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THE NONEXISTENCE OF CERTAIN INVARIANT MEASURES

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ABSTRACT. It is shown that there does not exist an uncountable group G and a nontrivial, σ-finite, countably additive measure defined on all subsets of G which is left-invariant.

The purpose of this note is to resolve a point left unclear in a recent paper of F. Terpe [1] and its review [2]. In [1], F. Terpe shows that the existence of a certain "maximal" integral is equivalent to the existence of a nontrival countably additive σ-finite measure m_I defined on all subsets of the interval $I = [0, 1)$ and invariant under translation mod 1. In his review [2] of this paper, J. C. Oxtoby points out that the proof given there for the nonexistence of such a measure tacitly presupposes that the σ-field $2^I \times 2^I$ of subsets of $I \times I$ generated by generalized rectangles is invariant under the shear map S, where $S(x, y) = (x + y, y)$ and addition is mod 1, and that by a theorem of Iwanik [3] this instance of Weil's measurability condition is satisfied if and only if all subsets of $I \times I$ belong to $2^I \times 2^I$. Thus, Terpe's reasoning actually established the nonexistence of m_I only under the hypothesis $2^{I \times I} = 2^I \times 2^I$. Finally, Oxtoby points out in his review that $2^{I \times I} = 2^I \times 2^I$ is implied by CH, but that CH makes the group argument unnecessary. Oxtoby ends his review by stating that the situation is unclear without CH.

We give a short argument below to show that no such hypothesis is needed.

THEOREM. *Suppose G is an uncountable group and μ is a σ-finite countably additive left-invariant measure defined on all subsets of G. Then μ is trivial.*

PROOF. Let M be a subgroup of G of cardinality $\aleph_1$. Let R be the family of all right cosets of M and let A be a subset of G which intersects each set in R in exactly one point.

Let $\mathcal{H} = \{mA: m \in M\}$. Then $\mathcal{H}$ is a family of $\aleph_1$ disjoint sets covering G and if H_1 and H_2 belong to $\mathcal{H}$, then H_2 is a left translate of H_1.

Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of sets of finite measure covering G. For each n, the sets of the form $K_n \cap H$, where $H \in \mathcal{H}$ form a decomposition of K_n and therefore there are not uncountably many H's with $\mu(K_n \cap H) > 0$.

Thus, there is a set H_0 in $\mathcal{H}$ with $\mu(K_n \cap H_0) = 0$ for each n. Therefore, $\mu(H) = 0$ for all $H \in \mathcal{H}$. This implies that $\aleph_1$ is a real-valued measurable

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cardinal. But, assuming the axiom of choice (which we are in this paper), it is known that $\aleph_1$ is not measurable [4].

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