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PROBLEMS AND RESULTS IN COMBINATORIAL ANALYSIS

RIASSUNTO. — Nei primi quattro paragrafi si discutono problemi estremali sui grafi e sugli ipergrafi. Qui si pone soltanto un problema: sia $|S| = n$, è vero che se $|A_i| = 3$, $A_i \subset S$, $i = 1, 2, \cdots, t = \varepsilon n^2$ è un arbitrario insieme di $t = \varepsilon n^2$ terne di S, esiste sempre un sottinsieme di k elementi di S che contiene $k - 3$ terne tra quelle fissate? Szemeredi ha dimostrato questa congettura (formulata da W. G. Brown, V. T. Sos e l'autore) per $k = 6$, ma per $k > 6$ il problema resta aperto.

Un paragrafo è dedicato a problemi combinatori sui sottinsiemi e l'ultimo paragrafo fornisce vari problemi e risultati (non collegati tra loro) considerati dall'autore e dai suoi collaboratori.

During the last few years I have written several papers on this and related topics. As much as possible I will try to avoid overlap with previous papers. As is always the case the choice of my problems is purely subjective I only discuss questions on which I worked and of course do not claim that these problems are more important then others which I neglected. In this paper I will mention several problems related to block designs, a topic about which I do not know too much, but my collaborators and I often used results obtained by others; perhaps some of the experts in this field will be able to settle some of the questions which baffled us so far.

In the first four sections I discuss some extremal problems on graphs and hypergraphs.

At the end of each section I give references, here is a list of my papers on combinatorial problems.

Problems and results in combinatorial analysis, « Proc. Symp. Pure Math. » XIX, « Amer. Mat. Soc. », 1971, 77–89.

Some unsolved problems, « Michigan Math. Journal », *4* (1957), 291-300 and « Publ. Math. Inst. Hung. Acad. Sci. », *6* (1961), 221–254.

Extremal problems among subsets of a set (*with D. J. Kleitman*), Proc. second Chapel Hill Colloquium 1970, 146–170 see also « Discrete Math. », *8* (1974), 281–294.

Problems and results in chromatic graph theory, proof techniques in graph theory 1969, Acad. Press 27–35.

Some unsolved problems in graph theory and combinatorial analysis, Combinatorial Math. and its Applications Oxford conference 1969, Acad. Press 97–109.

1. $G^{(r)}(n; m)$ denotes an r-graph of n vertices and m edges (i.e. r-tuples). $f(n; G^{(r)}(k, l))$ is the smallest integer for which every $G^{(r)}(n; f(n; G^{(r)}(k; l))$ contains a $G^{(r)}(k; l)$ as a subgraph. New and

interesting complications arise if we also prescribe the structure of our $G^{(r)}(k ; l)$. Recently several papers appeared on extremal graph problems. Perhaps the most interesting unsolved problem is the original problem of Turan which he formulated in 1940. Denote by $K^{(r)}(t)$ the complete r-graph of t vertices $\left(\text{and } \binom{t}{r} \text{ edges}\right)$. For the sake of convenience if we speak of ordinary graphs we will omit the upper index. Turan's problem states: Determine $f(n ; K^{(r)}(t))$ for every $t > r$ and also determine the structure of the extremal graphs i.e. the graphs $G^{(r)}(n ; f(n ; K^{(r)}(t) - 1)$ which do not contain a $K^{(r)}(t)$.

This problem was solved by Turan for $r = 2$ and every t but for $r > 2$ nothing definite is known, though Turan has several plausible conjectures. It is easy to see that for every r and t

$$(1) \qquad \lim_{n=\infty} f(n ; K^{(r)}(t)) \Big/ \binom{n}{r} = \alpha(t, r)$$

exists and that $\alpha(t, 2) = 1 - \frac{1}{t-1}$, but for $r > 2$ none of the values $\alpha(t, r)$ are known.

Let $r > 2, 2 < l < r + 1$. It would be very interesting to determine $f(n ; G^{(r)}(r + 1, l))$, (for fixed r and l there is only one $G^{(r)}(r + 1, l)$). It is again easy to see that

$$\lim_{n=\infty} f(n ; G^{(r)}(r + 1, l)) \Big/ \binom{n}{r} = \beta(r, l)$$

exists and is positive but none of the $\beta(r, l)$ are known.

The case $r = 2$ is trivial here, $r > 2, l = 2$ is not trivial, very likely

$$(2) \qquad f(n ; G^{(r)}(r + 1, 2)) = \frac{1}{r} \binom{n}{r-1} + O(1).$$

Put

$$g(3n) = n^3 + 1 \quad , \quad g(3n + 1) = (n + 1) n^2 + 1 ,$$
$$g(3n + 2) = n(n + 1)^2 + 1 .$$

Katona conjectured and Bollobas proved that every $G^{(3)}(n ; g(n))$ contains three edges e_1, e_2, e_3 so that e_1 contains the symmetric difference of e_2 and e_3. It is easy to see that is best possible. This is one of the few exact results on extremal problems on hypergraphs, the analogous questions for $r > 3$ are unsolved.

P. TURAN, *Eine Extremalaufgabe ausder Graphentheorie (in Hungarian)*, «Mat es Fiz Lapok» *48* (1941), 436–452 see also colloquium «Math.». *3* (1954), 19–30.

KATONA, NEMETZ, SIMONOVITS, *On a problem of Turan in the theory of graphs*, «Mat. Lapok», *15* (1969), 228–238 see also J. Spencer, Turan's theorem for k-graphs, «Discrete Math.», *2* (1972), 183–186.

