CHARACTERIZING CLIQUES IN HYPERGRAPHS

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Abstract

This paper deals with the set-theoretical analog to a problem from coding theory. Let S be a set of r-subsets of an n-set X. Suppose every (n-j)-subset of X contains some binomial number $\binom{x}{r}$ of members of S, where x depends on which (n-j)-set is considered. Then it is known that for n sufficiently large S must be exactly all the r-subsets of some k-subset of X. In this paper we consider some extremal problems of determining the best bounds and extremal configurations achieving them. In particular, we get bounds for n which depend on r and $\ell = |S|$, but essentially not on j. In the $r = 2$ case, we obtain much sharper results, characterizing in some cases all the exceptions.

1. *Introduction.*

Suppose H is a set of q^k points in affine n-space over GF(q). Further suppose that every hyperplane contains either 0, q^k or q^{k-1} points of H. Then MacWilliams [5] showed that H must be a k-dimensional subspace. We say that the subspace is characterized by its

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intersections with hyperplanes. Gleason generalized the problem, and asked whether one could characterize a set S of the appropriate number of r-subspaces by its intersections with $(n - j)$-subspaces. This was shown for $r = 1$ or $j = 1$ in [7]. In [6] the answer was shown to be true in general, for arbitrary r and j if n is sufficiently large, depending on k, r and j.

There are a number of related questions concerning similar characterizations of, for example, other geometric configurations. But one can also consider other systems (or more formally combinatorial geometries). In [6] such a generalization is made, and includes, in particular, the set-theoretic analog to the linear question. It is this question which we are concerned with here. In [6] an upper bound is found on the minimum value of n (depending on r, j and k) which implies that the appropriate structure, a clique (see below), is characterized by intersections. Because of the symmetry and simplicity of the Boolean algebra of sets, we are able to improve the bounds on n to be sharp, to weaken the conditions by eliminating the restriction on $|S|$, and to construct extremal examples where S is not characterized. We do this essentially uniformly in j. However, for particular j better results can be obtained. For $r = 2$, the case of ordinary graphs, we can "almost" completely characterize the exceptions for j very large or very small. (See Section 5 below.) The results are stated below and in Section 5.

An <u>r-uniform hypergraph</u>, $r > 1$, is a pair $(X,\mathcal{E})$, when X is a set and $\mathcal{E}$ a subset of $\binom{X}{r}$. ($\binom{X}{r}$ denotes the set of all r-subsets (subsets of r elements) of X). We use the term "hypergraph" throughout this paper to mean r-uniform hypergraph when r is understood. r is the <u>rank</u> of the hypergraph. A section subhypergraph, or just <u>subgraph</u>, is a pair $(Y, \mathcal{E}_Y)$, where $Y \subseteq X$ and $\mathcal{E}_Y = \mathcal{E} \cap \binom{Y}{r}$. $(Y, \mathcal{E}_Y)$ is said to be the section by Y. A k-clique is a hypergraph $(X, \mathcal{E})$ with $|X| = k$ and $\mathcal{E} = \binom{X}{r}$. In this paper we shall extend the terminology and refer to a hypergraph $(X, \mathcal{E})$ as a k-clique also when X has a subset Y of k elements with $\mathcal{E} = \binom{Y}{r}$. A clique is a hypergraph which is a k-clique for some k. This terminology is compatible with the usual terminology [2].

If $(X, \mathcal{E})$ is a k-clique of rank r, then it satisfies the following properties for any $j \geq 1$, which we denote by $P(n, \ell, r, j)$:

(i) $|X| = n$

(ii) $|\mathcal{E}| = \ell$

(iii) $(X, \mathcal{E})$ is r-uniform

(iv) For every $(n - j)$-subset $S \subseteq X$,

$$|\mathcal{E} \cap \binom{S}{r}| \in \left\{\binom{h}{r} : 0 \leq h \leq n\right\}.$$

If for some class of hypergraphs those satisfying $P(n, \ell, r, j)$ are cliques, then we say that $P(n, \ell, r, j)$ characterizes cliques for that class. In this case $(X, \mathcal{E})$ would be a k-clique with k determined by $\ell = \binom{k}{r}$.

Following is a special case of a result proved in [6]:

THEOREM 1.1. *If $n > r\ell + j - 1$ and $\ell = \binom{k}{r}$ for some k, then $P(n, \ell, r, j)$ characterizes k-cliques of rank r.*

The following examples show that $P(n, \ell, r, j)$ does not always characterize k-cliques, even when $\ell = \binom{k}{r}$.

Example A. Let $G = (X, \mathcal{E})$, where $|X| = n = \ell + r - 1$, $\mathcal{E} = \ell = \binom{k}{r}$, for some k, $X = \{x_1, x_2, \ldots, x_{r-1}, y_1, \ldots, y_\ell\}$, and $\mathcal{E}$ is the set of all edges $\{x_1, \ldots, x_{r-1}, y_i\}$, $1 \leq i \leq \ell$. G satisfies $P(n, \ell, r, j)$ for $j = \binom{k-1}{r-1}$.

Example B. Let $G = (X, \mathcal{E})$, where $|X| = \ell r$, $\mathcal{E}$ consists of ℓ disjoint r-subsets of X. Then G satisfies $P(n,r,\ell,j)$ for $n = \ell r$, and any $j > n - 2r$, ℓ arbitrary.

Example C. Let G be as in Example B above, except that we choose $\ell = \binom{k}{r} + 1$ for some k. Then G satisfies $P(n,\ell,r,1)$.

The principal result of this paper is the following:

THEOREM 1.2. $P(n,\ell,r,j)$ *characterizes cliques of rank* r *in each of the following cases:*

(a) $j = 1$, $\ell = \binom{k}{r}$ *for some* k.

(b) $n \geq \ell + r$, $1 < j < r$, $n - j \geq 2r$, $n \geq r/2\binom{r+2}{2} + 3r - 2$.

(c) $n \geq \ell + r$, $n - j \geq 2r$, $r \leq j$.

The lower bounds given in (c) are best possible, and in fact Examples A and B above are extremal cases not satisfying these conditions. In (a) the requirement that $\ell = \binom{k}{r}$ is needed in view of Example C. However, the last inequality on n in (b) is almost surely not necessary. We do not have best possible results in this case.

Note that we need not require ℓ to be a binomial value $\binom{k}{r}$ in (b) and (c). This follows from the theorem. If we do make this assumption then we can prove that $P(n,r,\ell,j)$ characterizes cliques:

THEOREM 1.3. *Let* $n \geq \ell + r$, $n - j \geq 2r$, *and assume* $\ell = \binom{k}{r}$ *for some* k. *Then* $P(n,\ell,r,j)$ *characterizes* k-*cliques of rank* r.

For the case $r = 2$ there are even stronger results in Section 5. There (in Theorem 5.1) we characterize all exceptions for $r = 2$, $j = 2$, and practically all of them for fixed j or for fixed $n - j$ (Theorems 5.2 and 5.3). The final details of this last result involves some diophantine alalysis.

2. PRELIMINARY RESULTS

Let $G = (X, \mathcal{E})$ be an r-uniform hypergraph with $|X| = n$, $|\mathcal{E}| = \ell \geq 1$. Throughout the remainder of the paper we assume G satisfies $P(n,\ell,r,j)$.

For any vertex $x \in X$, $d(x)$ is defined to be the number of edges in G containing x. $d(x)$ is called the degree of x. If $S \subseteq X$, then $D(S)$ is the number of edges of G contained in S. For any $x \in X$, G_x denotes the section of G by $X - \{x\}$, i.e. the subgraph obtained by deleting x. We will let $d = \min_{x \in X} d(x)$.

LEMMA 2.1. *If* $n \geq \ell + r$, *then at least* r *vertices of* G *have degree at most* $r - 1$.

Proof. If not, more than $n - r$ vertices have degrees at least r. Hence $\ell r = \Sigma_x d(x) \geq \Sigma_{d(x) \geq r} d(x) > (n - r)r \geq \ell r$, a contradiction.

LEMMA 2.2. *If* $n \geq \ell$, *then* $d < r$.

Proof. If $d \geq r$, then $\ell r = \Sigma_x d(x) \geq nr$, a contradiction.

LEMMA 2.3. *Let* $H = (Y, \mathfrak{F})$ *be an* r-*uniform hypergraph such that*

(i) $|\mathfrak{F}| = \binom{k}{r}$ *for some* k.

(ii) *If* $y \in Y$, *then* $d(y) = 0$ *or* $d(y) \geq \binom{k-1}{r-1}$.

Then H *is a clique.*

The proof is direct by counting (or see [6]).

LEMMA 2.4. *Let* S, S' *be subsets of* X *with* $|S| = |S'| = n - j$.

Then if $|D(S) - D(S')| < r$, *we have* $|D(S) - D(S')| \leq 1$. *Further*, $|D(S) - D(S')| = 1$ *implies that* $\{D(S), D(S')\} = \{0,1\}$.

Proof. This follows from $P(n,\ell,r,j)$ and the fact that distinct binomial coefficients of the form $\binom{m}{r}$ differ by at least r unless they are 1 and 0, respectively.

LEMMA 2.5. *If there are* j *vertices of degree* 0 *in* G, *then* G *is a clique*.

