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ON SPANNED SUBGRAPHS OF GRAPHS

The aim of this note is to prove some theorems of the following type:

We assume that C is a class of finite graphs satisfying certain assymptotic conditions saying that both G and its complement are large. Then we consider a class D of graphs and show that for all $G \in C$ and $H \in D$, H is isomorphic to a spanned subgraph of G provided the size of G is large enough compared to the size of H.

We have already considered problems of the above kind for infinite graphs in [2] and [3]. In those cases the conditions imposed on the elements of C were of "Ramsey type". To explain this expression we state a very easy result which is implicitly contained in [3].

PROPOSITION 1. Let $c > 0$ be a real number. Let C be the class of graphs G such that neither G nor its complement contains complete bipartite graphs $[A,B]$ of the size $|A|=|B|=c \log n$, where n is the number of vertices of G. Then for each graph H and for each $G \in C$ G contains H as a spanned subgraph provided $n > n_0(c,|H|)$.

(In fact, we can prove this for n^{ε} in place of $c \log n$ provided $\varepsilon = \varepsilon(H)$ is sufficiently small.)

We did not know for a while if the condition of Proposition 1 can be weakened to the following: C is the class of graphs G such that neither G nor its complement contains complete graphs of size $c \log n$. We are going to answer this problem affirmatively in §.4 of this paper. However our main aim is to investigate the new phenomenae which arise if the conditions imposed on C say that both G and its complement have many edges.

First we agree upon some notation. A graph $G = \langle V_G, E_G \rangle = \langle V, E \rangle$ contains no loops or multiple edges i.e. $E \subset [V]^2 = \{X \subset V : |X| = 2\}$. We put $n(G) = |V|$.

For $x \in V$, $A \subset V$ we put $D(x,A,G) = \{y \in A : \{x,y\} \in E\}$ and $d(x,A,G) = |D(x,A,G)|$. $d(x,A,G)$ is the degree of the vertex x for A in G. We briefly put $d(x,V,G) = d(x,G)$, $D(x,V,G) = D(x,G)$. We remind the reader that for any sets A,B $[A,B] = \{\{x,y\} : x \neq y \wedge x \in A \wedge y \in B\}$. Hence $[A,A] = [A]^2$ for any A.

Whenever $A,B \subset V$ for a given graph $G = \langle V,E \rangle$ we will denote by $e(A,B)$ the number of edges of G lying in $[A,B]$ i.e. $e(A,B) = |E \cap [A,B]|$

Especially, $e(A) = e(A,A)$ is the number of edges of $G(A) = \langle A, [A]^2 \cap E \rangle$, which we call the subgraph of G spanned by A.

It will be convenient to identify graphs to two partitions of $[V]^2$. In what follows if there is no danger of confusion, for any graph $G = \langle V,E \rangle$ we also denote E by E_0, $[V]^2 \setminus E$, by E_1, $G_i = \langle V, E_i \rangle$ for $i < 2$. Hence G_0 is G, G_1 is the complement of G. We briefly write

$$d_i(x,A,G_i) = d_i(x,A), \quad D(x,A,G_i) = D_i(x,A),$$

$$d(x,G_i) = d_i(x), \quad D(x,G_i) = d_i(x) \quad \text{for } i < 2,$$

$x \in V$, $A \subset V$.

When $[A,B] \subset E_i$ for some $i < 2$, $[A,B]$ is said to be *homogeneous* for G. The classes "$\mathcal{D}$" will usually consist of some homogeneous pieces.

§.1. Assumptions on the number of edges

DEFINITION 1. a) Let $c > 0$, $C_1(c)$ is the class of graphs G which satisfy $|E_i| \geq cn^2$ for $i < 2$ and $n=n(G)$.

b) Let D_1 be the class of graphs $H=\langle V,E\rangle$ satisfying the following conditions.

$V=A\cup B_0\cup B_1$, where A, B_0, B_1 are pairwise disjoint, $|A|=|B_0|=|B_1|=\frac{1}{3}n(H)$. A and B_i, $i < 2$ are homogeneous for H. Moreover, $[A,B_i]\subset E_i$ for $i < 2$.