B. BOLLOBAS, *Three graphs without two triples whose symmetric difference is contained in a third*, «Discrete Math»., *8* (1974), 21–24.

P. Erdös, *Extremal problems in graph theory*, Proc. Symp. Theory of Graphs. Smolenice, 1963, 29–36. Some recent results on extremal problems in graph theory. Theory of graphs. International Symposium Rome 1966, 117–130, *On some new inequalities concerning extremal properties of graphs*, " Theory of graphs Proc. Coll. held at Tihany Hungary ", 1966, 77–81.

M. Simonovits, *A method for solving extremal graph problems in graph theory*, stability preblems *ibid*, 279–319. *Extremal graph problems with conditions*, Comb. theory and its applications «Coll. Math. Soc. J. Bolyai», 1970, vol. III, 999–1012 (North Holland).

P. Erdös, M. Simonovits, *A limit theorem in graph theory*, «Studia Sci. Math. Hungar.», *1* (1969), 51–57.

2. In this section $r = 2$. We will discuss bipartite graphs. C_l denotes a circuit of l edges. First we discuss $f(n; C_4)$. Brown, Renyi, VT Sos and I proved that

$$f(n; C_4) = \left(\frac{1}{2} + o(1)\right) n^{3/2}.$$

We in fact showed (using finite geometries) that if $n = p^2 + p + 1$ where p is a power of a prime then

$$f(p^2 + p + 1; C_4) > ((p+1)\, p^2 + p\,(p+1))/2 = \frac{1}{2}(p^3 + p) + p^2.$$

It is not impossible that in fact

$$(1) \qquad f(p^2 + p + 1; C_4) = \frac{1}{2}(p^3 + p) + p^2 + 1$$

but we were unable to prove (1). Our attempts to prove (1) were not entirely wasted since they led us to discover the so called friendship theorem.

Our graph $G\left(p^2 + p + 1; \frac{1}{2}(p^3 + p) + p^2\right)$ contains many triangles. It is perhaps true that if $G(n)$ contains no C_4 and no C_3 then it has at most $(1 + o(1)) \frac{n^{3/2}}{2\sqrt{2}}$ edges. If true this is certainly best possible since Reiman and E. Klein constructed a bipartite $G\left(n; \frac{n^{3|2}}{2\sqrt{2}}(1 + o(1))\right)$ without a C_4.

A simple argument shows

$$(2) \qquad f(n; C_4) \leq \frac{1}{2} n^{3/2} + \frac{n}{4}(1 + o(1)).$$

(2) easily follows from the simple observation that if v_i is the valency (or degree) of the vertex x_i and $G(n)$ contains no C_4 then

$$(3) \qquad \sum_{i=1}^{n} \binom{v_i}{2} \leq \binom{n}{2}$$

(3) easily implies (2). It would be tempting to conjecture that

$$(4) \qquad f(n; C_4) = \frac{1}{2} n^{3/2} + \frac{n}{4} + o(n)$$

but I could only prove

$$f(n\,;\,C_4) = \frac{1}{2}\, n^{3/2} + o\,(n^{3/2-c}) \tag{5}$$

for certain $c > 0$. (5) follows easily by considering the smallest p satisfying $p^2 + p + 1 \geq n$. The following question might be of some interest here. Let $|S| = n$, determine subsets $A_k \subset S$, $|A_{k_1} \cap A_{k_2}| \leq 1$ $|A_k| = \sqrt{n} + o\,(1)$ so that as many pairs $(x\,,\,y)\,,\, x \in S\,,\, y \in S$ as possible should be contained in the A's. Is it in fact possible to find such a system A_k which contains all the pairs with a possible exception of cn of them? In view of the recent surprisingly strong results of Wilson the following problem could be asked: Let:

$$\alpha_1 \binom{k}{2} + \alpha_2 \binom{k+1}{2} + \alpha_3 \binom{k+2}{2} > \binom{n}{2} + C_k$$

$\alpha_i > \varepsilon n^2$, $i = 1\,,\,2\,,\,3$. Let $|S| = n$. Is it possible to find a family of subsets $\{A_j\}$ of S, $|A_{j_1} \cap A_{j_2}| \leq 1$ $|A_j| = k + i$, $0 \leq i \leq 2$ every pair is contained in one and only one A and there are at most α_i sets of size $k + i = 0\,,\,1\,,\,2)$.

More generally one could ask: Let

$$\Sigma_i \binom{u_i}{2} = \binom{n}{2}\cdot$$

What is the necessary and sufficient condition that one can find sets $|A_i| = u_i$ so that every pair of S should be contained in one and only one A_i? It is no doubt hopeless to find a good necessary and sufficient condition but perhaps useful necessary and useful sufficient conditions can be found. Also it might be often useful to try to find sets A_i, $|A_i| \leq u_i$ $|A_{i_1} \cap A_{i_2}| \leq 1$ so that all but $o\,(n)$ (or all but $o\,(n^2)$) of the pairs of S are contained in an A_i.