Proof. Let v be any vertex of positive degree in G, and let $V = \{v_1,\ldots,v_j\}$ be a set of j vertices of degree 0. Then $|\mathcal{E}| = D(X) = D(X - V) = \binom{k}{r}$ for some k by $P(n,\ell,r,j)$. If v has positive degree, then if $V' = \{v,v_2,\ldots,v_j\}$, we have $d(v) = \binom{k}{r} - D(X - V$ $= \binom{k}{r} - \binom{k'}{r}$ for some $k' < k$. Thus $d(v) \geq \binom{k-1}{r-1}$. Hence by Lemma 2.3 G is a clique.

LEMMA 2.6. *If* $\ell = \binom{k}{r}$ *for some* k, *and* $j = 1$, *then* G *is a* k-*clique of rank* r.

Proof. If $x \in X$ is a vertex of positive degree, then by $P(n,\ell,r,1)$, $D(X - \{v\})$ is at most $\binom{k-1}{r}$. Hence $d(x) \geq \binom{k-1}{r-1}$, and G is a clique by Lemma 2.3.

LEMMA 2.7. *If* $d = 0$, *and if either* $j = 1$ *or* $n - j \geq 2r$, *then* G *is a clique*.

Proof. Let $d(v) = 0$, $v \in X$. Then if $j = 1$, $D(X - \{v\}) = \ell = \binom{k}{r}$ for some k, by $P(n,\ell,r,1)$, and by Lemma 2.6 G is a clique.

Recall that we are assuming G satisfies $P(n,\ell,r,j)$, and now must prove if $d = 0$ and $n - j \geq 2r$, then G is a clique. The

$j = 1$ case of this statement is true. Assume, then, by induction that $j > 1$ and it holds for $j - 1$. Let x be a vertex of positive degree in G, and consider G_x.

Clearly G_x satisfies $P(n - 1, \ell', r, j - 1)$, where $\ell' = \ell - d(x)$, and $(n - 1) - (j - 1) \geq 2r$. Thus by the induction hypothesis, G_x must be a clique. Since this statement did not depend on the choice of x, G_x is a clique for each x of positive degree.

If G has only one edge, there is nothing to prove. So let e_1 and e_2 be two edges of G. If all vertices of positive degree are contained in $e_1 \cup e_2$, then since $n - j \geq 2r$, we can find j vertices of degree 0, and G is a clique by Lemma 2.5. So we assume that for every two edges, e_1, e_2 there is a vertex x of positive degree not in $e_1 \cup e_2$. Since G_x is a clique, the section by $e_1 \cup e_2$ is a clique. This is true for every two edges of G. Thus G is a clique.

The following development will show that under the assumptions of Theorem 1.2, we must have $d = 0$, thereby establishing Theorem 1.2, by the previous lemma. Note that we have part (a) $(j = 1)$ of the theorem already by Lemma 2.6. The remainder of this section is devoted to proving that $d = 1$ implies $j = 1$.

LEMMA 2.8. <u>If</u> $n - j \geq 2r$ <u>and</u> $d = 1$, <u>then for any vertex</u> x <u>either</u> $d(x) = 1$ <u>or</u> $d(x) \geq r$.

<u>Proof.</u> Let v be a vertex with $d(v) = 1$. Let $x \neq v$ be any vertex with $d(x) \leq r - 1$. Suppose there are two edges e_1 and e_2 containing x but not v. Let S be an $(n - j)$-set containing $e_1 \cup e_2$ but not v, and let $S' = (S - \{x\}) \cup \{v\}$. As $d = 1$, we have $1 \leq D(S) - D(S') \leq r - 1$. Since $D(S) \geq 2$, this contradicts

Lemma 2.4. Thus no such pair e_1, e_2 exist. If $d(x) > 1$, then there must be exactly one edge, e_1, containing x and not v, and we have $d(x) = 2$. If e is the edge containing x and v, and S is an $(n - j)$-set containing $e_1 \cup e$, then $D(S) \geq r + 1$ by $P(n,\ell,r,j)$ and the fact that $\binom{k}{r} > 1$ implies $\binom{k}{r} \geq r + 1$. Thus there is another edge e_3 not containing x. Let S'' contain $e_1 \cup e_3$ but not v. Let $S''' = (S'' - \{x\}) \cup \{v\}$. Then $D(S'') - D(S''') = 1$, but $D(S'') \geq 2$, contradicting Lemma 2.4. Thus $d(x) \leq 1$ and the proof is complete.

LEMMA 2.9. <u>If</u> $d = 1, j > 1, n - j \geq 2r$, <u>then no two vertices of degree</u> 1 <u>are contained in a common edge</u>.

<u>Proof</u>. Suppose u and v are two vertices of G of degree 1 on a common edge e. The graph G_u satisfies $P(n - 1, \ell - 1, r, j - 1)$, with $(n - 1) - (j - 1) \geq 2r$, and has a vertex of degree 0. By Lemma 2.7 G_u must be a clique, so $\ell = \binom{k}{r} + 1$ for some k. If $\ell = 1$, then d would be 0 by $n - j \geq 2r$. So $\ell \geq 2$. Let S be an $(n - j)$-set containing e with $D(S) \geq 2$. Now the section of G_u by $S - \{u\}$ must be a clique, hence $D(S) = \binom{t}{r} + 1$ for some $t \geq r$. This contradicts $P(n,\ell,r,j)$.

LEMMA 2.10. <u>Let</u> $d = 1, j > 1, n - j \geq 2r$, <u>and suppose there are two vertices of degree</u> 1 <u>in</u> G, <u>say</u> a,b, <u>with edges</u> e_a <u>and</u> e_b <u>containing them, respectively</u>, $e_a \neq e_b$. <u>Then</u> $(e_a - e_b) \cup (e_b - e_a) - \{a,b\} \subseteq$ <u>for every edge</u> $e \neq e_a, e_b$.

<u>Proof</u>. If $((e_a - e_b) \cup (e_b - e_a)) - \{a,b\} = \emptyset$, we are done. So assume it isn't empty. Let $x \in (e_a - e_b) - \{a\}$. Then G_x satisfies $P(n - 1, \ell', r, j - 1)$ where $\ell' = \ell - d(x)$, $(n - 1) - (j - 1) \geq 2r$, and the degree of a in G_x is 0. By Lemma 2.7 G_x is a clique, with

$d(b) = 1$. Thus G_x has only one edge, e_b. Hence all edges e in G other than e_a and e_b contain x. A similar argument works for the vertices $y \in (e_b - e_a) - \{b\}$.

LEMMA 2.11. If $d = 1$, $j > 1$, $n - j \geq 2r$, $n \geq \ell + r$, and if e_a, e_b contain vertices a,b of degree 1, respectively, then $|e_a - e_b| \geq 2$.

Proof. First note that $a \in e_a - e_b$, by Lemma 2.9. If $|e_a - e_b| = 1$, then $|e_a \cap e_b| = r - 1$. Let $e_a \cap e_b = \{c_1,\ldots,c_{r-1}\}$. If c is any other vertex of degree 1, and e_c is the edge containing c, then by Lemma 2.9 e_c is distinct from e_a and e_b. By Lemma 2.10 applied to a and c, $(e_c - e_a) - \{c\} \subseteq e_b$. This can only occur if $e_c = \{c,c_1,\ldots,c_{r-1}\}$. Let p denote the number of vertices of degree 1, and q the number of vertices distinct from the c_i and of degree at least r. Then $q = n - p - (r - 1)$ (by Lemma 2.8 and the remarks above). Counting degrees we get $r\ell \geq qr + p + (r - 1)p = r(p + q)$. This gives $n \leq \ell + r - 1$, a contradiction, completing the proof.

Now consider the hypergraph G, and let $n \geq \ell + r$, $n - j \geq 2r$, $j > 1$ and $d = 1$. Then $\ell \geq 2$, or there would be vertices of degree 0. Let $a_1,a_2,\ldots,a_x$ be all the vertices of degree 1 and let e_i be the edge containing a_i for each i, where $e_i \neq e_j$ if $i \neq j$ by Lemma 2.9. By Lemmas 2.8 and 2.1, $x \geq r$. But by Lemma 2.11 for each $i = 2,3,\ldots,x$, there is a vertex $u_i \neq a_1$ in each $e_1 - e_i$. By Lemma 2.10, $u_i \in e_j$ for every $j \neq i$. Thus $u_i \neq u_j$ for $i \neq j$, and e_1 contains a_1, u_i, $2 \leq i \leq x$, and $x \leq r$. Then $x = r$. Replacing e_1 by an arbitrary e_i above, we see that for each i, $1 \leq i \leq r$, there is a u_i contained in all e_j, $j \neq i$, and not contained in e_i. In fact, then

$e_i = \{a_i\} \cup \{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_r\}$ for each i. By Lemma 2.10, all the u_i are contained in every edge of G other than the e_i, the only possibility for which is therefore $\{u_1,\ldots,u_r\}$. But as $d = 1$, there can be no other vertices than the a_i and u_i. This contradicts $n - j \geq 2r$. So we have proved the following:

LEMMA 2.12. *If* $n \geq \ell + r$, $n - j \geq 2r$, *and* $d = 1$, *then* $j = 1$.

3. PROOF OF THEOREM 1.2

We recall that we are still assuming that G satisfies $P(n,\ell,r,j)$.

LEMMA 3.1. *If* $n \geq \ell + r$, $n - j \geq 2r$, *then* $d > 0$ *implies* $j \leq d$.

Proof. We prove this by induction on d. For $d = 1$ the results of the last section establish this result. So we assume $d > 1$ and that the lemma holds for all smaller values of d. Suppose $j > d$. We will obtain a contradiction. Let v be a vertex of degree d, and let $e_1,\ldots,e_d$ be the edges containing v.