THEOREM 1. There are functions $n_1(c)$, $c_1(c) > 0$ such that for all $c > 0$, for all $G\in C_1(c)$ with $n(G)=n > n_1(c)$ and for all $H\in D_1$ with $n(H) \leq c_1(c)\log n$, H is isomorphic to a spanned subgraph of G.

The following examples show that the theorem is in some sense best possible.

EXAMPLES 1. Let $G^1_{k,n}$ be a graph with $|V|=|V^1_{k,n}|=n$, $A \subset V$ $|A|=k$ and $E^1_{k,n}=[A]^2$.

Clearly, $G^1_{k,n}\in C_1(\frac{c^2}{2})$ if $k=[cn]$ and n is large enough. $G^1_{k,n}$ does not contain spanned subgraphs $H'=\langle V',E'\rangle$ of the following kind $V'=A'\cup B', A'\cap B'=\emptyset$ $[A',B']\subset E'_i$ and $[A']^2, [B']^2\subset E'_{1-i}$ for some $i < 2$, where A', B' are both large.

We will often use the following

LEMMA 1. There are functions $\tilde{c}(c',c'',\alpha) > 0$ and $\tilde{n}(c',c'',\alpha)$ such that for all $c',c'' > 0$, $0 < \alpha < 1$ and graphs $G=\langle V,E\rangle$ with $n=n(G) > \tilde{n}(c',c'',\alpha)$ and for all $A \subset V$ with $|A| \geq [c'\log n]$ and $|e(A,V-A)| \geq c''\log n\ n$ there are sets $A'\subset A$, $B'\subset V\setminus A$ with $[A',B']\subset E$, $|A'| \geq \tilde{c}(c',c'',\alpha)\log n$, $|B'| \geq n^{\alpha}$. For the lemma see e.g. [1].

PROOF OF THEOREM 1. Assume $G\in C_1(c)$, $n(G)=n$. Let $A_d = \{x\in V:\ d_i(x) > dn$ for $i < 2\}$ for $d > 0$. We claim

(1) $|A_d| \geq d$ if $d > 0$ is small enough, and n is large enough.

To see this, put $T_{i,d} = \{x\in V:\ d_i(x) \leq dn\}$ for $i < 2$. Consider the following inequalities: $T_{0,d} \cap T_{1,d} = \emptyset$ for $d < \frac{1}{2}$. Because of this $|T_{0,d}|\ |T_{1,d}| \leq nd(|T_{0,d}|+T_{1,d}|)$ By the assumption on the number of edges,

$$cn^2 \leq dn|T_{i,d}| + \frac{(n-|T_{i,d}|)^2}{2} \quad \text{for } i < 2.$$

These inequalities imply that $|T_{0,d} \cup T_{1,d}| \leq (1-d)n$ provided d is small and n is large enough.

We now fix some $d_1 > 0$ depending on c which satisfies (1). Using the fact that the partition relation

(2) $$n \to \left(\frac{\log n}{2\log 2}\right)^2_2$$

holds for sufficiently large n (see e.g. [4] or [5]), it follows that there exists a number $d_2 > 0$ such that A_{d_1} has a subset A_2, $|A_2| \geq d_2\log n$ which is homogeneous for G provided n is large enough. Note that $d_i(x,V\setminus A) \geq \frac{1}{2}d_1 n$ holds for $x\in A_2$ and $i < 2$. Applying Lemma 1 twice we get a number d_3 (depending on d_1,d_2 and α) such that there exist pairwise disjoint subsets $A\subset A_2$, $B_i'\subset V\setminus A_2$ for $i < 2$ satisfying the following conditions:

$$|A| \geq d_3 \log n, \quad |B_i'| \geq n^{\alpha} \text{ for } i < 2 \text{ and}$$

$[A,B_i']\subset E_i$ for $i < 2$ provided n is large enough. Applying (2) with n^{α} in place of n we get that there are $B_i\subset B_i'$ such that $|B_i| \geq \frac{\alpha}{2\log 2} n$ and B_i is homogeneous for G for $i < 2$ provided n is large enough.

Let $c_1(c)=3\min(d_3, \frac{\alpha}{2\log 2})$. The subgraph $H=G(A\cup B_0\cup B_1)$ establishes our claim.

