha(G)$ is a monochromatic subgraph of K_{n-1} isomorphic to a member of $\text{hom } \mathcal{G}$. The same argument applies to $\mathcal{H}$. This proves $r_c(\mathcal{G},\mathcal{H}) \geq n$.

We now prove the inequality in the other direction. It is easy to find finite subsets $\mathcal{G}_f \subseteq \mathcal{G}$ and $\mathcal{H}_f \subseteq \mathcal{H}$ with $r(\text{hom } \mathcal{G}_f, \text{hom } \mathcal{H}_f) = n$; just select one red subgraph belonging to $\text{hom } \mathcal{G}$ for each 2-coloring of $E(K_n)$ and a graph in $\mathcal{G}$ with this image, and similarly for $\mathcal{H}$.

Since $r_c(G,H) \le r_c(G_1,H_1)$, it suffices to show $r_c(G_1,H_1) \le n$. We do this by showing that if m is sufficiently large, $K(m,\ldots,m) \to (G_1,H_1)$ where the left-hand graph is a complete n-partite graph.

To see this, let the maximal independent sets of a two-colored $K(m,\ldots m)$ be $S_1,\ldots,S_n$, so that $|S_i| = m$. It is well known (and easy to show along the lines of the proof of Ramsey's theorem) that given any m', we can choose m so large that $S_1 \cup S_2$ induces a monochromatic $K(m',m')$. Discard the remaining points from S_1 and S_2 and an arbitrary set of the appropriate number of points from the other S_i to form $\{S_i'\}$ having m' points. Now, given any m'', we can choose m and m' so large that $S'_2 \cup S'_3$ induces in turn a monochromatic $K(m'',m'')$. Given any $\bar{m}$, we can choose m large enough so that we can repeat this operation $\binom{n}{2}$ times, finally yielding a $K(\bar{m},\ldots,\bar{m})$ with maximal independent sets $\bar{S}_i \subseteq S_i$ such that all $(\bar{S}_i,\bar{S}_j)$ edges have the same color α_{ij}. Choose $\bar{m} = \max|V(F)|$ and m large enough as above.

Now α_{ij} defines a 2-coloring of the edges of K_n and so by $n = r(\hom G_1, \hom H_1)$, some $G \in \hom G_1$ or $H \in \hom H_1$ occurs monochromatically in the desired color in K_n; suppose it is $G \in \hom G_1$. Let $G = \phi(G')$, $G' \in G_1$. Then we can find a monochromatic copy of G' spanned by $\bar{S}_1,\ldots,\bar{S}_n$, since certainly $\bar{m} \ge |V(G')|$.

The above shows $r_c(G_1,H_1) \le n$, completing the proof.

Corollary 1. $r_c(K_m,K_n) = r(K_m,K_n)$.

In fact K_m and K_n have no homomorphic images other than themselves. Indeed, if G is n-chromatic and contains a complete n-graph then

$$r_c(G) = r(K_n).$$

To see this, note that $r_c(G) \geq r_c(K_n) = r(K_n)$, since $K_n \subseteq G$, and $r_c(G) = r(\text{hom } G) \leq r(\text{hom } K_n) = r(K_n)$, since hom $G \supseteq$ hom $K_n = \{K_n\}$.

The above corollary was proved by Lin in [5], in fact for any number of colors. We note that Theorem 1 and corollary 2 below can also be easily generalized to more than two colors; but we will not pursue this.

Now, let $\mathcal{K}_r$ denote the class of all r-chromatic graphs.

Corollary 2. $r_c(\mathcal{K}_r,\mathcal{K}_s) = (r-1)(s-1) + 1$.

In fact, hom $\mathcal{K}$ consists of all graphs which are not (r-1)-chromatic. Since $E(K_{(r-1)(s-1)})$ can be 2-colored such that the two colors induce (r-1) - or (s-1)-chromatic graphs respectively but $E(K_{(r-1)(s-1)+1})$ cannot, the assertion follows.

Corollary 3. *If* $\chi(G) = 3$, *then* $r_c(G) = \begin{cases} 5 \text{ if } G \text{ is homomorphic to } C_5, \\ 6 \text{ otherwise.} \end{cases}$

Thus $r_c(C_{2n+1}) = 5$ if $n \geq 2$. We leave the straightforward proof to the reader. Similar finite characterisations can in principle be given in other cases, but this is probably impractical except perhaps for $r_c(G,H)$, where $\chi(G) = 4$, $\chi(H) = 3$.