Claim: If $x \notin e_j$ for some j, and $x \in \bigcup_{i=1}^{d} e_i$, $x \in e$ for every edge e in G, $e \neq e_j$. For consider G_x. It satisfies $P(n-1,\ell',r,j-1)$ where $\ell' = \ell - d(x)$. We have $(n - 1) \geq (\ell - 1) + r > \ell' + r$, $(n - 1) - (j - 1) \geq 2r$, and minimum degree $d' < d$, as v has degree less than d in G_x. But $j > d$ implies $(j - 1) > (d - 1) \geq d'$. Thus by induction the minimum degree of G_x must be 0. Then G_x is a clique, by Lemma 2.7. The degree of each vertex of positive degree in G_x is the same as the degree of v in G_x, which is at most $d - 1 \leq r - 2 < \binom{k-1}{r-1}$ for $k > r$. Hence G_x can contain only one edge, e_j, and x is contained in all other edges, as claimed.

Now let $Y = \bigcup_{i=1}^{d} e_i$, $S = \bigcap_{i=1}^{d} e_i$. By the Claim above, every edge of G other than the e_i contains $Y - S$. Since d is minimal, there must be at least one edge other than the e_i, or vertices in $Y - S$ would have smaller degree. So $|Y| - |S| \leq r$, and $|Y| \leq r + |S|$. On the other hand, since $d > 1$, we have $|Y| \geq r + 1$. If $|S| = 1$, we get $|Y| = r + 1$. But then $|Y - e_j| = 1$ for each j. By the Claim in the previous paragraph, the $Y - e_j$, $1 \leq j \leq d$, are disjoint. Thus $Y = S \cup \bigcup_{j=1}^{d}(Y - e_j)$, whence $|Y| = 1 + d$, and $d = r$. This contradicts $d \leq r - 1$ (Lemma 2.2). We conclude then that $|S| > 1$.

Let $Z = Y - S$, and let $|S| = s > 1$, $|Z| = z$. By the Claim above, we certainly have $Y = e_1 \cup e_2$, and $|Y| \geq r + 1$, $z \geq r + 1 - s$, and $z + 2s \leq 2r$. Let $x \in S$, $x \neq v$ (recall $s > 1$). G_x has the vertex v of degree 0, and satisfies $P(n - 1, \ell', r, j - 1)$, $\ell' = \ell - d(x)$. Since $(n - 1) - (j - 1) \geq 2r$, Lemma 2.7 implies that G_x is a clique. We can use the Claim above to conclude that Z is contained in every edge of G_x. Hence G_x contains at most one edge.

Let f be the number of edges different from the e_i and not contained in Z. (Since $Z \subseteq G_x$ which has at most one edge, we have $f = \ell - d$ or $\ell - d - 1$.) By the Claim, all vertices of Z are in all of these f edges. On the other hand, for each $x \in S$, $x \neq v$, at most one of the f edges does not contain x (as G_x has at most one edge).

Now we count edge-vertex incidences for the f edges. The total is rf. The vertices of Z account for zf of them, and the vertices of $S - \{v\}$ for at least $(s - 1)(f - 1)$ of them, by the remarks above. If there are t vertices not in Y, each has degree d at least, and each is clearly contained only in edges accounted for by f. Thus these vertices account for at least td incidences. We get:

$$rf \geq zf + (s - 1)(f - 1) + td.$$

This gives:

$$n = t + s + z \leq \frac{1}{d}((r - z)f - (s - 1)(f - 1)) + s + z$$

$$= \frac{1}{d}((r - z - s + 1)f + s - 1) + z + s.$$

Recalling that $z \geq r + 1 - s$ and $z + 2s \leq 2r$, we get

$$n \leq \frac{s - 1}{d} + z + s < 2r.$$

But this contradicts $n - j \geq 2r$, and completes the proof of the lemma.

COROLLARY 3.2. _If_ $n \geq \ell + r$, $n - j \geq 2r$, _and_ $j > d$, _then_ $d = 0$ _and_ G _is a clique_. _In particular, if_ $j \geq r$, _then_ G _is a clique_.

This establishes part (c) of Theorem 3.1. It remains only to prove (b). We consider the case $1 < j \leq d < r$, and show that it essentially cannot occur.

LEMMA 3.3. _Let_ $1 < j \leq d < r$, _and_ $n - j \geq 2r$. _If_ $n \geq r^2 - r + 1$, _then for every vertex_ x, _either_ $d(x) = d$ _or_ $d(x) \geq r$.

Proof. Let v be a vertex with $d(v) = d$, and x a vertex with $d(x) \leq r - 1$. Let S be an $(n - j)$-set excluding v, and including x and all edges of x not containing v. If there is only one such edge ($d(x) = d + 1$), take S to contain it and any other edge not containing v. Such a set S exists because the greatest number of vertices x could determine this way is $(r - 1)(r - 1) + 1 = (r^2 - r + 1) - (r - 1) \leq n - j$. (Or 2r if $d(x) = d + 1$). Let $S' = (S - \{x\}) \cup \{v\}$. Then $0 \leq D(S) - D(S') \leq d(x)$. By Lemma 2.4, either $d(x) = d$ or $D(S) = 1$. But if $d(x) > d$, $D(S) \geq 2$, as $d \geq j > 1$. Thus $d(x) = d$, and the proof is complete.

LEMMA 3.4. <u>Let</u> $2 = j \leq d < r$, $n \geq r/2\binom{r+2}{2} + 2r + 1$, $n \geq \ell + r$. <u>Then there are two vertices of degree</u> d <u>not on a common edge</u>.

<u>Proof</u>. Suppose not. Let v be a vertex of degree d. Let t be the number of vertices of degree d. Then since all of them are adjacent to v, we have $t \leq d(r-1) + 1 \leq r^2 - 2r + 2$.

Now choose another vertex u with $d(u) = d$. (It exists by Lemma 2.1 and Lemma 3.3.) Let $S = X - \{v,u\}$. Then $D(S) = \binom{k}{r}$ for some k, and $\ell = \binom{k}{r} + 2d - a$, where $a \geq 1$ is the number of edges u and v have in common. Since ℓ and d are independent of the choice of u and v, a will be also except possibly in case $\binom{k}{r} = 0$ or 1. Then an exception could occur if there is some pair u' and v' incident to all edges of G, and another pair incident to all but one edge of G, giving $\ell = 1 + 2d - a = 2d - (a-1)$, where $(a-1)$ is the number of edges common to u' and v'. But this implies that there are at most $2d$ edges, and $\ell \leq 2d$. Since $nd \leq \ell r$, we have $n \leq 2r$, which violates the assumptions that $n \geq 1/2\binom{r+3}{3} + 2$ and $r \geq 3$. Thus this situation cannot occur, and a is independent of the choice of u and v.

We claim now that there is some vertex of degree $c > d$. For if not, then every vertex would have degree d, every pair would be in a common edges, and the edges would form a balanced incomplete block design with standard parameters v,b,k,r,λ equal to n,ℓ,r,d,a respectively. But $n \geq \ell + r$ violates Fisher's inequality for such designs. Thus some vertex must have degree $c > d$.

Let w be a vertex such that $d(w) = c$, and let v and u' have degree d, as before. Let y be the number of edges common to v and w. If $S' = X - \{w,v\}$, $D(S') = \binom{h}{r}$ for some h. Then $\ell = \binom{h}{r} + d + c - y$. Recalling from above that $\ell = \binom{k}{r} + 2d - a$, we get

$\binom{h}{r} + d + c - y = \binom{k}{r} + 2d - a$, or $c = \binom{k}{r} - \binom{h}{r} + d - a + y$.

Fix v and u' of degree d. Then if w has $d(w) = c > d$, as above, there are three possibilities:

(i) $h > k$

(ii) $h < k$

(iii) $h = k$.

In case (i) we get $\binom{h}{r} - \binom{k}{r} = d - c - a + y \leq r - 3$, as $a \geq 1$, $y \leq d \leq r - 1$, and $c > d$. This only happens when $\binom{k}{r} = 0$ and $\binom{h}{r} = 1$. But just as we argued above, this implies v and u' are incident to all edges, and $n \leq 2r$, contradicting the assumptions of the lemma. Hence (i) doesn't occur for any w.

Let w satisfy (iii). Then $c = d + y - a$, and as $c > d$ and $a \geq 1$, we get $y \geq 2$. So each w satisfying (iii) has at least two edges in common with v. Let m denote the number of vertices satisfying (iii). The number of pairs (w,e) where e is an edge containing v and w a vertex, distinct from v is $d(r - 1)$. At least $2m$ of these pairs arise from vertices w satisfying (iii). Thus $m \leq \frac{d(r-1)}{2}$.

In case (ii) we have $c = \binom{k}{r} - \binom{h}{r} + d - a + y \geq \binom{k-1}{r-1} + 1$ (as $d > a$). Then counting edge-vertex incidences we get

$$r\ell \geq [td] + \left[(n - m - t)\left(\binom{k-1}{r-1} + 1\right)\right] + [m(d + 1)]$$

(the three terms come from the three kinds of verices, namely degree d, case (ii) and case (iii), respectively.)