Kovari V. T. Sos Turan and I proved that ((K $(k\,,\,l)$ denotes the complete bipartite graph of k black and l white vertices)

$$f(n\,;\,K\,(l\,,\,l)) \leq \left(\frac{1}{2} + o\,(1)\right) n^{2-1/l}. \tag{6}$$

It seems likely that (6) is in fact an asymptotic formula, this has been proved (as stated) for $l = 2$ but is open for $l > 2$. Brown proved

$$f(n\,;\,K\,(3\,,\,3)) > cn^{5/3}$$

but nothing is known for $l > 3$. The following finite geometry type construction would be needed: Let $|S| = n$, find $c_1\, n$ subsets of S of size $\geq c_2\, n^{1-1/l}$ so that the intersection of any l of them is $< l$. Such a set system would immediately give $f(n\,;\,K\,(l\,,\,l)) > cn^{2-(1/l)}$; in fact this is what Brown did for $l = 3$. A finer analysis might yield the asymptotic formula, but this worked only for $l = 2$.

Before closing this chapter I mention a few other extremal problems on bipartite graphs which we considered. Simonovits and I proved that every $G(n; [cn^{8/5}])$ contains a cube; it would be very interesting to decide if the exponent $8/5$ is best possible; by the way our proof is surprisingly difficult.

At first we thought that for every bipartite graph G $f(n; G)$ is of the form $cn^{1+(1/k)}$ or $cn^{2-(1/k)}$ but Simonovits and I showed that this is not so; we then modified our conjecture and guessed that for every G there is a rational α, $1 < \alpha < 2$ so that

$$f(n; G)/n^{\alpha} \to c(G) \tag{7}$$

and conversely for every α there is a corresponding graph for which (7) holds, we are very far from being able to decide this question. In the next chapter we will see that for hypergraphs $(r > 2)$ $f(n; G^{(r)})$ can have a much more complicated form.

It is known that $f(n, C_{2k}) < c_1 n^{1+(1/k)}$ (for a very much more general and thorough investigation see the forthcoming paper of Bondy and Simonovits, probably $f(n; C_{2k}) > c_2 n^{1+(1/k)}$ but this is known only for $k \leq 3$. and $k = 5$ (R. Singleton « Journal Comb. Theory », *1* (1966), 306–332; C. Benson, « Canad. J. Math », *18* (1966), 1091–1095). The general case could be settled if the following block design like structure would exist: $|S| = n$, $A_r \subset S$, $1 \leq r < c_3 n$, $|A_r| > c_4 n^{1/k}$. We now define a graph as follows: The vertices are the A_r, two vertices are joined if the corresponding sets have a non empty intersection. This graph should have girth $> k$ (i.e. it should contain no C_l for $l \leq k$).

It seems certain that for every $k > 1$

$$f(n; C_{2k})/n^{1+(1/k)} \to c_k.$$

Define $G - e$ as the graph from which the edge e has been omitted. I proved $f(n; K(r, r) - e) < cn^{2-1/(r-1)}$ and very likely

$$f(n; K(r, r) - e)/n^{1-(1/r-1)} \to c'_r,$$

but this is not even known for $r = 3$.

Simonovits and I investigated a few other special graphs. Define $G_{k,r}$ as follows: It has $1 + k + \binom{k}{r}$ vertices $x; y_1, \cdots, y_k; z_1, \cdots, z_{\binom{k}{r}}$. x is joined to all the y's and each z is joined to r y'a (distinct z's to distinct r–tuples). $G'_{k,r} = G_{k,r} - x$ (i.e. the vertex x and all edges (x, y_i) $1 \leq i \leq k$ are omitted from $G_{k,r}$). Estimate $f(n; G_{k,r})$ and $f(n; G'_{k,h})$ as accurately as possible. I proved, that $f(n; G_{3,2}) < cn^{3/2}$, $G_{3,2}$ is a cube with one vertex omitted and $G^{(1)}_{3,2}$ is C_6). Is it true that $f(n; G_{k,2}) < c_k n^{3/2}$? and more generally $f(n; G_{k,r}) < c_k n^{2-(1/r)}$? The first inequality may very well fail for $k > 3$ and the second for $k = 4$, $r = 3$. If $k = r + 1$ the inequality holds and is essentially a consequence of the result of Kovari and the Turans. Perhaps $f(n; G'_{k,r})$ is of the order of magnitude $n^{2-(1/r)-\varepsilon_k}$ with $\varepsilon_k \to 0$ as $k \to \infty$, but we could not settle this even for $r = 2$.

Simonovits and I in fact proved that every $G(n; cn^{3/2})$ contains a cube with one edge omitted (this graph of course contains $G_{3,2}$). An edge e of G is called inessential if

$$f(n; G - e) > cf(n; G)$$

in other words the omission of e does not decrease the order of magnitude of $f(n; G)$. (The same definition can of course be made for vertices). Clearly every graph which is not a tree has a subgraph without an inessential edge. It would be worth while to try to characterise these graphs e.g. is it true that every symmetric (bipartite) graph has this property? (symmetric heve means that the automorphism group is transitive).

It seems certain that every edge of $K(r, r)$, or the r-dimensional cube is essential but this is known only for $r = 2$. On the other hand it is easy to see that every vertex of valency one is inessential.

Let now G have chromatic number greater than two. Then perhaps it is more reasonable to define an edge (or vertex) whose omission does not change the chromatic number $K(G)$ to be inessential if

$$(8) \qquad f(n; G - e) - \frac{n^2}{2}\left(1 - \frac{1}{K(G) - 1}\right) > c\left(f(n; G) - \frac{n^2}{2}\left(1 - \frac{1}{K(G) - 1}\right)\right)$$

or in fact an edge is said to be strongly inessential if

$$(9) \qquad f(n; G - e) = f(n; G).$$

A theorem of Dirac and myself states that

$$(10) \qquad f(n; K(r) - e) = f(n, K(r - 1)).$$

$K(r) - e$ perhaps has as many inessential edges as possible. Simonovits remarks that this is false. If $r = 2s$ and we omit from K_r s independent edges, in the remaining graph every edge is inessential. The following conjecture very likely holds: Assume G has r vertices and $e_1, \cdots, e_{r-1}$ are edges of G then $f(n; G - e_1 -, \cdots, - e_{r-1}) < f(n; G)$ and perhaps even $f(n; G - e_1 -, \cdots, - e_{r-1}) < (1 - c) f(n; G)$. (Simonovits disproved these conjectures).

