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SOME PROBLEMS AND RESULTS ON THE IRRATIONALITY OF THE SUM OF INFINITE SERIES

by P. Erdös

It is usually extremely difficult to decide whether the sum of a convergent infinite series is irrational or not. $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ was proved by Euler to be a polynomial in π and is thus transcendental, but $\sum_{k=1}^{\infty} \frac{1}{n^3}$ seems intractable.

The situation is a little better if the series converges very fast. I proved that if $n_k^{1/2^k} \to \infty$ then $\sum_{n=1}^{\infty} \frac{1}{n_k}$ is irrational, Straus and I [1] proved the following theorem which is somewhat deeper:

Let lim sup $n_k^2/n_{k+1} \leqslant 1$ and further assume that

$$\limsup \frac{N_k}{n_{k+1}} \left(\frac{n_{k+1}^2}{n_{k+2}} - 1 \right) \leqslant 0; \tag{1}$$

then $\sum_{n+1}^{\infty} \frac{1}{n_k}$ is irrational except if $n_{k+1} = n_k^2 - n_k + 1$ for all $k \geqslant k_0$ where N_k is the least common multiple of $n_1, \ldots, n_k$. It is possible that our theorem remains true without the assumption (1) but we have not been able to prove this.

In this paper I prove the following:

THEOREM 1 Let $n_1 < n_2 < \ldots$ be an infinite sequence of integers satisfying

$$\lim_{k=\infty} \sup n_k^{1/2^k} = \infty \tag{2}$$

and

$$n_k > k^{1+\epsilon} \tag{3}$$

for some fixed $\epsilon > 0$ and $k > k_0(\epsilon)$. Then

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{n_k}$$

is irrational.

The proof will not be entirely trivial. Theorem 2 is much simpler.

THEOREM 2 Assume that (3) holds and that for every t

$$\lim_{k=\infty} \sup n_k^{1/t^k} = \infty . \tag{4}$$

Then α is a Liouville number.

It is easy to see that Theorem 1 is best possible. It is well known and easy to see that for every A there is a sequence n_k satisfying $n_k > A^{2^k}$ for every $k > 0$ but $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is rational. (3) is also best possible. Let

$$f(k) \to \infty, \quad \log f(k)/\log k \to 0.$$

There is a sequence n_k satisfying (2) and $n_k > k\,f(k)$ for all k; but $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is rational. We leave the details to the reader.

$\sum_{k=1}^{\infty} \frac{1}{2^{t^k}}$ is not a Liouville number, thus (4) is best possible; but I think if (4) holds then a much weaker condition than (3) will ensure that α is a Liouville number, but I have not yet succeeded in clearing this matter up.

Before I prove the Theorems, I state a few unsolved problems. Let $n_1 < n_2 < \ldots$, $\lim\sup n_k/k = \infty$. Is it true that $\sum_{k=1}^{\infty} \frac{n_k}{2^{n_k}}$ is irrational? I cannot prove this even if $n_{k+1} - n_k \to \infty$ is assumed, but I have no counterexample if we only assume that $\lim\sup (n_{k+1} - n_k) = \infty$. In other words I have no example of a series $\sum_{k=1}^{\infty} \frac{n_k}{2^{n_k}}$ whose sum is rational, but $\lim\sup (n_{k+1} - n_k) = \infty$. I would guess that such a series exists.

Is it true that for every integer a there is a finite sequence of integers $a < m_1 < \ldots < m_k$ for which

$$\frac{a}{2^a} = \sum_{i=1}^{k} \frac{m_i}{2^{m_i}} \, ?$$

Let $n_1 \leqslant n_2 \leqslant \ldots, n_k \to \infty$, $d(n)$ denotes the number of divisors of n. Straus and I proved that

$$\sum_{k=1}^{\infty} \frac{d(k)}{M_k}, \quad M_k = \prod_{i=1}^{k} n_i \tag{5}$$

is irrational [2]. Very likely $n_k \to \infty$ (without assuming monotonicity) suffices for the irrationality of (5). I find it frustrating that I cannot prove the irrationality of $\sum_{n=2}^{\infty} \frac{1}{n! - 1}$. For further problems see [3].