We obtain a much more difficult question that that of Corollary 2 if we consider

$$M_r = \min_{G \in K_r} r_c(G).$$

Clearly,

$$r_c(G) \geq r_c(K_r) = (r-1)^2 + 1,$$

whence $$M_r \geq (r-1)^2 + 1.$$

Conjecture 1. $\min r_c(G) = (r-1)^2 + 1.$

This conjecture would follow from

Conjecture 2. $\chi(G \times H) = \min(\chi(G), \chi(H))$; or, equivalently, if $\chi(G) = \chi(H) = r$ then $\chi(G \times H) = r$.

It is clear that $\leq$ holds here.

We will prove a weakened version of conjecture 2, whence conjecture 1 will follow for $r = 3, 4$.

THEOREM 2. *Let* $\chi(G) = \chi(H) = r$ *and suppose each point of* H *is contained in a complete* $(r-1)$*-graph. Then* $\chi(G \times H) = r$.

We remark this proves conjecture 2 in case $r = 3$.

Proof: If α is any r-coloring of G then α' defined by $\alpha'(x,y) = \alpha(x)$ is an r-coloring of $G \times H$. Thus $\chi(G \times H) = r$.

We prove now that $G \times H$ is not $(r-1)$ chromatic. Suppose indirectly that $G \times H$ has an $(r-1)$-coloring α. Let us label the points of H by $1,\ldots,n$. We may assume this ordering is such that for each i, H contains a complete $(r-1)$-graph whose vertices are $i, i+1, \ldots, i+\nu-1$ for some $1 \leq \nu \leq r-1$ and $r-\nu-1$ points less than i.

For each $y \in V(G)$, $\alpha_y(x) = \alpha(y,x)$ cannot be a legitimate coloring of H, i.e., there are two points $i,j \in V(H)$ such that $\alpha_y(i) = \alpha_y(j)$, $(i,j) \in E(H)$. Select, for a fixed y, such a pair (i_y,j_y) with i_y minimal and set $\alpha_y(i_y) = \alpha_y(j_y) = \beta(y)$. Now β cannot be a legitimate coloring of G, whence there are $y, z \in V(G)$ such that $(y,z) \in E(G)$ and $\beta(y) = \beta(z)$. Thus we know

(1) $\quad \alpha(y,i_y) = \alpha(y,j_y) = \alpha(z,i_z) = \alpha(z,j_z)$,

and $(y,z) \in E(G)$, $(i_y,j_y) \in E(H)$, $(i_z,j_z) \in E(H)$. We cannot have $i_y = i_z$ since then (y,i_y) would be adjacent to (z,j_z) in contradiction with (1). So we may suppose $i_y < i_z$. By the definition of the ordering of $V(H)$, there is a complete $(r-1)$-graph $K \subseteq H$ whose vertices are i_y, $i_y + 1, \ldots, i_y + \nu - 1$ and $r-\nu-1$ points less than i_y $(1 \le \nu \le r-1)$. Since (y,i_y) and (z,i_z) are non-adjacent by (1), $i_z \notin V(H)$ and so, $i_z \geq i_y + \nu$.

Now the points (z,k) $(k \in V(K))$ are all adjacent to (y,i_y) (for $k \neq i_y$) or to (y,j_y) (for $k = i_y$). Therefore these $r-1$ points only get $r-2$ colors, since none of them can be colored $\alpha(y,i_y) = \alpha(y,i_z)$. Thus some two of them, say (z,k_1) and (z,k_2), must have the same color: $\alpha(z,k_1) = \alpha(z,k_2)$. Since $k_1,k_2 < i_z$, this contradicts the choice of i_z.

THEOREM 3. *Conjecture 1 is valid for* $r = 4$.