Finally it is possible that (8) and (9) is possible for a bipartite graph only if it has vertices of valency one. Let G be bipartite and $f(n; G) > cn^{3/2+\varepsilon}$ then perhaps every vertex of valency 2 is inessential.

W. G. BROWN, *On Graphs that do not contain a Thomsen graph*, «Canad. Math Bull.», *9*, (1966), 281–285.

P. ERDÖS, A. RENYI, V. T. SOS, *On a problem of graph theory*, «Studia Sci. Math. Hungar», *1* (1966), 215–235.

I. REIMAN, *Über ein Problem von K. Zarankiewicz*, «Acta Math., Acad. Sci. Hungarica», *9* (1958), 269–278, the proof of E. Klein is given in P. Erdös, *On sequences of integers no one of which divides the product of two others and on some related problems*, «Tomsk Gos Univ. Ocen. Zap.», *2* (1938), 74–82, see also *On some applications of graph theory to number theoretic problems*, «Publ. Ramanujan Inst. Number», *1* (1969), 131–136.

P. ERDÖS, M. SIMONOVITS, *Some extremal problems in graph theory*, «Combinatorial Theory and Applications Coll. Math. Soc. J. Bolyai», *1* (1969), 377–390 (Academic Press).
J. A. BONDY and M. SIMONOVITS, *Cycles of even length in graphs*, «Journal Comb. Theory», Ser. B, *16* (1974), 97–105.
T. KOVARI, V. T. SOS, P. TURAN, *On a problem of K. Zarankievicz* «Coll. Math.», *3* (1954), 50–57.
P. ERDÖS, *On an extremal problem in graph theory*, «Coll. Math.», *13* (1965), 251–254.
P. ERDÖS, *On some extremal problems in graph theory*, «Israel J. Math.», *3* (1965), 113–116.
R. M. WILSON, *An existence theory for pairwise balanced designs*. I. *Composition theorems for mophisms*. II *The Structure of PBD–closed sets and the existence conjectures*. «J. Comb. Theory Ser.», A *13* (1922), 220–245, 246–273. *An existence theory for pairwise balanced designs*. III. *Proof of the existence conjectures*, «ibid», *18*, (1975), 71–79.

3. Now we discuss some extremal problems on hypergraphs. Brown, V. T. Sos and I conjectured that

$$f(n\,;\,G^{(3)}(6\,,\,3) = o(n^2)\,, \tag{1}$$

in fact we thought it likely that it is less than n^{2-c} for a certain $c > 0$. Szemeredi recently proved (1) but I. Ruzsa proved ($r_k(n)$ denotes the cardinality of the largest set of integers not exceeding n which does not contain an arithmetic progression of k terms).

$$f(n\,;\,G^{(3)}(6\,,\,3)) > cn\,r_3(n) > c_1\,n^2/\exp\,(\mathrm{logn})^{1/2} \tag{2}$$

where the second inequality of (2) follows from a well known result of Behrend.

This is the first example of an extremal problem on hypergraphs where the asymptotic formula is certainly not of the form $c_1\,n^\alpha$. It is not known if this Ruzsa-Szemeredi phenomenon can also occur for $r = 2$.

About a year ago Szemeredi proved $r_k(n) = o(n)$, his paper will appear in «Acta Arithmetica», one of his decisive lemmas used in his proof also is needed for the proof of (1). This connection was certainly quite unexpected for all of us. More generally one can conjecture that for every $k > 6$

$$f(n\,;\,G^{(3)}(k\,,\,k-3)) = o(n^2)\,. \tag{3}$$

At the moment of my writing these lines this is still open for $k > 6$ but Ruzsa proved

$$f(n\,;\,G^{(3)}(7\,,\,4)) > cn\,r_4(n)$$

and perhaps

$$f(n\,;\,G^{(3)}(k\,,\,k-3)) > c_k\,nr_{k-3}(n)\,,$$

but Ruzsa proof does not seem to work in general. Ruzsa and Szemeredi will write a joint paper about their results.

A related problem is the following one: Is it possible to find a Steiner system for every $n > n_0(k)$ so that for every $3 < r \leq k$ the system should not contain a $G^{(3)}(r\,;\,r-2)$. Doyen informs me that he can do this for

$k = 6$ and infinitely many n. Instead of a Steiner system one could ask: For which k and c can one find a $G^{(3)}(n; cn^2)$ which does not contain a $G^{(3)}(r; r-2)$ for $3 < r \leq k$?

It is quite possible that many other new types of problems could be found with equally unexpected answers.