This gives:

$$t \geq \frac{m(d + 1) + (n - m)\left(\binom{k-1}{r-1} + 1\right) - r\ell}{\binom{k-1}{r-1} + 1 - d}$$

To simplify this we wish to use the inequality $k \geq r + 3$. We obtain this by observing again $nd \leq \ell r$, or $n \leq \left(\frac{\binom{k}{r} + 2d - a}{d}\right) r \leq \frac{r}{2}\binom{r+2}{2} + 2r$ if $k \leq r + 2$, contradicting $n \geq \frac{r}{2}\binom{r+2}{2} + 2r + 1$.

We also recall $m \leq \frac{d(r-1)}{2}$ and $t \leq r^2 - 2r + 2$ from above. Then our expression gives

$$t \geq (n - m)\left(\frac{\binom{k-1}{r-1} + 1}{\binom{k-1}{r-1} + 1 - d}\right) + \frac{m(d+1) - r\ell}{\binom{k-1}{r-1} + 1 - d}$$

$$> n - m - \frac{r\ell}{\binom{k-1}{r-1} + 1 - d} \geq \ell + r - \frac{d(r-1)}{2} - \frac{r\ell}{\binom{k-1}{r-1} + 2 - r}$$

$$\geq \ell\left[1 - \frac{r}{\binom{k-1}{r-1} + 2 - r}\right] + r - \frac{(r-1)^2}{2}$$

$$> \binom{k}{r}\left[1 - \frac{r}{\binom{k-1}{r-1} + 2 - r}\right] + \frac{4r - 1 - r^2}{2}$$

$$\geq \binom{r+3}{3}\left[1 - \frac{r}{\binom{r+2}{3} + 2 - r}\right] + \frac{4r - 1 - r^2}{2}$$

$$\geq \binom{r+3}{3} - (r+3)\left(\frac{\binom{r+2}{3}}{\binom{r+2}{3} + 2 - r}\right) + \frac{4r - 1 - r^2}{2}$$

$$\geq \binom{r+3}{3} - 2(r+3) + 2r - \frac{1 + r^2}{2}.$$

Together with $t \leq r^2 - 2r + 2$ this gives $r^3 - 3r^2 + 23r - 45 \leq 0$, a contradiction as $r \geq 3$.

This contradiction finally confutes the original supposition that the lemma was false, and thus completes the proof of Lemma 3.4.

LEMMA 3.5. *If* $2 = j \leq d < r$, $n \geq \ell + r$, *and* $n \geq r/2\binom{r+2}{2} + 2r + 1$, *then* $\ell = \binom{k}{r} + 2d$ *for some* k.

Proof. This follows directly from Lemma 3.4.

LEMMA 3.6. *If* $2 = j \leq d < r$, $n \geq \ell + r$, *and* $n \geq r/2\binom{r+2}{2} + 2r + 1$, *then no two vertices of degree* d *have a common edge.*

Proof. Suppose $d(x) = d(y) = d$, and they have a edges in common. Then if $S = X - \{x,y\}$, we have $D(S) = \binom{h}{r}$, for some h, and $\ell = \binom{h}{r} + 2d - a$. By Lemma 3.5, $a = \binom{h}{r} - \binom{k}{r}$, an impossibility unless $h = k$, since $k > r$. Thus $a = 0$, and the lemma is proved.

LEMMA 3.7. *Let* $2 = j \leq d < r$, $n \geq \ell + r$, $n \geq \frac{r}{2}\binom{r+2}{2} + 2r + 1$. *If* t *is the number of vertices of degree* t, *we have* $t \leq \frac{\ell}{d}$.

Proof. This follows directly from the previous Lemma.

Let $2 = j \leq d < r$, $n \geq \ell + r$, $n \geq r/2\binom{r+2}{2} + 2r + 1$. We consider any vertex u with $d(u) > d$. There are two possibilities:

(a) Some vertex v exists with $d(v) = d$ and no common edge with u.

(b) u has an edge in common with every vertex v of degree d.

In case (a), by considering $S = X - \{u,v\}$, $P(n,\ell,r,2)$, and Lemma 3.5, we have for some h, $\binom{h}{r} = \binom{k}{r} + 2d - d - d(u)$, where k comes from Lemma 3.5. Thus we have

$$d(u) = \binom{k}{r} - \binom{h}{r} + d \geq \binom{k-1}{r-1} + d.$$

Since all vertices of type (b) must be on an edge common to v, we have at most $d(r-1)$ of these vertices. Further, $d(u) \geq t$ in this case, by Lemma 3.6. Now let a and b denote the numbers of vertices of types (a) and (b) respectively. We have:

$$n = a + b + t.$$

Counting incidences we have:

$$td + bt + a\left(\binom{k-1}{r-1} + d\right) \le r\ell.$$

Using $a = n - t - b \ge n - \frac{\ell}{d} - d(r-1) \ge \ell\,\frac{d-1}{d} + r - (r-1)^2$ we get:

$$r\ell \ge \left(\frac{\ell}{2} - (r-1)^2 + r\right)\left(\binom{k-1}{r-1} + d\right) + t(d+b)$$

$$\ge \binom{k-1}{r-1}\left(\frac{\ell}{2} - (r-1)^2\right) + t(d+b).$$

Then

$$\left(1/2\binom{k-1}{r-1} - r\right)\ell \le \binom{k-1}{r-1}(r-1)^2 - t(d+b)$$

$$= \frac{(\ell - 2d)r}{k}(r-1)^2 - t(d+b)$$

$$\le \frac{\ell r(r-1)^2}{k} - t(d+b)$$

$$< \frac{\ell r(r-1)^2}{k}.$$

So

$$1/2\binom{k}{r}r - kr < r(r-1)^2$$

$$\binom{k}{r} < 2(r-1)^2 + 2r.$$

This gives a contradiction for $k \ge r + 3$, and hence for $n \ge \frac{r}{2}\binom{r+2}{2} + 2r + 1$, as assumed. Thus we have proved the next lemma.

LEMMA 3.8. <u>If</u> $n \ge \ell + r$, $n \ge r/2\binom{r+2}{2} + 2r + 1$, <u>and</u> $2 \le d < r$, <u>then</u>

$P(n,\ell,r,2)$ *cannot be satisfied.*

LEMMA 3.9. *If* $n \geq \ell + r$, $n \geq r/2\binom{r+2}{2} + 2r + j - 1$, *and* $2 \leq j \leq d < r$, *then* $P(n,\ell,r,j)$ *cannot be satisfied.*

Proof. We use induction on j. For $j = 2$ this is Lemma 3.8. Assume that it holds for $j - 1$ and consider the j case. Suppose G is a hypergraph with the minimum n such that G satisfies $P(n,\ell,r,j)$ and $n \geq \ell + r$, $n \geq r/2\binom{r+2}{2} + 2r + j - 1$ and $2 < j \leq d < r$. Let x have degree d. Then G_x satisfies $P(n - 1,\ell - d,r,j - 1)$, and we still have $(n - 1) \geq (\ell - d) + r$ and $(n - 1) \geq r/2\binom{r+2}{2} + 2r + (j - 2)$. Then by induction we must have the minimum degree d' in G_x satisfying $d' < j$. But by Corollary 3.2, G_x is a clique, and $d' = 0$. This means that all vertices of degree d in G must have all their edges in common. But by Lemma 2.1, there are at least r such vertices, and hence $d = 1$, a contradiction. This proves the lemma.

COROLLARY 3.10. *If* $n \geq \ell + r$, $n \geq r/2\binom{r+2}{2} + 3r - 2$, $2 \leq j \leq d < r$,

This completes the proof of Theorem 1.2.

4. PROOF OF THEOREM 1.3

With the assumptions of the theorem we can conclude that G is a clique by Corollary 3.2 if $j > d$. By Lemma 2.2 and Lemma 2.6, we see that it suffices to consider only the case $1 < j \leq d < r$. Thus let $\ell = \binom{k}{r}$ for some k. By Lemma 2.1 we can find some set of j vertices of degree at most $r - 1$. By $P(n,\ell,r,j)$, the number of edges not containing any of these is at most $\binom{k-1}{r}$, while the number containing at least one is at most $j(r-1) \leq (r-1)^2$. Thus $(r-1)^2 + \binom{k-1}{r} \geq \binom{k}{r}$. This is impossible for $k \geq r+3$. Hence the only cases of the theorem to prove are $\ell = 1$, $\ell = r+1$, and $\ell = \binom{r+2}{2}$.

In the first case, $\ell = 1$, there is nothing to prove, as G is a clique. If $\ell = r+1$, let S be an $(n-j)$-set containing a vertex v of degree $d \geq 2$ and at least two edges to which v belongs. (We can do this as $n - j \geq 2r$.) By $P(n,\ell,r,j)$ S must contain all $r+1$ edges. Thus the j vertices not in S have degree 0, contradicting $d \geq 2$. So this case is also impossible.