Proof: Let us 2-color the complete $((r-1)^2+1)$-graph in all possible ways; let $\alpha_1,\ldots,\alpha_N$ be these 2-colorings. At least one of the two colors must form a graph with chromatic number $\geq r$; let H_1 be a monochromatic subgraph of G_i with chromatic number $\geq r$. Consider

$$G = H_1 \times \ldots \times H_N.$$

Then $H_1, \ldots, H_N \subset \hom G$ and so,

$$r_c(G) = r(\hom G) \leq (r-1)^2 + 1,$$

by the choice of the graphs $H_1, \ldots, H_N$. So if G has chromatic number $\geq r$ we are finished. This follows immediately if Conjecture 1 is valid, but, making use of the freedom we still have in the choice of $H_1, \ldots, H_N$ we can prove $\chi(G) = r$ for $r = 4$.

By Theorem 2 it suffices to prove

LEMMA 1. *If the edges of the complete 10-graph are 2-coloured, there exists a monochromatic subgraph* H *with chromatic number 4 such that each vertex of* H *is contained in a triangle of* H.

Proof. Let the edges be colored red and blue, and let G_1 and G_2 denote the subgraphs formed by red and blue edges, respectively.

If $\chi(G_1) = 3$ then 3-coloring the points of G_1 one color class will have at least 4 points. This gives a K_4 in G_2, and we can take this complete 4-graph as H.

Thus we may suppose $\chi(G_1) \geq 4$ and similarly $\chi(G_2) \geq 4$. If each point of G_1 is contained in a triangle of G_1 we can take $H = G_1$. So we may suppose there exists a point x_1 contained in no red triangle and similarly, there is a point x_2 contained in no blue triangle.

Suppose first $x_1 = x_2$. Then there are ≥ 5 edges adjacent to x_1 in one of the two colors, say in red. The points z with (z, x_1) red must form a blue complete graph and we can take this as H.

Finally, suppose $x_1 \neq x_2$, and let the edge (x_1,x_2) be red (say). The points z with (z,x_1) red span a blue complete graph containing x_2, hence their number is at most 2. The number of blue edges adjacent to x_2 is at most 3 (otherwise we could find a red K_4 as before) and if (z,x_1) is red $(z \neq x_2)$ then (z,x_2) is blue. Thus there are at least 5 points which are all connected to x_1 by blue edges and to x_2 by red edges. If these 5 points induce a monochromatic triangle we have a monochromatic K_4 again; otherwise, there is a red 5-cycle here which forms a red 5-wheel together with x_2. Then we can take this 5-wheel as H. This completes the proof of Lemma 1, and hence of Theorem 3.

3. Minimum and Maximum Degrees

We now consider $\delta(F)$ and $\Delta(F)$, the minimum and maximum degrees of F. It is necessary to make a definition. We say F is (G,H)-irreducible if $F \rightarrow (G,H)$ but $F' \not\rightarrow (G,H)$ for any proper subgraph F' of F; we make the obvious definition of G-irreducibility.

J. Nešetřil conjectured that there are infinitely many non-isomorphic K_r-irreducible graphs for any fixed r. We are going to describe a construction which will answer this conjecture in the affirmative, but first we consider δ and Δ.

Among other things the next theorem gives a new way to give a lower bound to the classical Ramsey numbers.

THEOREM 4. $2^{r/2} \leq \min \Delta(G) = r(K_r)-1$, *where the minimum is taken over all* G *for which* $G \rightarrow K_r$.

Proof. The right-hand equality is obvious by corollary 1 to Theorem 1. The lower bound can be proved as follows. Construct a hypergraph H on E(G) whose edges are sets of edges of complete r-subgraphs of G. Then $G \to K_r$ means H is not 2-chromatic. By Theorem 3 in [6], it follows that some edge (x,y) of G is contained in at least

$$\frac{2^{\binom{r}{2}-3}}{\binom{r}{2}} \quad \text{complete r-graphs.}$$

Let d be the degree of x; then

$$\frac{2^{\binom{r}{2}-3}}{\binom{r}{2}} \le \binom{d}{r-1},$$

whence

$$d \ge 2^{r/2}$$

if $r \ge 6$. The cases with $r \le 5$ can be settled by direct considerations. (They also follow by our next theorem.)

THEOREM 5. $\min\ \delta(G) = (r-1)\ (s-1)$,

where the minimum is taken over all (K_r,K_s)*-irreducible graphs.*

Proof. Suppose first that $G \to (K_r,K_s)$ and some $x \in V(G)$ has degree $< (r-1)\ (s-1)$.