Obviously $\sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = 1$. This led Straus and me to the following question: A sequence $n_1 < n_2 < \ldots$ is said to have property P if for every

$m_k > 0$, $m_k \equiv 0 \pmod{n_k}$ $\sum_{k=1}^{\infty} \frac{1}{m_k}$ is irrational. In particular we wondered if $n_k = 2^{2^k}$ has property P. I will prove this conjecture. By the way property P is only interesting if $\lim n_k^{1/2^k} < \infty$ and in fact I cannot prove that such a sequence with property P exists if $(n_i, n_j) = 1$ is also assumed. I do not know if there is a sequence n_k with property P for which n_k does not tend to infinity very fast.

To prove our Theorems we first need the following simple lemma:

LEMMA. Let $n_1 < \ldots$ satisfy (3) for every k. Then

$$\sum_{i=1}^{\infty} \frac{1}{n_{k+i}} < \frac{c_\epsilon}{n_{k+1}^{\epsilon/1+\epsilon}}.$$

The proof is very easy. First of all it is clear from (3) that the number of $n_i < x$ is at most $x^{1/1+\epsilon}$; thus from (3) we easily obtain

$$\sum_{i=1}^{\infty} \frac{1}{n_{k+i}} < \frac{1}{n_{k+1}} n_{k+1}^{1/1+\epsilon} + \sum_{T > n_{k+1}^{1/1+\epsilon}} \frac{1}{T^{1+\epsilon}} < \frac{c_\epsilon}{n_{k+1}^{\epsilon/1+\epsilon}}$$

which proves the Lemma.

Our Lemma almost immediately implies Theorem 2. To prove that α is a Liouville number it suffices to show that for every s there is a k so that

$$\left|\alpha - \sum_{i=1}^{k} \frac{1}{n_i}\right| < \frac{1}{M_k^s}, \quad \left(M_k = \prod_{i=1}^{k} n_i\right). \tag{6}$$

To prove (6) let $t = t(\epsilon, s)$ be sufficiently large and choose k such that

$$n_{k+1}^{1/t^{k+1}} > n_j^{1/t^j} \quad \text{for every} \quad j \leqslant k. \tag{7}$$

Such a k exists by (4). Thus by (7)

$$M_k < n_{k+1}^{1/t-1}. \tag{8}$$

(8) and our Lemma immediately give (6) for sufficiently large t which proves Theorem 2.

The proof of Theorem 1 will be more complicated. First of all assume that for every l there is a k so that

$$n_{k+1} > M_k^l. \tag{9}$$

(9) easily implies the irrationality of $\alpha = \sum_{i=1}^{\infty} \frac{1}{n_i}$. Assume $\alpha = \frac{a}{b}$. Multiply both sides by bM_k. We obtain that $bM_k \sum_{i=1}^{\infty} \frac{1}{n_{k+i}}$ is a positive integer and therefore $\geqslant 1$. From (9) and Lemma 1 we thus obtain

$$bn_{k+1}^{1/l} n_{k+1}^{-\epsilon/1+\epsilon} \geqslant 1$$

which is clearly false for $l > \frac{1+\epsilon}{\epsilon}$ and sufficiently large k. This contradiction proves the irrationality of α.

Henceforth we can assume that there is an l so that for every k

$$n_{k+1} < M_k^l. \tag{10}$$

(10) implies by induction that for every k

$$n_k < 2^{(l+1)^k}. \tag{11}$$

To prove Theorem 1 we now distinguish two cases. Assume first that for every $k > k_0$

$$n_k > 2^k. \tag{12}$$

(12) implies $\sum_{n_k < x} 1 < \frac{\log x}{\log 2} + 0(1)$. Thus by the same argument as used in the proof of our Lemma we obtain that (12) implies that for some absolute constant c and every k

$$\sum_{i=1}^{\infty} \frac{1}{n_{k+i}} < \frac{c \log n_k}{n_k}. \tag{13}$$

Put $n_k^{1/2^k} = L_k$. By (1) $\lim_{k=\infty} \sup L_k = \infty$. Thus it is easy to see that for infinitely many k

$$L_{k+1} > \left(1 + \frac{1}{k^2}\right) \max_{1 \leqslant j \leqslant k} L_j. \tag{14}$$

If (14) would hold for only a finite number of values of k let k_0 be the largest such k and then for every $r > k_0$

$$L_r \leqslant \max_{1 \leqslant k \leqslant k_0} L_k \prod_{k \leqslant k_0} \left(1 + \frac{1}{k^2}\right) < c$$

which contradicts (1). As far as I know this simple and useful idea was first used by Borel, but I cannot give an exact reference.