The only remaining case is $\ell = \binom{r+2}{r} = \frac{(r+2)(r+1)}{2}$. In this case the equation $(r-1)^2 + \binom{k-1}{r} \geq \binom{k}{r}$ from above implies $r \geq 5$. In this case consider any j vertices. By $P(n,\ell,r,j)$ they must meet at least $\binom{r+2}{r} - \binom{r+1}{r} = \binom{r+1}{r-1} = \binom{r+1}{2}$ edges. The total number of incidences of edges and j-sets is thus at least $\binom{n}{j}\binom{r+1}{2}$. On the other hand, each edge meets exactly $\binom{n}{j} - \binom{n-r}{j}$ j-sets. Hence there are exactly $\binom{r+2}{r}\left(\binom{n}{j} - \binom{n-r}{j}\right)$ such indicences, and

$$\binom{r+2}{r}\left(\binom{n}{j} - \binom{n-r}{j}\right) \geq \binom{r+1}{2}\binom{n}{j}.$$

Simplifying this gives:

$$\begin{aligned}
\frac{r+2}{2} &\leq \frac{n(n-1)\cdots(n-j+1)}{(n-r)\cdots(n-r-j+1)} \\
&\leq \left(\frac{n-j+1}{n-r-j+1}\right)^j \\
&\leq \left(\frac{\ell-j+1+r}{\ell-j+1}\right)^j \quad (\text{as } n \geq \ell + r) \\
&\leq \left(\frac{\ell+2}{\ell+2-r}\right)^{r-1} \quad (\text{as } j \leq r-1)
\end{aligned}$$

$$= \left(1 + \frac{r}{\ell + 2 - r}\right)^{r-1} = \left(1 + \frac{2r}{r^2 + r + 6}\right)^{r-1}$$

$$= \left(1 + \frac{2}{(r + 1 + 6/r)}\right)^{r-1} \leq \left(1 + \frac{2}{r - 1}\right)^{r-1}$$

$$\leq e^2.$$

This implies that $r \leq 2e^2 - 2 < 13$. But in fact considering the intermediate inequality

$$\frac{r + 2}{2} \leq \left(1 + \frac{2r}{r^2 + r + 6}\right)^{r-1},$$

we can check that this fails for $r \leq 12$ as well. Thus we get a contradiction when $k = r + 2$ also. This completes the proof of Theorem 1.3.

5. THE CASE $r = 2$, ORDINARY GRAPHS

In Theorem 1.2 we found a bound sufficient to guarantee that $P(n,\ell,r,j)$ characterizes cliques. Examples A, B and C showed the necessity of the two bounds $n \geq \ell + r$ and $n - j \geq 2r$, if the conditions must be uniform for all j,r. However, for any fixed r or j, the conditions may not be necessary. This also applies to the case where we require $\ell = \binom{k}{r}$ (Theorem 1.3). For example, for $j = 1$, we have from Theorem 1.2 that no bounds are needed if $\ell = \binom{k}{r}$. In this section we investigate the case $r = 2$. As usual, the $r \geq 3$ cases seem to be much more difficult.

In particular, we consider two cases: First, we assume j is fixed, $\ell = \binom{k}{2}$ for some k, and show that for k sufficiently large, $P(n,\binom{k}{2},j,2)$ characterizes cliques. This means that $P(n,\binom{k}{2},j,2)$ always characterizes cliques except for a finite number of graphs for each j. Characterizing these exceptions can be done in the smallest cases, as we illustrate below. Otherwise, obtaining good lower bounds on k to guarantee that $P(n,\binom{k}{2},2,j)$ characterizes cliques seems extremely hard. The second situation we consider is where $n - j$ is fixed. In this case we show that $P(n,\ell,2,j)$ characterizes cliques in all but a finite number of "types" of cases, which are more or less classified.

We now consider the case $r = 2$, $j = 2$. Example A above for $k = 3$, $n = 4$ is the "star", $K_{1,3}$, which satisfies $P(4,3,2,2)$ but is not a clique. Similarly, the path, P_3, consisting of vertices v_i, $1 \leq i \leq 4$, and edges $\{v_i,v_{i+1}\}$, $1 \leq j \leq 3$, is not a clique but satisfies $P(4,3,2,2)$. We note first that for $n \leq 5$, these are the only exceptions for $\ell = \binom{k}{2}$ for some k. Hence we will only need to consider the $n \geq 6$ case.

THEOREM 5.1. $P(n,\binom{k}{2},2,2)$ _characterizes cliques except when_ G _is one of the examples_ $K_{1,3}$ _or_ P_3 _above._

Proof. We prove this in several steps. Assume that G is not a clique, that it satisfies $P(n,\binom{k}{2},2,2)$, and therefore that $k > 2$.

1. $n > k > 2$.

This is clear since $n < k$ is impossible by counting edges and degrees, and $n = k$ implies G is a clique.

2. If $n \geq 6$, then the minimum degree $d > 0$.

This follows from Lemma 2.7. We note that this implies $k \geq 4$, for otherwise with $n \geq 6$ we would have only three disjoint edges, which doesn't satisfy $P(6,3,2,2)$.

3. Some pair of vertices meets exactly $k - 1$ edges.

If this were not true, then $P(n,\ell,2,2)$ $(\ell = \binom{k}{2})$ would imply that each pair must meet at least $(2k - 3)$ edges. Counting edge-pair incidences we get $(2k - 3)\binom{n}{2} \leq \binom{k}{2}(2n - 3)$, which contradicts step 1.

4. The minimum degree d satisfies $d \leq \frac{k-1}{2}$ if $n \geq 6$.

Let v be a vertex of minimum degree and let v' be a vertex of minimum degree among the remaining vertices. By step 3, $d + d' = k - 1$ or k, and the latter only if v and v' are joined by an edge. Thus $d > (k - 1)/2$ only if $d = d' = k/2$ and v and v' are joined. Then suppose u is another vertex adjacent to v' but not to v. $\{u,v\}$ meets one more edge than $\{u,v'\}$, violating $P(n,\ell,2,2)$, unless u is contained in every edge except $\{v,v'\}$ and $\{v,u'\}$ for some other vertex u', by Lemma 2.4. But then $d \geq 2$, and there can be no other vertices than these four, for they would have degree 1. Since $d(u) \geq d \geq 2$, we must have at least 4 edges, namely $\{v,v'\}$, $\{u',v\}$, $\{u,v'\}$ and $\{u,v\}$ or $\{u,u'\}$. But then $k = 4$ and G must be a clique, a contradiction.

So no such vertex u exists. This means either $d = 1$, contradicting $k > 2$, or every vertex joined to either v or v' is joined to both. There must be $\frac{k}{2} - 1$ of these. Now since $d = k/2$, there can't be any vertex of degree d not adjacent to v and v'. There are at least $(k+1) - (k/2+1)$ vertices w which are not adjacent to v and v' and not of minimum degree. By $P(n,\ell,2,2)$ these must have degree at least $(2k-3)$ $k/2$, by considering $\{w,v\}$. This gives for the sum of the degrees at least $\left(\frac{3k}{2} - 3\right)(k/2) + (k/2 + 1)(k/2) = k^2 - k = 2\binom{k}{2}$. This forces the vertices adjacent to v to be of degree $k/2$, other vertices to be of degree $3k/2 - 3$, and these other vertices to be only $k/2$ in number. But then there are only $(k/2 + 1)$ vertices of degree $k/2$, and $k/2$ of degree $\left(\frac{3k}{2} - 3\right)$, giving a total of $k + 1$ vertices, a contradiction. Hence $d \leq \frac{k-1}{2}$.

5. $d = \frac{k - 1}{2}$ if $n \geq 6$.

By step 3, $d \leq \frac{k - 1}{2}$. Suppose $d(v) = d < \frac{k - 1}{2}$. If u is any other vertex, $d(u) \geq k/2$. We claim we can't have $d(u) = k/2$. For if so, then by $P(n,\ell,2,2)$ we must have $d = k/2 - 1$, k must be even, and by step 2, $k \geq 4$. Let w be any other vertex which is not adjacent to u. There must be at least $n - 2 - k/2 \geq k/2 - 1 \geq 1$ of these. Recalling that X denotes the vertex set of G, we let $S = X - \{u,w\}$; $S' = X - \{v,w\}$. We have $1 \leq D(S') - D(S) < 2$. This can only happen if $D(S) = 0$, $D(S') = 1$, or $D(S) = 1$, $D(S') = 3$. It can easily be checked that neither of these cases can occur for $d(u) = k/r$, $d(v) = k/2 - 1$, and w and u not joined. Hence $d(u) > k/2$.

Now consider two vertices u and u' distinct from v. By $P(n,\ell,2,2)$ and the remarks above we must have $d(u) + d(u') \geq 2k - 3$. Suppose $d(u) \leq k - 2$. Then $d(u') \geq k - 1$ for all choices of u', and taking any two such choices, we get $d(u'') + d(u') \geq 3k - 6$. Thus the sum of the degrees for vertices distinct from u and v is at least $\binom{n-2}{2}(3k - 6)\frac{1}{n - 3}$. Hence $\binom{n-2}{2}\frac{3k - 6}{n - 3} < k(k - 1)$, which gives a contradiction for $n > k$ and $n \geq 6$. Hence $d(u) \geq k - 1$ for all vertices distinct from v. The sum of the degrees is then at least $k(k - 1) + 1$, a contradiction. This completes the proof of step 5.

6. $n \leq 2k$ if $n \geq 6$.

This follows from $nd \leq k(k - 1)$.

7. If M is the maximum degree, then $M \leq 2k - 3$, if $n \geq 6$.

Let $d(v) = M$. Adding the degrees of v and all adjacent vertices we get $M + Md \leq k(k - 1)$, which gives the result.

8. No two vertices of degree $d = \frac{k - 1}{2}$ are adjacent.

This follows from $P(n, \ell, 2, 2)$.

9. Let u be any vertex, v and v' vertices of degree d and $n \geq 6$. Then u is adjacent either to both v and v' or to neither.