Denote by $K_k^{(r)}(l)$ the r-graph of kl vertices and $\binom{k}{r} l^r$ edges. The vertices are divided into k disjoint classes of size l and every r-tuple whose vertices are in different classes in an edge of our graph I proved that for every r and k there is an $\varepsilon_{k,r} > 0$ so that for $n > n_0(k, r, \varepsilon_l)$ every $G^{(r)}(n; n^{r-\varepsilon_{k,r}})$ contains a $K_r^{(r)}(l)$. This is an extension of the result of Kovari and the Turans stated in Chapter 2. For $n > 2$ nothing seems to be known about the best possible values of the exponents $\varepsilon_{k,r}$. Every $G^{(3)}(n, cn^{11/4})$ contains a $K_3^{(3)}(2)$ but it is not known whether the exponent $11/4$ could not be decreased.

W. G. BROWN, P. ERDÖS, V. T. SOS, *Some extremal problems on r-graphs*, New directions in the theory of graphs, Proc. third conference on graph theory at « Ann Arbor Acad. press », 1973, 53–63, *On the existence of triangulated spheres in 3-graphs and related problems*, « Studia Sci. Math. Hungar. ».

P. ERDÖS, *On extremal problems of graphs and generalized graphs*, « Israel J. Math », *2* (1965), 183–190.

F. BEHREND, *On sets of integers which contain no three terms in an arithmetic progression*, « Proc. Nat. Acad. Sci. USA », *32* (1964), 331–332.

4. Some remarks on a theorem of Stone and myself. Stone and I proved that for $n > n_0(\varepsilon, k, l)$ every $G\left(n; \frac{n^2}{2}\left(1 - \frac{1}{k-1} + \varepsilon\right)\right)$ contains a $K_k^{(2)}(l)$ (for $k = 2$ this is again a weaker form of the Kovari–Sos, Turan theorem). Our original proof did not give a very good dependence of n on l and ε.

A very much sharper result in this direction was just published by Bollobas and myself, a further improvement which is nearly best possible has recently been obtained by Bollobas, Simonovits and myself: Chvatal and Szemeredi obtained a further very significant improvement.

Recently I succeeded to extend this theorem to r-graphs as follows: To every r, ε, t and l there is an $n_0 = n_0(\varepsilon, r, t, l)$ so that every $G^{(r)}\left(n; (\alpha(t, r) + \varepsilon)\binom{n}{r}\right)$ contains a $K_t^{(r)}(l)$ where $\alpha(t, r)$ is defined by (1) of chapter 1. Here we do not yet have a good estimate of n in terms of ε, k and l (unlike for $r = 2$).

The following problem is open and seems very challenging to me: Let $G^{(r)}(n_i)$ $i = 1, 2, \cdots, n_i \to \infty$ be a sequence of r-graphs of n_i vertices. We say that the family has subgraphs of edge density $\geq \alpha$ if there is a sequence of subgraphs $G(m_i)$ of $G(n_i)$, $m_i \to \infty$, so that $G(m_i)$ has at least $(\alpha + \sigma(1))\binom{m_i}{r}$ edges. The theorem of Stone and myself implies that every

$G\left(n ; \frac{n^2}{2}\left(1-\frac{1}{l}+\varepsilon\right)\right)$ contains a subgraph of density $1-\frac{1}{l+1}$ and it is easy to see that this is best possible. Thus the possible maximal densities of subgraphs are of the form $1-\frac{1}{l}$, $1 \leq l < \infty$. Now it may be true that for $r > 2$ there are also only a denumerable number of possible values of the maximal densities of subgraphs. As stated at the end of the previous chapter I proved that every r-graph of density ε contains a subgraph of density $\geq \frac{r!}{r^r}$. The simplest unsolved problem states: Is there a constant $\alpha_r > 0$ so that every r-graph of n vertices (n large) and $\left(\frac{r!}{r^r}+\varepsilon\right) n^r$ edges contains a subgraph of density $\geq \frac{r!}{r^r}+\alpha_r$. This is unsolved even for $r=3$. Perhaps every $G^{(3)}(3n ; n^3+1)$ contains either a $G^{(3)}(4 ; 3)$ or a $G^{(3)}(5 ; 4)$, $(1,2,3),(1,2,4),(1,2,5),(3,4,5)$ or a $G^{(3)}(5,5),(1,2,3),(1,2,4)$, $(1,3,5),(2,4,5),(3,4,5)$.

The same unsolved problems on the possible maximal densities arise on multigraphs and digraphs as stated in a recent paper of Brown, Simonovits and myself.

By the methods of probabilistic graph theory it is easy to prove that to every ε and $0<\alpha<1$ there is a $C=C(\varepsilon, \alpha)$ so that for $n > n_0(C, \varepsilon, \alpha)$ there is a $G^{(r)}\left(n, \alpha\binom{n}{r}\right)$ so that for every $m > C(\log n)^{1/(r-1)}$ every spanned subgraph of it m vertices has more than $(\alpha-\varepsilon)\binom{m}{r}$ and less than $(\alpha+\varepsilon)\binom{m}{r}$ edges and it follows from the results of my paper on graphs and generalized graphs that this result is best possible (« el Journal Math. », *2* (1965), 183–190).

P. ERDÖS, A. STONE, *On the structure of linear graphs*, « Bull. Amer. Math. Soc. », *52* (1946), 1087–1091.

B. BOLLOBAS, P. ERDÖS, *On the structure of edge graphs*, « Bull. London Math. », *15* (1937), 317–321.

B. BOLLOBAS, P. ERDÖS and M. SIMONOVITS, *On the structure of the edge graphs* II, « J. London Math. Soc. », *12* (1976), 219–224.

P. ERDÖS, *On some extremal problems on r-graphs*, « Discrete Math. », *1* (1971), 1–6.

W. G. BROWN, P. ERDÖS, M. SIMONOVITS, *Extremal problems for directed graphs*, « J. Comb. Theory », ser. B. *15* (1973), 77–93.