(11) and (14) easily imply the irrationality of α. Assume $\alpha = \frac{a}{b}$ and let k satisfy (14) and be sufficiently large. As before we obtain that

$$bM_k \sum_{i=1}^{\infty} \frac{1}{n_{k+i}} \geqslant 1. \tag{15}$$

Thus by (13) and (15)

$$bcM_k \frac{\log n_{k+1}}{n_{k+1}} \geqslant 1. \tag{16}$$

By (14)

$$n_{k+1} > M_k \left(1 + \frac{1}{k^2}\right)^{2^{k+1}}. \tag{17}$$

Thus from (16) and (17)

$$n_{k+1} > \exp\left[\left(1+\frac{1}{k^2}\right)^{2^k}\Big/bc\right]$$

which contradicts (11) for sufficiently large $k > k_0(l)$; hence α is irrational.

Thus finally we can assume that for infinitely many k

$$n_k \leqslant 2^k\,. \tag{18}$$

As in the previous cases to prove the irrationality of α we show that

$$\liminf_{k=\infty} M_k \sum_{i=1}^{\infty} \frac{1}{n_{k+i}} = 0\,. \tag{19}$$

To prove (19) we shall show that for every $\epsilon > 0$ there is a $k = k_\epsilon$ so that

$$M_k \sum_{i=1}^{\infty} \frac{1}{n_{k+i}} < \epsilon\,. \tag{20}$$

To prove (20) we will use (18), (10), (11) and (2). Let $A = A(\epsilon)$ be sufficiently large and let k_1 be the smallest integer for which

$$L_{k_1} > \max_{k<k_1} L_k > A\;(L_k = n_k^{1/2^k})\,. \tag{21}$$

By (2) such a k exists. From our Lemma we have

$$\sum_{i=1}^{\infty} \frac{1}{n_{k_1+i}} < \frac{1}{n_{k_1}^{\epsilon/1+\epsilon}} < \frac{1}{(A^{\epsilon/1+\epsilon})^{2^k}}\,. \tag{22}$$

Let k_2 be the greatest integer not exceeding k_1 satisfying (18). By our assumption such a k_2 exists. From (13) and (22) we have for every $k_2 < k < k_1$

$$\sum_{i=1}^{\infty} \frac{1}{n_{k+i}} < \frac{c \log n_{k+1}}{n_{k+1}} + \frac{1}{n_{k_1}^{\epsilon/1+\epsilon}}\,. \tag{23}$$

Observe that $n_{k_2}^{1/2^{k_2}} \to 1$, $n_{k_1}^{1/2^{k_1}} \to \infty$. Thus as in (14) there is a $k_2 \leqslant k \leqslant k_1$ for which

$$L_{k+1} > \left(1+\frac{1}{k^2}\right) \max_{k_2 \leqslant j \leqslant k} L_j\,. \tag{24}$$

Let in fact k_0 be the smallest k satisfying (24). [Observe that from (10) and (11) it is easy to see that $L_{k_0} < C$ and $k_1 - k_0 \to \infty$]. From (24) it follows as in (17) that

$$n_{k_0+1} > M_{k_0}\left(1+\frac{1}{k_0^2}\right)^{2^{k_0}} M_{k_2}^{-1}\,.$$

But from $n_{k_2} < 2^{k_2}$, $M_{k_2} < 2^{k_2^2}$. Thus

$$n_{k_0+1} > M_{k_0}\left(1+\frac{1}{k_0^2}\right)^{2^{k_0}} 2^{-k_2^2} > M_{k_0}\left(1+\frac{1}{2k_0^2}\right)^{2^{k_0}}\,. \tag{25}$$