Suppose to the contrary that u and v are adjacent but not u and v'. Then $D(X - \{u,v\}) - D(X - \{u,v'\}) = 1$. By Lemma 2.4 we get exactly one edge in $X - \{u,v\}$, and this must contain v' and some other vertex x. Now $d = \frac{k - 1}{2} > 1$, as $k \geq 4$, and thus there must be another edge containing v'. The only choice is $\{v,v'\}$, violating step 8. Thus step 9 is proved.

10. Any vertex u with $d(u) > d$ has $d(u) \geq \frac{3k - 5}{2}$, if $n \geq 6$.

We consider a vertex v with $d(v) = d$, and apply $P(n, \ell, 2, 2)$ to $X - \{u,v\}$.

11. If $n \geq 6$ and $d(u) \neq d$, then u is adjacent to all vertices of degree d, and $d(u) = \frac{3k - 3}{2}$.

If u is not adjacent to some vertex of degree d, then it is adjacent to none by step 9, and summing the degrees of u and the vertices adjacent to it we get by step 10, at least $\left(\frac{3k - 5}{2}\right) + \left(\frac{3k - 5}{2}\right)^2$. This is larger than $k(k - 1)$ for $k \geq 4$, a contradiction.

Let $d(v) = d$, $d(u) > d$, where we now know u and v must be adjacent. Then $d(u) = \frac{3k - 3}{2}$ by step 11.

12. For $n \geq 6$ we get a contradiction.

If we have two vertices u, u' of degree $\frac{3k - 3}{2}$, then together they meet $3k - 3$ or $3k - 2$ edges, violating $P(n, \ell, 2, 2)$. Thus we have exactly one large vertex, and from step 8 we get $d = 1$, contradicting $k \geq 4$.

By the remarks preceeding this theorem, this completes the proof.

We next consider the case of arbitrary j. We can't describe exactly what the examples are which satisfy $P(n, \binom{k}{2}, 2, j)$ and are not cliques, as we did with $j = 2$ above, but we do show that there are only a finite number of them.

THEOREM 5.2. <u>For each j $P(n, \binom{k}{2}, 2, j)$ characterizes cliques for all but a finite number of values of n and k.</u>

<u>Proof</u>. For $j = 1$ and $j = 2$ this follows from Theorems 1.2 and 5.1. For the general case we prove that for each j there is an $f(j)$, depending only on j, such that if G satisfies $P(n, \binom{k}{2}, 2, j)$ and $k \geq f(j)$, then G is a clique. By Theorem 1.3 we know that if G is not a clique, we must have $n < \binom{k}{2} + 2$, or $n < 4 + j$. Thus the existence of $f(j)$ is sufficient to establish the result. We do this in steps also. Let G be a graph satisfying $P(n, \binom{k}{2}, 2, j)$ which is not a clique. We can assume $j \geq 3$, and that the theorem is true for all $j' < j$.

1. $n < jk$.

Consider j vertices of smallest degree, $d(v_1) \leq \cdots \leq d(v_j)$. If $n \geq kj$, then the average degree of vertices in $X - \{v_1, \ldots, v_{j-1}\}$ is at most $\frac{k(k-1)}{jk - j + 1} < \frac{k}{j}$. Thus it is at most $\frac{k-1}{j}$ as it must be an integer. We have $d(v_i) \leq \frac{k-1}{j}$ for all i. By $P(n, \binom{k}{2}, 2, j)$ there are only two possibilities, $d(v_i) = 0$ for all i or $d(v_i) = \frac{k-1}{j}$ for all i. The first is impossible by Lemma 2.5. The second implies that the average degree is $\frac{k-1}{j}$ and that $d(v) = \frac{k-1}{j}$ for all vertices v. But then consider any j of them. These must meet at least $k - 1$ edges. Thus no two of the vertices can be joined. As the j vertices were chosen arbitrarily, there can be no edges in G at all, a contradiction.

2. For some $\varepsilon > 0$ and all k sufficiently large, $n > (1 + \varepsilon)k$.

Suppose this is false. For each $\varepsilon > 0$ and k we let $G = G(k,\varepsilon)$ have $n \leq (1 + \varepsilon)k$. Then for k sufficiently large there are at least j vertices of degree at least $k - 2\varepsilon k$. Call these large vertices. Any set of j large vertices must meet at least $(j - 1)k - \binom{j}{2}$ edges if ε is small, and any set of j vertices at all must meet at most $j(1 + \varepsilon)k < (j + 1)k - \binom{j+2}{2}$ edges if k is large. Thus every j meet precisely $jk - \binom{j+1}{2}$ edges.

Let v_1 and v_2 be large vertices. Since $n \leq (1 + \varepsilon)k$, there are some $j - 1$ vertices $u_1,\ldots,u_{j-1}$ joined to both. Let the two sets $\{u_1,\ldots,u_{j-1},v_1\}$ and $\{u_1,\ldots,u_{j-1},v_2\}$ meet e_1 and e_2 edges, respectively. Then $e_1 - e_2 = d(v_1) - d(v_2)$. Since v_1 and v_2 are large, this difference is at most $4\varepsilon k$. But both e_1 and e_2 are less than $(j + 1)k - \binom{j+2}{2}$, as we observe above. Thus e_1 and e_2 must differ by 0 or by at least $k - j$, by $P(n,\binom{k}{2},2,j)$. Hence $d(v_1) = d(v_2)$, and all large vertices have the same degree.

Let v_1, v_2 and v_3 be large vertices with v_1 and v_2 joined by an edge, but v_1 and v_3 not joined. Take any $j - 2$ vertices joined to all three, say $u_1,\ldots,u_{j-2}$. Then $\{u_1,\ldots,u_{j-2},v_1,v_2\}$ and $\{u_1,\ldots,u_{j-2},v_1,v_3\}$ differ by one in the number of edges they meet, a contradiction of $P(n,\binom{k}{2},2,j)$ unless $\{u_1,\ldots,u_{j-2},v_1,v_3\}$ meets all edges. But by step 1 $d(u_i)$ and $d(v_i)$ are less than jk. Hence no j-set can meet more than j^2k edges. If k is large, then the j-set $\{u_1,\ldots,u_{j-2},v_1,v_3\}$ can't meet $\binom{k}{2}$ edges. Thus either all large vertices are joined, or all are independent. They have degree either $k - 1$ or $k - \frac{j+1}{2}$, respectively.

Let u be a vertex with $d(u) < k - 2\varepsilon k$. Let $v_1,\ldots,v_j$ be large vertices. Then $\{v_1,\ldots,v_{j-1},u\}$ meets $y = jk - \binom{j+1}{2} - x + d(u) - a$, where x is the number of edges not joining v_j to other v_i, and a is the number of edges joining u to $\{v_1,\ldots,v_{j-1}\}$. We have $x = k - \frac{j+1}{2}$ or $x = k - j$, and $0 \le a \le j - 1$. Thus y is at most

$$jk - \binom{j+1}{2} - (k - j) + k - 2\epsilon k$$
$$= (j - 2\varepsilon)k - \binom{j}{2}.$$

Hence y is at most

$$(j - 1)k - \binom{j}{2}.$$

But $v_1,\ldots,v_{j-1}$ themselves meet at least $(j - 1)k - \binom{j}{2} - \frac{j}{2}$. Thus $d(u)$ can be at most $(3/2)j$. This means that there can't be j of these "small" vertices, unless they all have degree 0. This means that the large vertices form a clique, as they can't be joined only to small vertices. But then, since they have degree $k - 1$, they form a complete graph on k vertices, a contradiction. This completes step 2.

3. For any $\beta > 0$, and some integer i, $1 \le i \le j - 1$, the minimum degree d satisfies $(1 - \beta)\frac{i}{j}k \le d \le (1 + \beta)\frac{i}{j}k$ if k is large enough.

For any M, and any sufficiently small $\delta > 0$, we can find at least M vertices of degree at most $(1 - \delta)k$, if k is sufficiently large. For the average degree of any $n - m$ vertices, by step 2, must be at most $\frac{k(k-1)}{k(1+\varepsilon) - M} \le \frac{k(k-1)}{k(1+\varepsilon/2)} < (k - 1)(1 - \frac{\varepsilon}{3})$ for k sufficiently large. Thus if we consider the M vertices of smallest degree, we see their degrees must be at most $k(1 - \varepsilon/3) \le k(1 - \delta)$ for δ sufficiently small.

Let $\eta > 0$ be arbitrary, and M sufficiently large that there is some c, $0 \le c \le 1 - \varepsilon/3$ such that there are at least $j + 1$ vertices with degrees in the interval $[c(1 - \eta)k, c(1 + \eta)k]$. The total number of edges any j of these vertices meet is at most $j c k(1 + \eta)$, and at least $j c k(1 - \eta) - \binom{j}{2} \ge j c k(1 - \eta/2)$ for k sufficiently large. By $P(n, \binom{k}{2}, 2, j)$, they must meet exactly $ik - \binom{i+1}{2}$ for some i, $0 \le i \le j - 1$. Thus $j c k$ must be in an interval $[ik - \binom{i+1}{2} - 2\eta k, ik - \binom{i+1}{2} + 2\eta k]$ for some i, for k sufficiently large. By considering all different j-subsets of the vertices with degrees in $[c(1 - \eta)k, c(1 + \eta)k]$, we see that each j-set must meet exactly $ik - \binom{i+1}{2}$ edges, and thus no two such vertices have degrees differing by more than j. Thus the degrees of these vertices must all be in the interval $\left[\frac{ik}{j}(1 - \alpha), \frac{ik}{j}(1 + \alpha)\right]$ for k sufficiently large, depending on α, which is arbitrary.