§.2. Assumptions on the degree of vertices

DEFINITION 2. a) Let $\delta > 0$ and let $C_2(\delta)$ be the class of graphs satisfying $d_i(x) \geq (\frac{1}{4} + \delta)n$ for $i < 2$ and $x\in V$.

b) Let D_2 be the class of complete bipartite graphs $[k,k]$ and their complements i.e.

$$H=\langle V,E\rangle\in D_2 \text{ iff } V=A\cup B, \quad A\cap B=\emptyset,$$

$|A|=|B|$ and $[A]^2,[B]^2\subset E_i$ $[A,B]\subset E_{1-i}$ for some $i < 2$ and $A,B\subset V$.

THEOREM 2. There are functions $n_2(\delta)$, $c_2(\delta) > 0$ such that for all $\delta > 0$, for all $G\in C_2(\delta)$ with $n(G)=n > n_2(\delta)$ and for all $H\in D_2$ with $n(H) \leq c_2(\delta)\log n$, H is isomorphic to a spanned subgraph of G.

Before we prove the theorem we show that it is best possible.

EXAMPLES 2. Let n be even and $0 < \alpha < 1$. Let $G^2_{n,\alpha}$ be a graph satisfying the following conditions:

$G^2_{n,\alpha} = \langle V,E \rangle$, $|V|=n$, $V=A\cup B$, $|A|=|B|=\frac{n}{2}$.

Choose $E\cap[A,B]$ to be "sufficiently random" on $[A,B]$. We can certainly find an E with $d_i(x,A); d_i(y,B) \geq \frac{n}{2} - \sqrt{n}\log^2 n$ for $x\in B$, $y\in A$, $i < 2$ for sufficiently large n.

It is well-known that for $0 < \alpha < 1$ there is a $k(\alpha)$ and a graph $G'_{n,\alpha}$ on $\frac{n}{2}$ vertices for sufficiently large n, such that the degree of each vertex is not less than n^α and $G'_{n,\alpha}$ contains no $[A',B']$ with $|A'|=|B'|=$ $=k(\alpha)$.

See e.g. [5] for a method to prove this.

Now choose $G^2_{n,\alpha}$ to be a copy of $G'_{n,\alpha}$ on A and to be a copy of the complement of $G'_{n,\alpha}$ on B

It is easy to see that $d_i(x) \geq \frac{1}{4}n+n^\alpha$ for $i < 2$ $\frac{1}{2} < \alpha < 1$ and $x\in V$, and still there is some $k'(\alpha)$ such that no $G^2_{n,\alpha}$ contains a copy of an element H of D_2 with $n(H) \geq k'(\alpha)$.

PROOF OF THEOREM 2. In what follows in the proof if we choose constants and claim that all $G\in C_2(\delta)$ satisfy certain properties we always mean that all $G\in C_2(\delta)$ with large enough $n(G)$ have these properties.

Let $G\in C_2(\delta)$ and $n(G)=n$

Frist we claim that there is a $\delta' > 0$ such that

(1) For all disjoint pairs $A_0,A_1\subset V$, satisfying

$e_i(A_i) \leq 2\delta' n^2$ for $i < 2$, $|V\setminus(A_0\cup A_1)| > \delta' n$ holds.

This comes from the inequality

$$|A_o||A_1| \geq (\tfrac{1}{4} + \delta)(|A_o|+|A_1|)-2(e_1(A_1)+e_2(A_2))-|V-(A_o\cup A_1)|n.$$

We now fix a number $0 < \alpha < 1$ for the rest of the proo

The next claim to prove is

(2) There is a number $d_1 > 0$ such that either G has a spanned subgraph $H\in D_2$ with $n(H) \geq 2d_1\log n$ or for all $A\subset V$, $|A| > \delta' n$ there are $B\subset A$ and $i < 2$ such that $|B| \geq n^\alpha$ and there is no $C\subset B$, $|C| \geq \log n$ with $[C]^2\subset E_i$.