G being (K_r,K_s)-irreducible, the edges of G-x can be 2-colored such that no complete r-graph or s-graph is monochromatic in the appropriate color and consider such a 2-coloring with red and blue, say. Let S denote the set of neighbors of x, and let $T_1,\ldots, T_k$ be a maximal set of disjoint complete red (r-1)-graphs spanned by S. By $|S| < (r-1)(s-1)$

we have $k \leq s-2$. Let us color now all edges connecting x to $T_1,\ldots,T_k$ blue, all edges connecting x to $S-T_1-\ldots-T_k$ red. We claim that no appropriate monochromatic complete graph arises. In fact, each blue complete graph containing x can contain at most one point of each $T_1,\ldots,T_k$, whence it has at most $k+1 \leq s-1$ points. Each red complete graph containing x has at most $r-2$ points in $S-T_1-\ldots-T_k$ (by the maximality of $\{T_1,\ldots,T_k\}$), thus the same conclusion holds again.

The coloration defined above shows $G \not\to (K_r,K_s)$, a contradiction. Thus x must have degree $\geq (r-1)(s-1)$.

The construction of a K_r-irreducible graph G with $\delta(G) = (r-1)(s-1)$ will depend on a lemma.

LEMMA 2. *For any $r,s \geq 3$ there exists a graph* G *and two independent edges* e *and* f *of* G *such that*

(a) $G \not\to (K_r,K_s)$,

(b) *for any 2-coloring of* $E(G)$ *such that no* K_r *or* K_s *is monochromatic in the appropriate color,* e *and* f *have different colors.*

A graph G with properties a, b will be called a *negative (e,f)-signal sender*.

If G_1 is a negative (e_1,f_1)-signal sender and G_2 is a negative (e_2,f_2)-signal sender then by identifying f_1 and e_2 we get a positive (e_1,f_2)-signal sender: a graph G such that $G \not\to (K_r,K_s)$, but if we color $E(G)$ red and blue such that no appropriate monochromatic complete graph arises then e_1 and f_2 must have the same color. By forming chains of signal senders analogously we can construct positive and negative (e,f)-signal senders such that the distance between e and f is arbitrarily large.

Before proving Lemma 2 let us complete the proof of Theorem 5. Without loss of generality, we may assume that $r \le s$. Let $S_1,\ldots,S_{s-1}$ be disjoint sets with $|S_i| = r-1$. Let $x \notin S_1 \cup \ldots \cup S_{s-1} \cup \{x\}$ span a complete graph K. Form an edge e disjoint from K. If f is an edge of K such that f connects two points in different S_i's, construct a positive (e,f)- signal sender in which e and f have distance ≥ 3 and which has no other point in K. All of these signal senders should have no points in common except at their end-edges. Call the union of K and all the signal senders G. Then $G \to (K_r,K_s)$. To see this, suppose that $E(G-x)$ is colored red and blue so that no red K_r or blue K_s occur. Then, since $r \le s$, the signal senders force that all the edges connecting different S_i's must be blue (except that colors can be reversed if $r = s$). Consequently, all edges within each S_i must be red. There is now no way to color the edges emanating from x without forming a red K_r or a blue K_s.

Also $G-x \not\to (K_r,K_s)$; for we can color all edges spanned by any S_i red and the edges connecting different sets S_i blue. This coloring extends to the signal senders so that no K_r or K_s is monochromatic in the appropriate color. Since the distance of e and f is ≥ 3 and the signal senders are nearly disjoint, there is no other K_r or K_s in G than these in K and in the signal senders. Therefore, no red K_r or blue K_s occurs by the above coloring.

If the signal senders are minimal in the sense that removal of an edge from one destroys its properties, then G is (K_r,K_s)-irreducible. To see this observe first that $(G-e) \not\rightarrow (K_r,K_s)$, where $e \in E(K)$. Moreover, consider the removal of an edge e not in K. This permits the color of some edge between two S_i's to go free, and it is easy to see that in this case again $(G-e) \not\rightarrow (K_r,K_s)$. This completes the proof of Theorem 5.

A similar argument yields

THEOREM 6. *If* $r,s \geq 3$, *there are infinitely many non-isomorphic* (K_r,K_s)-*irreducible graphs.*

Proof. Let one of the signal senders used in the above construction have arbitrarily large distance between e and f.