Let $d(v) \le k(1 - \delta)$, and let $u_1, \ldots, u_{j-1}$ have degrees in $\frac{ik}{j}(1 \pm \alpha)$. Then the j-set they form meets x edges, where $x \le \frac{j-1}{j} ik + (j - 1)\alpha k + k - \delta k < (i + 1)k - \binom{i+2}{2}$ and $x \ge \frac{j-1}{j} ik - (j - 1)\alpha k - \binom{j-1}{2} > (i - 1)k - \binom{i}{2}$, for k sufficiently large, and α and δ small. This forces x to be exactly $ik - \binom{i+1}{2}$, and the degree of v to be in the interval $\frac{ik}{j} \pm j\alpha k$. If α is small, this means $d(v)$ is in $\left[\frac{ik}{j}(1 - \beta), \frac{ik}{j}(1 + \beta)\right]$. This completes the proof, as we could take v to have minimum degree. This completes step 3, and in fact proves step 4:

4. If $\beta > 0$, $\delta > 0$, then there is some i, $0 \le i \le j - 1$ such that if $d(v) \le k(1 - \delta)$, then $d(v)$ is in the interval $\left[\frac{i}{j}k(1 - \beta), \frac{i}{j}k(1 + \beta)\right]$

if k is sufficiently large.

Call vertices with degree $d(v) \leq k(1 - \delta)$ "small" vertices. From the proof of step 3 we know that every j of them meet exactly $\frac{i}{j}k - \binom{i+1}{2}$ edges, for some fixed $i \leq j - 1$ and k large. If u is a vertex with $d(u) > k(1 - \delta)$, J a set of j small vertices (which exist for k large, as we saw in the proof of step 3), and v a vertex of J, then $d(u) - d(v) \geq \frac{j - i}{j}k - (\delta + \beta\frac{i}{j})k > j$ for k large. The set $(J - \{v\}) \cup \{u\}$ must meet $(i + h)k - \binom{i + h + 1}{2}$ edges then, and thus $d(u) - d(v) > hk - \binom{i + h + 1}{2}$. Then for any $\gamma > 0$, and k sufficiently large, depending on h, we have $d(v)$ in the interval $\left[\left(\frac{i}{j} + h\right)k(1 - \gamma), \left(\frac{i}{j} + h\right)k(1 + \gamma)\right]$. Since $h \leq j$, by step 1, we have the following:

5. Let $\gamma > 0$. Then for some i, $0 \leq i \leq j - 1$ and every vertex v, $d(v) \in [(i/j + h)k(1 - \gamma), (\frac{i}{j} + h)k(1 + \gamma)]$, where h depends on v, but $0 \leq h \leq j - 1$, for k large enough.

Next let $\delta > 0$, and recall that v is "small" if $d(v) \leq k(1 - \delta)$.

6. All small vertices have the same degree if k is sufficiently large.

In step 3 we showed there have to be M small vertices if k is large. Now if M is large enough, then by Ramsey's Theorem there must be either a clique or an independent set of size $j + 1$. Let J be a subset of j of them. In step 3 we saw that this set meets exactly $ik - \binom{i+1}{2}$ for some $i \leq j - 1$. We claim any other small vertex v either is joined to all vertices in J, or is joined to none of them. For suppose v and u are joined, v and w are not, and $u, w \in J$. Then $(J - \{u\}) \cup \{v\}$ and $(J - \{w\}) \cup \{v\}$ meet numbers of edges differing by 1. This is a contradiction, as it implies by Lemma 2.4 that one of these sets meets all the edges, making the degrees too large.

Thus every small vertex is joined completely to J or not at all.

Let K be a maximum clique of small vertices if J is a clique, or a maximum independent set if J is independent. We have at least $j + 1$ vertices in K. Since each j-subset meets exactly $ik - \binom{i+1}{2}$, every two vertices of K have the same degree. Every other small vertex is joined to no vertex if K is a clique, or to every vertex of K if K is independent, by maximality of K and the remarks above. Hence the small vertices not in K must all have the same degree (by considering $j - 1$ vertices of K together with one outside K.)

If K is a clique, the degrees of vertices of K must be $\frac{ik}{j} - \frac{1}{j}\binom{i+1}{2} + \frac{j-1}{2}$. If K is independent, then the degrees of vertices of K will be $\frac{ik}{j} - \frac{1}{j}\binom{i+1}{2}$. The small vertices not in K will have degree $\frac{ik}{j} - \frac{1}{j}\binom{i+1}{2} - \frac{j-1}{2}$ or $\frac{ik}{j} - \frac{1}{j}\binom{i+1}{2} + j - 1$, respectively. In either case, if we form a j-set from $j - 2$ vertices of K and two not in K, then by counting degrees and edges contained in the j-set, we see that the j-set meets a number of edges differing from $ik - \binom{i+1}{2}$ by x, where $1 \leq x \leq 5$. This is a contradiction. Thus all but at most one small vertex is in K.

Now if K is a clique, there can be at most $\frac{i}{j}(1 + \gamma)k + 1$ small vertices, by step 5. Then considering any vertex v which is not small, its degree must be at least $\left(\frac{i}{j} + 1\right)(1 - \gamma)k$. For k sufficiently large, we can take γ so small that v must be adjacent to $k(1 - a)$ "large" (i.e. not small) vertices for any given $a > 0$. But if γ and a are small enough, the sum of the degrees of these points must be at least $k(1 - a)\left(\frac{i}{j} + 1\right)k(1 - \gamma) > 2\binom{k}{2}$, a contradiction.

Thus K must be an independent set. If there is a small vertex not in K, then since it is adjacent to all vertices of K, there can be at most $\frac{i}{j}k(1 + \gamma) + 1$ small vertices, and the same argument as in the previous paragraph leads to a contradiction. This shows that all small vertices form an independent set, and all have degree $\frac{i}{j}k - \frac{1}{j}\binom{i+1}{2}$. This proves step 6 and also the next step:

7. The small vertices all have degree $\frac{i}{j}k - \frac{1}{j}\binom{i+1}{2}$ and form an independent set, for k large enough.

Let K denote the set of small vertices.

8. All large vertices are joined to all vertices of K, for k sufficiently large.

Let v be joined to x but not to y, $x, y \in K$. Taking $u_1, \ldots, u_{j-2}$ vertices from K different from x and y, and considering $\{u_1, \ldots, u_{j-2}, x, v\}$ and $\{u_1, \ldots, u_{j-2}, y, v\}$, we see that $\{u_1, \ldots, u_{j-2}, y, v\}$ must meet all edges of G, by Lemma 2.4. This is impossible for k large, since the sum of the degrees of these j vertices is certainly less than $jn < j^2 k < \binom{k}{2}$.

So we see that if a large vertex is joined to any vertex of K, it is joined to all. But it can't be joined to none of them, or by adding the degrees of the vertices adjacent to it (which would be large) we would get at least $((1 + i/j)(1 - \gamma)k)^2 > 2\binom{k}{2}$ for k sufficiently large and γ small. This completes step 8.

9. All large vertices have the same degree if k sufficiently large.

By step 8 there are exactly $(i/j)k - \frac{1}{j}\binom{i+1}{2}$ large vertices. Since they all are joined to all small vertices, they can only differ in the large vertices they are joined to. Thus if u and v are large, we have $(d(u) - d(v)) \leq i/j\ k - \frac{1}{j}\binom{i+1}{2}$. By step 5 there is some h, $1 \leq h \leq j - 1$ such that all large vertices have degrees in $[(h + i/j)k(1 - \gamma),$ $(h + i/j)k(1 + \gamma)]$. Combining v with $j - 1$ small vertices yields a j-set meeting $d(v) - (j - 1) + (j - 1)\left(\frac{i}{j}k - \frac{1}{j}\binom{i+1}{2}\right)$ edges. But this number must be in the interval $(h - i)k \pm 2\gamma k$ for k large. Hence, by $P(n,\binom{k}{2},2,j)$ it must be exactly $(h + i)k - \binom{h + i + 1}{2}$. This gives $d(v) = (h + i/j)k - \binom{h + i + 1}{2} + \frac{j - 1}{j}\binom{i + 1}{2} + j - 1$, and completes step 9.

10. This situation is impossible.

For consider a j-set with $j - 2$ small vertices and two large ones. Then this set meets exactly

$$(h + i)k - \binom{h + i + 1}{2} - \left(\frac{i}{j}k - \binom{i + 2}{j} - 1\right)$$

$$+ (h + \frac{i}{j})k - \binom{h + i + 1}{2} + \binom{j - 1}{j}\binom{i + 1}{2} + j - 1 - (j - 2) - x$$

where x is 0 or 1. But this number is in the interval $(2h + i)k$ $(2h + i)k - \binom{2h + i + 1}{2} + i/j\ k \pm \frac{1}{10j}k$ if k is large. This violates $P(n,\binom{k}{2},2,j)$ unless $i = 0$. But this means there are no large vertices, and G has no edges. This, then completes step 10 and the proof of Theorem 5.2.

Theorem 5.2 is concerned with the case of fixed j. We now turn our attention to the case of fixed $t = n - j$, and let ℓ and n be arbitrary. We do not require $\ell = \binom{k}{2}$ here. This case has a different spirit to it, as the examples below illustrate. In particular there may be infinitely many graphs G which are not cliques. However, when this is true, all but a finite number will be seen to fall into two or three easily described classes. The result is thus almost as satisfactory as the previous one.