Let $A\subset V$, $|A| > \delta n$. Let $A_i=\{x\in A_i: d_i(x,A) > \frac{1}{4}|A|\}$ for $i < 2$. Clearly, there is an $i < 2$ such that $|A_i| > > \frac{1}{4}|A| > \frac{1}{4}\delta' n$. We may assume that $|A_o| > \frac{1}{4}\delta' n > n^\alpha$. Now either $A_o=B$ satisfies our second claim with $i=1$, or else there is a $C\subset A$, $|C| \geq \log n$ such that $[C]^2\subset E_1$. By Lemma 1,there is a number $0 < d_1 < 1$, depending on α and δ' only, such that there are $B'\subset C$, $A'\subset A\setminus B'$, $|B'| \geq d_1\log n$, $|A'| \geq n^\alpha$ and $[A',B']\subset E_o$. Again, either $A'=B$ satisfies our second claim with $i=1$ or else there is an $A''\subset A'$, $|A''| \geq d_1\log n$ with $[A''] \subset E_1$. However, in this case $H=G(A''\cup B')$ shows that the first part of (2) is true.

We may assume in the rest of the proof that the second part of (2) is true.

The following is our main lemma

(3) There is a number $d_2 > 0$ such that either G has a spanned subgraph $H\in D_2$ with $n(H) \geq 2d_2\log n$ or else for all $A,B\subset V$, $|A|$, $|B| \geq n^\alpha$ and for all $i < 2$ the condition that no $C\subset A$ or $C\subset B$, $|C| \geq \log n$ satisfies

$[C]^2 \subset E_i$ implies that the inequality

$$e_i(A,B) \leq \delta'|A||B| \quad \text{holds as well.}$$

We are going to use the following corollary of a theorem of Erdös and Szekeres(see [5]).

(4) For all $c' > 0$ there is a $c'' > 0$ such that

$n \to (c'\log n,\ c''\log n)^2$ holds for sufficiently large n.

To prove (3) let now $A,B \subset V$, $|A|,|B| \geq n^\alpha$ and $i < 2$ be given, and assume that $[C]^2 \subset E_i$, $C \subset A$ or $C \subset B$ implies $|C| < \log n$.

Assume that the second claim of (3) is false i.e.

$$e_i(A,B) \geq \delta'|A||B|$$

Clearly there is a number $d_3 > 0$ such that there is a subset $A_1 \subset A$, $|A_1| \geq d_3|A|$ and satisfying $d_i(x,B) \geq d_3|B|$ for all $x \in A_1$.

By (4), we can choose a number $d_4 > 0$ such that $d_3 n^\alpha \to (\log n,\ d_4\log n)^2$ holds. Then there is a subset $A_2 \subset A$, $|A_2| = [d_4\log n]$ such that $[A_2]^2 \subset E_{1-i}$. Now we choose a number $0 < \beta < \alpha$. By Lemma 1, there is a number $d_5 > 0$ such that there are $A_3 \subset A_2$, $B_1 \subset B \setminus A_2$ satisfying $|A_3| \geq d_5\log n$, $|B_1| \geq n^\beta$ and $[A_3,B_1] \subset E_i$. By (4) again, there is a number $d_6 > 0$ such that $n^\beta \to (\log n,\ d_6\log n)^2$. There is a subset $B_2 \subset B_1$ such that $|B_2| \geq d_6\log n$ and $[B_2]^2 \subset E_{1-i}$. Let $d_2 = \min\{d_5,d_6\}$. $G(A_3 \cup B_2)$ shows that d_2 satisfies the first claim of (3). In the rest of the proof we may assume that the second claim of (3) holds. We now derive a contradiction. This will yield that Theorem 2 is true with $c_2(\delta) = 2\min\{d_1,d_2\}$.

Let $\{B_j : j < m\}$ be a maximal pairwise disjoint system of subsets of V satisfying the following conditions:

$|B_j| \geq n^\alpha$ for $j < m$; for each $j < m$ there is an $i(j) < 2$ such that $|C| < \log n$ for all $C \subset B_j$, $[C]^2 \subset E_{i(j)}$.
Put

$$A_i = \cup \{B_j : i(j)=i \wedge j < m\} \quad \text{for} \quad i < 2.$$

Let now $i < 2$. By (3), $e_i(B_j,B_\ell) \leq \delta'|B_j||B_\ell|$ holds for all pairs with $i(j)=i(\ell)=i$, $j,\ell < m$. It follows that

$$e_i(A_i) \leq \sum \{e_i(B_j,B_\ell) : i(j)=i \wedge j, \ell < m\} \leq$$

$$\leq \delta' \sum \{|B_i||B_j| : i(j)=i(\ell)=i \wedge j, \ell < m\} \leq 2\delta'|A_i|^2 \leq 2\delta' n^2$$