We note that by elaborating on the ideas of Theorem 5 it is possible to construct a (K_r,K_s)-irreducible graph with arbitrarily many points of degree $(r-1)(s-1)$, incidently yielding another proof of Theorem 6.

We are now ready to consider Lemma 2 again, but we do so by proving another lemma which yields Lemma 2 almost immediately.

LEMMA 3. *If* $r,s \geq 3$, *there exists a graph* G *with two adjacent edges* e *and* f *satisfying (a) and (b) of Lemma 2.*

Proof. Let $m = r(K_r,K_s)-2$. Then $E(K_{m+1})$ has some colorings with red and blue such that no K_r or K_s is monochromatic in the appropriate color. Let $0 \leq a_1 < \ldots < a_q \leq m$ be the possible numbers of red edges adjacent to any point in such 2-colorings of $E(K_{m+1})$.

Claim 1. $a_1 \neq 0$. Suppose there is a 2-coloring of $E(K_{m+1})$ with no appropriate monochromatic K_r or K_s such that a certain point x is adjacent to blue edges only. Take a new point x', connect it by blue edges to all points of $K_{m+1}-x$ and by a red edge to x. Then we obtain a 2-coloring of $E(K_{m+2})$ having no red K_r or blue K_s, which contradicts the definition of m.

Similarly it follows that $a_q \neq m$.

Claim 2. There exists an m-uniform hypergraph H and two points x, y x, y $\in$ V(H) such that

(a) The vertices of H can be 2-colored such that each edge contains $a_1, a_2, \ldots,$ or a_q red vertices;

(b) For any 2-coloring of V(H) for which each edge contains $a_1, \ldots,$ or a_q red vertices x and y have different colors;

(c) H contains no circuits shorter than 4.

To prove this claim let us consider a hypergraph H_o which contains no circuits shorter than 4, and which is 3-chromatic. The existence of such an H_o has been proved by probabilistic methods in [7], and constructively in [8]. Then by $a_1 \neq 0$, $a_q \neq m$, H_o has the property

(*) For any 2-coloring of $V(H_o)$ there is an edge $E \in E(H_o)$ such that the number of red points in E is different from $a_1, \ldots, a_q$. In fact, the number of red points in E is 0 or m. Consider now a hypergraph H_o with property (*) and with minimum number of edges. Let $E = \{x_1, \ldots, x_m\} \in H$ and take m points $y_1, \ldots, y_m$ not in H. Let $H_i = H - \{E\} + \{E_i\}$ where $E_i = \{y_1, \ldots, y_i, x_{i+1}, \ldots, x_m\}$.

Clearly H_o has property (*) while H_m fails to have this property. So there is a $0 \leq i \leq m$ such that H_i does not have property (*) but H_{i-1}

does. We claim $H = H_i$, $x = x_i$, $y = y_i$ satisfy the conditions of Claim 2. In fact, (a) is satisfied since H_i does not have property (*). (b) is satisfied since if H_i had a 2-coloring where x_i and y_i are colored alike and each edge contains $a_1,\dots,$ or a_q red vertices then identifying x_i and y_i the same coloring would show that H_{i-1} does not satisfy (*). Finally, (c) holds trivially.

We now construct our graph as follows. Let $V(G) = V(H) \cup \{v\}$. Connect v to all points of $V(H)$; connect two points of $V(H)$ if they belong to the same edge of H. Note that (c) implies that every K_r or K_s in this graph is contained in a set of $E \cup \{v\}$, $E \in E(H)$, and each edge of G spanned by $V(H)$ belongs to a unique edge of H.

Now $G \not\to (K_r, K_s)$ follows by (a) easily: Suppose $V(H)$ is 2-colored such that each edge E of H contains $a_1, a_2, \dots,$ or a_q red points. Let us color the edge (v,u) $(u \in V(H))$ with the color of u. Then for each edge E of H, the edges of the complete $(m+1)$-graph K spanned by $E \cup \{v\}$ are partially 2-colored; the number of red edges adjacent to v is a_i, $1 \le i \le q$ and hence by the definition of a_i, we can complete this 2-coloring of $E(K)$ without producing a red K_r or blue K_s. Doing so for each $E \in E(H)$ we obtain a 2-coloring of $E(G)$. This produces no red K_r or blue K_s, since each K_r or K_s in G is contained in some set $E \cup \{v\}, E \in E(H)$.