Now from (25) and (23)

$$M_{k_0} \sum_{i=1}^{\infty} \frac{1}{n_{k_0+i}} < \left(1 + \frac{1}{2k_0^2}\right)^{-2^{k_0}} c \log n_{k_0+1} + M_{k_0} n_{k_1}^{-\epsilon/1+\epsilon}. \tag{26}$$

Now by $L_{k_0} < C$, $M_{k_0} < 2^{k_0^2}$ we have

$$M_{k_0} < C^{2^{k_0+1}} 2^{k_0^2} < (2C)^{2^{k_0+1}}.$$

But $n_{k_1} > A^{2^k}$. Thus for sufficiently large A

$$M_{k_0} \sum_{i=1}^{\infty} \frac{1}{n_{k_0+i}} < \left(1 + \frac{1}{2k_0^2}\right)^{-2^{k_0}} c \log n_{k_0+1} + 2^{-2^{k_0}}. \tag{27}$$

(27) and (13) implies (20) and (19) and thus our proof of the irrationality of α is complete.

It is easy to prove by the same method that if $\liminf_{k=\infty} n_k^{1/2^k} > 1$ and $\lim_{k=\infty} n_k^{1/2^k}$ does not exist then $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is irrational.

Now we prove

THEOREM 3 We have $n_k \equiv 0 (\text{mod } 2^{2^k})$, $n_k > 0$ and wish to prove that $\alpha = \sum_{k=1}^{\infty} \frac{1}{n_k}$ is irrational. Observe that we did not assume that the sequence $\{n_k\}$ is monotonic. Reorder it as a monotonic sequence $m_1 \leqslant m_2 \leqslant \ldots$. We evidently have $m_k \geqslant 2^{2^k}$. Thus we can assume

$$\limsup_{k=\infty} m_k^{1/2^k} = C < \infty \tag{28}$$

for otherwise the irrationality of α immediately follows from Theorem 1. (28) and $m_k \geqslant 2^{2^k}$ imply as in the proof of our Lemma that

$$\sum_{i=1}^{\infty} \frac{1}{m_{k+i}} < \frac{c}{m_{k+1}}. \tag{29}$$

At least two of the m_i's, $1 \leqslant i \leqslant k$ are divisible by $2^{2^{k-1}}$. Thus

$$M_k \leqslant N_k \, 2^{-2^{k-1}} \tag{30}$$

where N_k is the least common multiple of the m_i, $1 \leqslant i \leqslant k$ and M_k is their product. Let now m_{k_r} be a sequence satisfying

$$m_{k_r} > (C - \epsilon_r)^{2^k}, \epsilon_r \to 0 \text{ as } k_r \to \infty. \tag{31}$$

To prove the irrationality of α it clearly suffices to show that

$$\lim_{r=\infty} M_{k_r-1} \sum_{i=0}^{\infty} \frac{1}{m_{k_r+i}} = 0. \tag{32}$$

Thus by (29) it suffices to show that

$$\lim_{r=\infty} M_{k_r-1}/m_{k_r} = 0. \tag{33}$$

By (28), (31) and (30 we obtain by a simple computation that for every $\delta > 0$ if $r > r_0(\delta)$

$$m_{k_r} > N_{k_r-1}(1+\delta)^{-2^k} > M_{k_r-1}2^{2^{k-2}}(1+\delta)^{-2^k}$$

which implies (32) and therefore Theorem 3 is proved.

I cannot decide whether there is a sequence u_k having property P and satisfying $u_k^{1/2^k} \to 1$, or $u_k > C^{2^k}$, $(u_i, u_j) = 1$. I would tentatively guess that such sequences exist.

References

[1] Erdös, P. and Straus, E. G. (1968): On the irrationality of certain Ahmes series, *J. Indian Math. Soc.*, **27**, 129–133.

[2] Erdös, P. and Straus, E. G. (1971): Some number theoretic results, *Pacific J. Math.*, **36**, 639–646; a second paper on the same topic will soon appear in the *Pacific Journal*.

[3] For some further problems see my papers: On arithmetical properties of Lambert series, *J. Indian Math. Soc.*, **12** (1948), 63–66; On the irrationality of certain series, *Nederl. Akad. Wetensch (Indigationes Math.)*, **60** (1957), 212–219; Sur certaines séries a valeur irrationnelle, *Enseignment., Math.* **4** (1958), 93–100.