5. In this chapter I discuss various combinatorial problems on subsets. First of all I call attention to my paper with Kleitman quoted in the introduction. Here I mainly discuss problems not considered in our survey paper.

First we consider some problems related to a result of Ko, Rado and myself. Let $|S| = n$, $A_i \subset S$ $|A_i| = k$. Denote by $t(n ; k, r, \alpha)$ the size of the largest family A_j, $1 \leq j \leq t(n ; k, r, \alpha)$ satisfying $|A_{j_1} \cap A_{j_2}| \leq r$ and every element is contained in at most $\alpha t(n ; k, r, \alpha)$ of the A's. $t(n ; k, r, <\alpha)$ is the size of the largest subfamily with the same properties but now every element is contained in fewer than $\alpha t(n\,;k\,,r\,,<\alpha)$ of the A's. Ko, Rado and I proved that for $n \geq 2k$

$$(1) \qquad t(n\,;k\,,1\,,1) = \binom{n-1}{k-1}.$$

For $n > 2k$ equality holds only if all the A's have a common element. For $n > n_0(k\,,r)$ we further proved

$$(2) \qquad t(n\,;k\,,r,1) = \binom{n-r}{k-r}.$$

Our estimation for $n_0(k\,,r)$ is probably very poor, but Min observed that (2) does not hold for all $n \geq 2k$. We conjectured that

$$(3) \qquad t(4l\,;2l\,,2\,,1) = \frac{1}{2}\left\{\binom{4l}{2l} - \binom{2l}{l}^2\right\}$$

(3) if true is best possible. We state in our paper several other problems most of which has been settled since then, but as far as I know (3) has not been settled as yet.

Hilton and Milner proved that for $n \geq 2k$.

$$(4) \qquad t(n\,;k\,,1\,,<1) = 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}.$$

Equality in (4) occurs if (and no doubt only if $n > n_0(k\,,r)$), A_1 is an arbitrary k–tuple x_1 is not in A_1. All the other A's contain x_1 and have a non-empty intersection with A_1.

Observe that for fixed k

$$t(n\,;k\,,1\,,1) = (1 + o(1))\, n^{-1} \binom{n}{k},$$

but from (4)

$$t(n\,;k\,,1\,,<1) = (1 + o(1))\, n^{-2} \binom{n}{k}.$$

Now Rothschild, Szemeredi and I took up this investigation. We first of all showed that for $\alpha = 2/3$

$$(5) \qquad t(n\,;k\,,1\,,2/3) = 3\binom{n-2}{k-2} - 2\binom{n-3}{k-3}.$$

Equality if and only if (until further notice n is supposed to be large), there are three elements and the A's contain at least two of them.

We further proved

$$t(n\,;k\,,t < 2/3) = (1 + o(1))\, cn^{-3} \binom{n}{k}.$$

The extremal family is obtained as follows: give three elements $x_1\,,x_2\,,x_3$ and a set A_1 not containing any of them. All the other A's meet A_1 and contain at least two of the x's.

Let now $\varepsilon > 0$ be sufficiently small. We are fairly sure that a family of size $t(n;k,1,2/3-\varepsilon)$ is obtained as follows: Let $x_1,\cdots,x_5$ be five elements, the A's contain three or more of them and $t(n;k,1,\alpha)$ is constant between $1/2$ and $3/5$. There seem to be only a finite number of values of $t(n;k,1,\alpha)$ for $3/7 < \alpha < 2/3$. $t(n;k,1,3/7)$ is probably obtained as follows: Consider a set $B \subset S$, $|B| = 7$ and the 7 Steiner triples of B. The A's are all the sets which meet B in a set which contains at least one of these triples. We also are fairly sure that

$$t(n;k,1,<3/7) < \frac{c}{n^4}\binom{n}{k}.$$

More generally we conjecture that

$$t\left(n;k,1,<\frac{l}{l^2-l+1}\right) < \frac{c}{n^{l+1}}\binom{n}{k}.$$

If there is a finite geometry on l^2-l+1 elements then it is easy to see that

$$t\left(n;k,1,\frac{l}{l^2-l+1}\right) = \frac{c}{n^l}\binom{n}{k}$$

but if there is no such finite geometry we conjecture that

$$t\left(n;k,1,\frac{l}{l^2-l+1}\right) < \frac{c}{n^{l+1}}\binom{n}{k}.$$

Needless to say these last two conjectures are very speculative. See a forthcoming paper of A. J. W. Hilton on this subject.

Kneser made the following pretty conjecture: Let $|S| = 2n+k$ define a graph $G_{n,k}$ as follows: Its vertices are the $\binom{2n+k}{n}$ n-tuples of S. Two vertices are joined if the corresponding n-sets are disjoint. Denote by $K(G)$ the chromatic number of G. Kneser conjectured $K(G_{n,k}) = k+2$. $K(G_{n,k}) \leq k+2$ is immediate but the opposite inequality seems to present great and unexpected difficulties. Szemeredi proved (unpublished) that $K(G_{n,k})$ tends to infinity uniformly in k. Hajnal and I and no doubt many others tried to attack this problem by the following extension of our theorem with Ko and Rado. Let $|S| = n \geq 2k+1$, $A_i \subset S$, $B_j \subset S$ $1 \leq i \leq t_1$, $1 \leq j \leq t_2$, the sets $A_1,\cdots,B_1,\cdots$ are all distinct, $A_{i_1} \cap A_{i_2}$ $1 \leq i_1 < i_2 \leq t_1$, and $B_{j_1} \cap B_{j_2}$ $1 \leq j_1 < j_2 \leq t_2$ are all non empty. Is it true that

$$t_1 + t_2 \leq \binom{n-1}{k-1} + \binom{n-2}{k-1}. \tag{6}$$

Equality in (6) if all the A's contain 1 and all the B's contain 2 but not. 1. A.J.W. Hilton proved that (6) does not hold in general. For the applications $t_1 + t_2 < \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$ would suffice.