A similar argument shows that for each 2-coloring of $E(G)$ in which no K_r or K_s is monochromatic in the appropriate color the edges (v,x) and (v,y) have different colors. This completes the proof.

Proof of Lemma 2. Take three copies G_i, $i = 1,2,3$, of the graph G of Lemma 3, with edges e_i, f_i as in that lemma. Identify f_1 with e_2 and f_2 with e_3 in such a way that e_1 is disjoint from e_3 and f_3 and f_1 is disjoint from f_3 and such that the G_i's have no points in common not implied by the above identifications. Then it is clear that the resulting graph satisfies the requirements of Lemma 2.

Also of interest is how large $\Delta(G)$ can be for a (K_r,K_s)-irreducible graph. The following result answers this question, which once again proves Nešetřil's conjecture.

THEOREM 7. *If* $r,s \geq 3$, *then there exist* (K_r,K_s)*-irreducible graphs with arbitrarily large* Δ.

Proof. We will be concise. It is sufficient to show that there exist minimal signal senders with arbitrarily large Δ, since then one can construct the desired graphs in many ways. To show the existence of such signal senders, consider the proof of Lemma 3.

That proof involved a hypergraph H_o which was 3-chromatic and had no circuits shorter than 4. Replace that hypergraph with one which is 3-chromatic and which has no circuits shorter than n, where n is any integer ≥ 4. Again, that this can be done is guaranteed by [7] or [8].

The hypergraph H inherits this property. Examining the rest of the proof, one sees that the graph G formed as in that proof has a cycle not containing v having at least n points and that v therefore has degree $\geq$ n. It is clear moreover that in any graph formed from G by removing edges and retaining the properties of G, v must still have degree $\geq n$. Proceeding now as in the proof of Lemma 2, we construct a minimal signal sender having maximum degree $\geq n$, completing the proof.

We note that this construction can be used in the other results of this section. For instance, there exist (K_r,K_s)-irreducible graphs G with $\delta(G) = (r-1)(s-1)$ and with $\Delta(G)$ arbitrarily large.

A question we have not answered is that of how large δ can be in a (K_r,K_s)-irreducible graph. Very likely it, too, can be arbitrarily large.

The methods of this chapter also let us consider the connectivity κ of (K_r,K_s)-irreducible graphs.

THEOREM 8. *If* $r,s \geq 3$, *then*

$$\min \kappa(G) = \begin{cases} 2 & \text{if } r \neq s \\ 3 & \text{if } r = s. \end{cases}$$

where the minimum is taken over all (K_r,K_s)*-irreducible G.*

Proof. It is trivial that $\min \kappa(G) \geq 2$. Now assume r=s and suppose G is a (K_r,K_s)-irreducible graph with $\kappa(G) = 2$. Then G is the union of two graphs G_1 and G_2 with only two points x,y in common. Since G is (K_r,K_s)-irreducible, $G_i \not\rightarrow (K_r,K_s)$, i=1,2. If G does not contain the edge (x,y) it is immediate that $G \not\rightarrow (K_r,K_s)$, a contradiction. If G contains (x,y), note that G_1 and G_2 may each be separately colored so that (x,y) is red and neither contains a red K_r or a blue K_s, since if any such coloring yields a blue (x,y) the colors may be reversed. Combining these two colorings in G, we see that again $G \not\rightarrow (K_r,K_s)$, a contradiction.

To see than $\min \kappa(G) \leq 3$, take three copies G_i, i = 1,2,3, of a minimal graph G having the properties of Lemma 3 with edges e_i, f_i as in that lemma. Form a graph H by identifying f_1 with e_2, f_2 with e_3, and f_3 with e_1 in such a way that the edges form a $K_{1,3}$ and such that the G_i's have no other points in common. It is clear that $\kappa(H)=3$ and that H is (K_r,K_s)-irreducible. Finally, assume without loss of generality that $r < s$. We construct a (K_r,K_s)-irreducible graph G with $\kappa(G)=2$. Form a complete r-graph K and a disjoint edge e. For every $f \in E(K)$ connect e and f with a positive signal sender in which e and f have distance ≥ 3 and which has no other point in K, while also assuring that no two signal senders have a point in common except at e and possibly K. It is clear that the resulting graph H can be 2-colored so that is contains no red K_r or blue K_s; but K will be monochromatic and therefore blue. Take two copies H' and H" of H with distinguished complete r-graphs K' and K". Connect an edge of K' to one of K" with a negative signal sender to form a graph F. Clearly $F \rightarrow (K_r,K_s)$ and $\kappa(F)=2$. Furthermore, F certainly retains the latter property when it is converted to a (K_r,K_s)-irreducible graph by possibly removing edges. This completes the proof.