First, then, we consider small cases. For $t = 1,2$ clearly every G satisfies $P(n,\ell,2,n - t)$. For $t = 3$ Example B can be generalized to the union of disjoint cliques. It is also easy to see that these are the only examples other than cliques. For $t = 4$ the only examples besides cliques are the stars (a set of ℓ edges all joined to a common vertex) and the pentagon (or cycle of length five). Finally, for $t = 5$, we see by examining the cases systematically that the only examples other than cliques are the two 3-regular graphs with 6 vertices. For $t = 5$ the stars are not examples. However for any x, and $t = \binom{x}{2} + 1$, the stars are examples. There are cases where the complete bipartite graphs $K_{2,m}$ work, for instance $t = 16$ or $t = 659988946$ and all $m \geq t - 2$. Whether this (or similar examples) works depends on the solutions to diophantine equations. Finally, consider the graphs $K'_{a,b}$ consisting of a clique of a vertices, an independent set of size b, and all ab edges joining them. Then $K'_{2,m}$ is an example for all $m \geq t - 2$ and $t = 29$ or 947 but not for all t.

THEOREM 5.3. For $t \geq 4$, and n sufficiently large, depending on t let G satisfy $P(n,\ell,2,n - t)$, but assume G is not a clique. Then G must be one of the following types of graphs: $K_{1,m}$, $K_{2,m}$, $K'_{2,m}$ or $K'_{3,m}$. The last type can only satisfy $P(n,\ell,2,n - t)$ for finitely many t, while the other three types can occur infinitely often.

Proof. Suppose G is not a clique, and G satisfies $P(n,\ell,2,n - t)$. For n sufficiently large, by Ramsey's Theorem, there is a set S of t vertices forming either a clique or an independent set.

First let S be a clique, and let K be a maximum clique. If $v \notin K$, $u \in K$, and $\{u,v\}$ an edge, then some vertex $w \in K$ must not be joined to v, by maximality of K. Let T be a set of $t - 2$ other vertices of K. Then $D(T \cup \{v\} \cup \{u\}) - D(T \cup \{v\} \cup \{w\}) = 1$, contradicting Lemma 2.4. Thus no vertex $v \notin K$ can be joined to any vertex of K. But some edge, say $\{v_1,v_2\}$ must occur outside K, as G isn't a clique. Then taking T as above, $T \cup \{v_1\} \cup \{v_2\}$ violates $P(n,\ell,2,n - t)$, as $t \geq 4$. Thus S is independent.

Let K denote a maximum independent set. We have from the previous paragraph that the largest clique in G must have at most $t - 1$ vertices. Let $v \notin K$, $u,w \in K$, and let v be joined to u but not to w. If T is any $t - 2$ other vertices of K, we see, as above, that $D(T \cup \{v\} \cup \{w\}) = 0$, which is a contradiction unless v is joined to only a single vertex of K. So each vertex in $X - K$, which we call $A = X - K$ from now on, is joined either to one vertex of K or to all of them. Suppose v_1 is joined to one vertex, w, in K, and $v_2 \in A$ is joined to all K. Letting $\{u_1,\ldots,u_{t-2}\}$ be any other $t - 2$ vertices of K, we get $D(\{u_1,\ldots,u_{t-2},v_1,v_2\}) = t - 2$, while $D(u_1,\ldots,u_{t-1},v_2) = t - 1$. This contradicts Lemma 2.4. Thus either all vertices of A are completely joined to K, or they are all joined by single edges.

Consider the case where all are joined by single edges to K. Then there must be a single $u \in K$ to which all are joined. For let $v_1,v_2 \in A$ be joined respectively to u_1 and u_2 in K. If v_1 and v_2 are not joined, and if $u_3,\ldots,u_{t-1}$ are other vertices of K, we have $D(\{v_1,v_2,u_1,u_2,u_3,\ldots,u_{t-2}\}) = 2$, a contradiction. Similarly, if v_1 and v_2 are joined, then $D(\{v_1,v_2,u_2,u_3,\ldots,u_{t-1}\}) = 2$, a contradiction. Thus all vertices of A are joined to some vertex $u \in K$. Further, A must be a clique. For if v_1 and v_2 are in A, then $D(\{v_1,v_2,u,u_4,\ldots,u_t\}) \geq 2$, where $u_4,\ldots,u_t$ are other vertices of K. Hence v_1 and v_2 must be joined so that $D(\{v_1,v_2,u,u_4,\ldots,u_t\}) = 3$, a binomial coefficient. But this means G is a clique, a contradiction. Thus the assumption that vertices in A are joined to single vertices of K is wrong.

This means that each vertex of A is joined to all vertices of K.

Now A must be either a clique or an independent set. For suppose $\{w,v\}$ is an edge in A, and $y,z \in A$ but not joined. Then if $u_1,\ldots,u_{t-2}$ are vertices of A, we have $D(\{u_1,\ldots,u_{t-2},u,v\})$ and $D(\{u_1,\ldots,u_{t-2},y,z\})$ differing by one, a violation of Lemma 2.4, as $t \geq 4$.

Since the maximum clique can be at most of size $t - 1$, this implies that G must be either $K_{a,b}$ or $K'_{i,m}$, $2 \leq i \leq t - 1$. It remains to show that the cases $a > 2$ and $i > 3$ are impossible. To do this let G consist of two "parts", an independent set B with $m \geq t$ elements, and a set A of a elements, forming either a clique or an independent set, and assume G contains all am edges joining A and B. For integers x, let S_x denote a t-set with x vertices in A and $t - x$ vertices in B. Then we must have $D(S_1) = \binom{e}{2}$, $D(S_2) = \binom{f}{2}$, $D(S_3) = \binom{c}{2}$ and $D(S_4) = \binom{d}{2}$. This is clearly equivalent to $P(n,\ell,2,n - t)$ for such G.

First let $G = K_{a,m}$, a bipartite graph, and let $a \geq 3$. Then we have $D(S_1) = t - 1$, $D(S_2) = 2t - 4$, $D(S_3) = 3t - 9$. These lead to the equations $2(2e - 1)^2 - (2f - 1)^2 = 17$, and $3(2e - 1)^2 - (2c - 1)^2 = 50$. Looking at the first equation modulo 10 shows that $(2e - 1)^2 \equiv 1 \pmod{10}$. The second equation gives $(2c - 1)^2 \equiv 3 \pmod{10}$, an impossibility. Hence $a \leq 2$ if G is $K_{a,m}$.

Now suppose $G = K'_{a,m}$ and let $a \geq 4$. Then $D(S_1) = t - 1$, $D(S_2) = 2t - 3$, $D(S_3) = 3t - 6$, and $D(S_4) = 4t - 10$. These yield the three simultaneous equations $2(2e - 1)^2 - (2f - 1)^2 = 9$, $3(2e - 1)^2 - (2c - 1)^2 = 26$, and $4(2e - 1)^2 - (2d - 1)^2 = 51$. The last equation is $(2(2e - 1) - (2d - 1))(2(2e - 1) + (2d - 1)) = 51$, which implies $e = 3$, $t = 4$, and $D(S_2) = 5$, an impossibility. Thus $a < 4$, and the first statement of the theorem is proved.

Now $K_{1,m}$ is an example for exactly those t with $t = \binom{x}{2} + 1$, as we observed above. We also saw examples of t for which $K_{2,m}$ works, and t for which $K'_{2,m}$. As we saw above, these will be examples if and only if $t = \binom{e}{2} + 1$, and $2(2e - 1)^2 - (2f - 1)^2 = 17$ or $2(2e - 1)^2 - (2f - 1)^2 = 9$, respectively, has a solution for some integer f. These are Pell's equations, and since there is one solution to each, there are infinitely many, and these can in fact be explicitly determined [4]. Thus there are infinitely many t for which the class $K_{2,m}$ is an example, and infinitely many for which $K'_{2,m}$ is an example.

The only remaining thing to prove is that $K'_{3,m}$ is an example for only finitely many t. But this also follows from number theory, and in particular, by using the method of Baker and Davenport [1], it can be shown that $K'_{3,m}$ works for no t greater than a specific bound (which turns out to be about $10^{10^{500}}$)*. There are no examples for $t \geq 4$ less than t about 10^6, and we suspect that there are in fact none. C. Grinstead [3] has developed a method which could check this on a large computer in several hours.

6. A FEW FURTHER REMARKS AND QUESTIONS

Although for Theorem 5.2 we required $\ell = \binom{k}{2}$, it may be that this condition can be removed and replaced by some easily described list of exceptions, as in the case of Theorem 5.2. So far, however, we have not been able to do this, even in the $j = 1$, $r = 2$ case. In this case we observe that any s-regular graph with $\binom{x}{2} + s$ edges satisfies $P\left(\frac{x(x - 1)}{s} + 2, \binom{x}{2} + s, 2, 1\right)$. This suggests that possibly some constraint on the regularity of G might be sufficient. However examples

*We thank C. Grinstead for showing this.

of the following type show that we can have many degrees. Let G consist of a 5 regular graph H on 272 vertices, and three vertices connected respectively to 49, 91, and 132 vertices of H, all distinct. G satisfies P(275, 952, 2, 1) and has four distinct degrees. Such examples can be proliferated.

It might be possible to fill in some of the space between the large values of j (Theorem 5.3) and the small values (Theorem 5.2) in the $r = 2$ case.

Finally, of course, the sharper results of Section 5 might be extendable to $r \geq 3$. This may be difficult.

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