Knesers conjecture can be extended to r-graphs. Let $|S| = rn + k$. The vertices of our r-graph are the k-tuples of S. The edges are the sets

$$A_{i_1}, \cdots, A_{i_r} \quad ; \quad |A_{i_j}| = k \quad , \quad 1 \le j \le r$$

and any two of the r k-sets are disjoint. The chromatic number or this r r-graph should be $k + 2$.

B. Grunbaum asked the following geometric question:

Let there be given n points in the plane, join any two of them by a line. What are the possible number of lines one gets. The number of lines is clearly at most $\binom{n}{2}$ and it can never be $\binom{n}{2} - 1$ and $\binom{n}{2} - 3$. I showed that there is an absolute constant c so that every $cn^{3/2} < t < \binom{n}{2} - 3$ can occur as the number of lines determined by an n-set. It follows from a result of Kelly and Moser that the order of magnitude $cn^{3/2}$ is best possible but the exact value of c is not known.

In this connection the following combinatorial problem is of interest. Let $|S| = n$, define I_r as a set of integers, with the following property: $t \in I_r$ if there is a family of subsets $A_k \subset S$ $1 \le k \le t$ so that every r-tuple of S is contained in one and only one of the A's. Let us first investigate the $r = 2$. Clearly all integers in I_2 are $\le \binom{n}{2}$, $1 \in I_2$ and $\binom{n}{2} - 1$ and $\binom{n}{2} - 3$ is not in I_2. A Theorem of de Bruijn and myself states that no integer $1 < t < n$ is in I_2. Trivially $n \in I_2$ and $\binom{n}{2} - 2 \in I_2$. I showed without much difficulty that there are absolute constants c_1 and c_2 so that every integer $n + c_1 n^{c_2} < t < \binom{n}{2} - 3$ is in I_2. It seems likely that $c_2 = 1/2$. If $n = p^2 + p + 1$ (i.e. if there is a finite geometry) it is easy to see that every $p^2 + 2p + c\sqrt{p} = n + 2\sqrt{n} + cn^{1/4} < t < \binom{n}{2} - 3$ belongs to I_2.

On the other hand A. Bruen recently proved that if $n = k^2$ then $t \notin I_2$ if $k^2 < t < k^2 + k$.

It seems that the results of A. Bruen and Bridges will give that there is an absolute constant $c > 0$ so that for every n there is a t not in I_2 which is $> n + c\sqrt{n}$.

It was observed by Hanani that the smallest nontrivial value of I_3 is $cn^{3/2}$ and it follows from the existence of Mobius (or inversive) planes that I_3 contains all integers t, $\left(1 + o(1)\, n^{3/2} < t \le \binom{n}{3}\right.$ except the integers $\binom{n}{3} - i$, where i is not of the form $\left.\sum_{j \ge 4} \alpha_j \left(\binom{j}{3} - 1\right)\right.$, $\alpha_j \ge 0$.

For $r > 3$ it is much more difficult to get sharp results for I_r. It is easy to see that if $t > 1$, $t \in I_r$ then $t > cn^{r/2}$. This follows from the fact that not many of the sets A_k can be larger than $(1 + \varepsilon)\, r^{1/2} n^{1/2}$ (for otherwise $|A_i \cap A_j| \ge r$), (see e.g. Hylten-Cavallius, on a combinatorial problems, "Colloq. Math." *6* (1958), 59–65). But it seems hard to prove that I_r contains an integer $1 < t < Cn^{r/2}$. The problem is to find $c_1 n^{r/2}$ sets A_k of size of the order of magnitude $n^{1/2}$ so that every r-tuple of our set $|S| = n$ should be contained in one and only one of the A_k's. Such a construction in known for $r = 2$ and $r = 3$, but it is open for $r > 3$.

Before closing this section I state one of the many unsoved problems in our survey paper with Kleitman: Let $|S| = n$, $A_i \subset S$, $1 \leq i \leq t$, assume that for no three distinct A's $A_i \cap A_j = A_k$ or $A_i \cup A_j = A_k$. We conjectured that for even n $\max t = \binom{n}{\left[\frac{n}{2}\right]} + 1$. Clements observed that this conjecture if true is best possible.

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6. In this last chapter I state a few miscellaneous problems which my colleagues and I considered recently.

Let $G(n; l)$ be a graph of n vertices and l edges. Goodman, Posa and I proved that the edges of our graph can be covered by at most $\left[\frac{n^2}{4}\right]$ edge disjoint cliques where the cliques are in fact all edges or triangles. The complete bipartite graph shows that $\left[\frac{n^2}{4}\right]$ is best possible.

It is not quite clear what is the best possible result if we want to cover $G\left(n; \left[\frac{n^2}{4}\right] + l\right)$ by edge disjoint cliques, though Lovasz has some results here.

Gallai and I conjectured that every $G(n; k)$ can be covered by at most cn edge disjoint circuits and edges. We could only prove this with cn logn instead of cn.