Another significant graphical parameter is the edge connectivity λ. Very likely min $\lambda(G) = (r-1)(s-1)$, where the minimum is taken over all (K_r,K_s)-irreducible graphs, but we have not been able to prove this. Certainly $(r-1)(s-1)$ is an upper bound by Theorem 5.

We have determined in Section 2 the minimum value of χ; in [9] it is shown that for (K_r,K_s)-irreducible graphs there is no maximum. Other interesting parameters are the point and edge covering and independence numbers. The construction in Theorem 6 shows that all four of these parameters can be made arbitrarily large in a (K_r, K_s)-irreducible graph. Very likely the minima of these parameters are determined by K_t, where $t=r(K_r, K_s)$. That the minimum number of points and edges are determined by K_t follows from the definition of t and from Corollary 1 to Theorem 1 respectively.

Finally, we note that all the questions we have studied in this section could be asked for (G,H)-irreducible graphs in general.

4. Explicit Characterizations.

In this section we discuss the few G for which it has been possible to characterize those F satisfying $F \to G$. Our first result is for stars; the argument is essentially due to U.S.R. Murty (personal communication).

THEOREM 9. *A necessary and sufficient condition that* $G \to K_{1,n}$ *is that* $\Delta(G) \geq 2n-1$ *or, if* n *is even, that* G *has a component which is regular of degree* 2n-2 *and which has an odd number of points.*

Proof. It is clearly only necessary to consider connected G. Clearly if $\Delta(G) \geq 2n-1$, then $G \to K_{1,n}$. Suppose n is even and G is regular of degree 2n-2 and has an odd number of points. Then $G \to K_{1,n}$, for if not, then G is the union of two graphs, each regular of degree n-1, which is impossible since n-1 is odd. This proves sufficiency.

To prove necessity, first consider a graph G which is regular of degree 2n-2 and suppose that either n is odd or G has an even number of points. Then G has an eulerian circuit, which necessarily has an even number of edges. Color the edges of this eulerian circuit alternately red and blue, yielding a 2-coloring of G. The red and blue graphs are each regular of degree n-1, so $G \not\to K_{1,n}$.

Finally, let G be a graph which is not regular of degree 2n-2 and for which $\Delta(G) \leq 2n-2$. We can clearly add edges to G in such a way that the resulting graph either has exactly two points of odd degree or has one point of degree $< 2n-2$, which is necessarily even, while preserving the property $\Delta \leq 2n-2$. In the latter case add one more point and join it to the point of degree $< 2n-2$. Thus in any case $G \subseteq G'$, where $\Delta(G') \leq 2n-2$ and G' has exactly two points of odd degree. Join these two points by an eulerian trail, and color its edges alternately red and blue. Then each of the monochromatic graphs in the resulting 2-coloring has maximum degree $\leq$ n-1. Thus $G' \not\to K_{1,n}$ and hence $G \not\to K_{1,n}$. This completes the proof.

The problem of characterizing those G for which $G \to (K_{1,m}, K_{1,n})$ seems difficult, even in the case m=2, n=3. For instance, the fact that $G \not\to (K_{1,2}, K_{1,3})$ when G is a bridgeless cubic graph is equivalent to Petersen's theorem [10]. Although some work has been done [11], a complete characterization does not seem to have been published of those cubic graphs G with bridges for which $G \to (K_{1,2}, K_{1,3})$, and even this would not be enough, since there exist $(K_{1,2}, K_{1,3})$-irreducible graphs with points of degree two. (Clearly there can be no point of degree one in such a graph).

One other graph that can be dealt with is $2K_2$, that is, a graph consisting of 2 disjoint edges.