Perhaps every $G(n; l)$ can be covered by at most $f(n; K(r)) - 1)$ edge disjoint edges and $K(r)$'s, if correct this is perhaps not hard to prove

($f(n\,;\,K(r))$ is Turan's function introduced in chapter one). This conjecture was in fact proved by B. Bollobás.

Let now $G^{(r)}(n\,;l)$ be an r-graph of n vertices and l edges. Sauer and I conjectured that its edges can be covered by at most $f(n\,;\,K^{(r)}(r+1)) - 1$ edge disjoint $K^{(r)}(r+1)$'s and edges (i.e. $K^{(r)}(r)$'s). For $r = 3$ already this conjecture seems difficult (if true).

Another problem of Sauer and myself states. Determine or estimate the smallest $g(t\,,n)$ so that every $G(n\,;g(t\,,n))$ contains a regular subgraph of valency t. Clearly $g(2\,,n) = n$ but we have no idea of the value of or even the order of magnitude of $g(3\,,n)$.

At the meeting in Rome R. Guy told me that I conjectured that the vertices of every tree of n vertices can be numbered by the integers $1\,,\cdots,n$ so that the integers corresponding to two vertices which are joined are relatively prime. This seems a nice conjecture which is perhaps not very difficult. I certainly do not remember having ever stated it. The conjecture in fact is due to Entringer.

Is it true that to every $\varepsilon > 0$ there is a c_ε so that every $G(n\,;[n^{1+\varepsilon}])$ contains a subgraph which is not planar and has at most c_ε vertices?

An old conjecture of Hajnal and myself states that there is a function $f(k\,,l)$ so that every graph of chromatic number $\geq f(k\,,l)$ contains a subgraph of chromatic number k and girth $\geq l$ (the girth of G is the length of its shortest circuit. This is unsolved even for $l = 4$.

Another conjecture of Hajnal and myself states that if G is k-chromatic

$$\Sigma \frac{1}{n_i} > c \log k$$

where $n_1 < n_2 < \cdots$ are the C_{n_i} contained in G. We can not even prove that $\Sigma \frac{1}{n_i}$ tends to infinity together with k.

V. T. Sos and I observed that if $|S| = n$ and $A_i \subset S\,,|A_i| = 3$, $1 \leq i \leq n+1$ then there always are two A's which have exactly one common element. The proof is easy. We then made the following more difficult conjecture. Let $n > n_0(k)$. Is it true that if $A_i \subset S\,|A_i| = k$, $1 \leq i \leq \binom{n-2}{k-2} + 1$ then there are two A's which have exactly one common element? This conjecture was proved by Katona for $k = 4$ but is open for $k > 4$. It is clearly related to the Theorem of Ko, Rado and myself discussed in the previous chapter.

The following problem can be stated here whose solution would be useful in n-dimensional geometry (see D. G. Larman and A. Rogers, The realisation of distances within sets in Euclidean space, "Mathematika" *19* (1972), 1-24).

Let $\eta > 0$ be given. Prove that there is an $\varepsilon = \varepsilon(\eta) > 0$ so that if $|S| = n\,,n > n_0(\varepsilon\,,\eta)$ and $A_i \subset S\,,1 \leq i \leq t\,,t > (2-\varepsilon)^n$ are subsets of S then for every r, $\eta n < r < (1/2 - \eta)\,n$ these are two A's whose intersection has exactly r elements.

Let $|S| = 2n$, $A_i \subset S$, $1 \leq i \leq t$. Is it true that if the number of indices i_1, i_2 with $A_{i_1} \cap A_{i_2}$ empty is at least 2^{2n} then $t > (1 - \varepsilon) 2^{n+1}$? More generally for given n and t determine or estimate the maximum number of pairs i_1, i_2, $1 \leq i_1 < i_2 \leq t$ for which $A_{i_1} \cap A_{i_2} = \varnothing$.

A well known theorem of Van der Waerden states that if one splits the integers into two classes at least one of them contains an arbitrarily long arithmetic progression. As stated in section 3 Szemeredi proved $r_k(n) = o(n)$ which is a very significant strengthening of Van der Waerden's theorem.

Graham and Rothschild conjectured that if one splits the integers into two classes there always is an infinite sequence of integers $n_1 < n_2 < \cdots$ so that all the sums

$$(1) \qquad \sum_i \varepsilon_i n_i, \qquad \varepsilon_i = 0 \quad \text{or} \quad 1, \qquad \sum_i \varepsilon_i < \infty$$

are in the same class. This conjecture was recently proved by Hindman. A simpler proof was very recently found by Baumgartner. Both paper appeared in the «Journal of combinatorial theory». I then conjectured that if $a_1 < a_2 < \cdots$ is an infinite sequence of integers of positive density there always is another infinite sequence $n_1 \cdots$ and a t so that all the integers (1) translated by t are a's. Straus found an easy counterexample. But perhaps it is true that there is an infinite sequence $n_1 \cdots$ and a t so that all the integers $t + \Sigma \varepsilon_i n_i$, $\Sigma_i \varepsilon_i = 1$ or 2 are a's.

Finally I state a conjecture of Faber, Lovasz and myself which seems very fascinating to me:

Let $|A_k| = n$, $1 \leq k \leq n$; $|A_{k_1} \cap A_{k_2}| \leq 1$, $1 \leq k_1 < k_2 \leq n$. Is it true that one can color the elements of $\bigcup_{k=1}^{n} A_k$ by n colors so that each A_k contains an element of each color?

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