THEOREM 10. $G \to 2K_2$ *if and only if* G *contains three disjoint edges or a 5-cycle.*

Proof. Clearly $G \to 2K_2$ if $3K_2 \subseteq G$ or $C_5 \subseteq G$. Now consider any graph G such that $G \to 2K_2$. Let v be the point of largest degree in G. Now $2K_2 \subseteq G-v$, since otherwise we may color the edges of $G-v$ red and the rest blue, showing that $G \not\to 2K_2$. Every edge incident with v may be assumed to have a point in common with this $2K_2$, since otherwise $3K_2 \subseteq G$. Hence $\deg(v) \leq 4$. If $\deg(v)=4$, G contains $K_1 + 2K_2$, that is two triangles with a point in common. But $K_1 + 2K_2 \not\to 2K_2$, and adding any edge to $K_1 + 2K_2$, with or without a new point, leads to a graph containing C_5 or $3K_2$ respectively.

Consequently we may assume $\deg(v) \leq 3$. But if $\deg(v) = 3$, we see that G contains a graph G_1 consisting of a triangle $(v\ v_1\ v_2)$ connected to a path $(v\ v_3\ v_4)$. Now $G_1 \not\to 2K_2$, so extra edges are necessary. But (v,v_4) is disallowed and (v_1,v_4) or (v_2,v_4) lead to C_5. Moreover (v_1,v'), (v_2,v'), or (v_4,v'), where v' is a new point, lead to a $3K_2$. The only other edges that can be added are (v_1,v_3), (v_2,v_3) or (v_3,v'), where v' is a new point. But adding any number of these edges alone still gives a graph G_2 such that $G_2 \not\to 2K_2$. So we may assume $\deg(v) \leq 2$. But this leads immediately to $3K_2 \subseteq G$ or $C_5 \subseteq G$. This completes the proof.

Thus there are only two $2K_2$-irreducible graphs. Also by Theorem 9 there is only one $K_{1,n}$-irreducible graph when n is odd. However it seems very likely that for almost all G and H, there are infinitely many (G,H)-irreducible graphs, as in the case where G and H are complete graphs. The number is finite nevertheless in some further cases, as the next theorem shows.

THEOREM 11. *For any m and n, the number of* (mK_2,nK_2)*-irreducible graphs is finite.*

Proof. We will show that if G is (mK_2,nK_2)-irreducible, then $\Delta(G) \le 2m+2n-3$. This is sufficient to show that the number of such graphs is finite, since certainly such a graph has no more than m+n-2 independent edges as well, and no isolates. In fact, one can see immediately that any such graph has rather less than $2(m+n)^2$ points.

Assume the contrary, that there is an (mK_2,nK_2)-irreducible G with a point u of degree $\ge 2m+2n-2$. Let v be any point adjacent to u. Since G is (mK_2,nK_2)-irreducible, G-uv has at least one edge-coloring with no red mK_2 and no blue nK_2; consider any such coloring. This coloring must give both a red $(m-1)K_2$ and a blue $(n-1)K_2$, neither matching using u or v, for otherwise uv could be colored in addition so as to give no red nK_2 and no blue nK_2. Fix such a pair of monochromatic matchings, which certainly use no more than 2m+2n-4 points.

Since $\Delta(G) \ge 2m+2n-2$, there is a point w adjacent to u which is not used in either matching. But now we see that the edge uw could not have been colored successfully in G-uv, a contradiction. This completes the proof.

With only a little more effort one can show that $\Delta(G) \le 2m+2n-4$, but in fact it probably must be much smaller still. Moreover, it seems likely that an (mK_2,nK_2)-irreducible graph can have no more than $2m+2n-2$ points.

It would be interesting to get some more exact characterizations such as those of Theorems 9 and 10. This might be possible for star forests in general, but it seems difficult even for such a simple graph as P_4, the path on four points. In [9], which pursues the ideas of this paper further, it is shown that for any $n \geq 3$ there are infinitely many P_n-irreducible graphs. The constructions used shed little light on exact characterizations. It would be of interest to determine more cases where the number of irreducible graphs is finite or infinite. Perhaps the cases we have found are the only finite cases.

As a final observation, we note that all the concepts and questions considered in this paper can be generalized to more than two colors, as can some of the results, especially those of Section 2.

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