Then, by (1), $|V \setminus (A_0 \cup A_1)| > \delta' n$. (2) implies that there is a set $B \subset V \setminus (A_0 \cup A_1)$, $|B| \geq n^\alpha$ and an $i < 2$ with $|B| \geq n^\alpha$ such that $|C| < \log n$ holds for all $C \subset B$ $[C]^2 \subset E_i$. This contradicts the maximality of the system $\{B_j : j < m\}$.

§.3. Strongly c,k - universal graphs

DEFINITION 3. a) Let $G=\langle V,E\rangle$ be a graph, $n(G)=n$, For each $X \in [V]^k$ and $\varphi : X \to \{0,1\}$ we write

$$K(X,\varphi)\ (=K(X,\varphi,G)) = \{v \in V \setminus X : \forall u \in X\ (\{u,v\} \in E_{\varphi(u)})\}$$

Note that for $k=1$, $|X|=1$, $X=\{u\}$ we have

$K(X,\varphi) = D(u, E_{\varphi(u)})$, and $K(\emptyset,\emptyset)=V$ in case $k=0$.

b) We say that G is strongly c,k-universal for some $c > 0$, $k \geq 1$ if for all $X \in [V]^k$ and for all $\varphi : X \to \{0,1\}$ we have

$$|K(X,\varphi)| \geq cn.$$

c) $C_3(c,k)$ is the class of strongly c,k-universal graphs. Clearly, $C_3(c,k) \neq \emptyset$ can hold only if $c < \frac{1}{2^k}$.

The concept defined above is a generalization of k-superuniversal graphs introduced in [7]. We only mention a few results and problems concerning this concept.

DEFINITION 4. Let $H = \langle V,E \rangle$ be a graph, $n(H)=k$. Let $\ell \geq 1$. $H' = \langle V', E' \rangle$ is said to be an *ℓ multiple of* H if there are pairwise disjoint sets $A_1,\ldots,A_k$, and an enumeration $x_1,\ldots,x_k$ of the vertices of H such that,

$V = \bigcup_{j=1}^{k} A_j$, $|A_j| = \ell$, A_j is homogeneous for H' for $1 \leq j \leq k$ and

$$[A_j,A_\ell] \subset E_i \text{ iff } \{x_j,x_\ell\} \in E'_i \text{ for } 1 \leq j < \ell \leq k,\ i < 2.$$

Using a computation similar to the one needed for the proof of Lemma 1, one can prove

THEOREM 3. Assume $c > 0$, $k,\ell \geq 1$. Then for all graphs H with $n(H)=k+1$ and for all $G \in C_3(c,k)$, G contains a spanned subgraph isomorphic to some ℓ-multiple of H provided

$n=n(G)$ is large enough $(n > n(c,k))$.

We omit the proof.

We only mention a class of $c,2$-universal graphs.

EXAMPLES 3. Let $H=\langle V',E'\rangle$ be a 2-super universal graph. i.e. a graph such that for all $X\in[V']^2$ and for all $\varphi : X \to \{0,1\}$, $K(X,\varphi,H)\neq\emptyset$. Assume $n(H)=2k$.

Let $G^3(H,n)=\langle V,E\rangle$ be a graph satisfying the following conditions.

$V = \bigcup_{i=1}^{2k} A_j$, where the A_i are pairwise disjoint, and $|A_j|=\frac{n}{2k}$ for $1 \le j \le 2k$. Let $V'=\{x_j: 1 \le j \le 2k\}$ be an ennumeration of the vertices of H. We put $[A_j]^2\cap E=\emptyset$ for $1 \le j \le 2k$;

$[A_j,A_\ell]^2\subset E_i$ iff $\{x_j,x_\ell\}\in E_i'$ for $1 \le j < \ell \le k$, $i < 2$

provided $\ell\neq k+j$.

Choose E to be a "sufficiently random graph" on $[A_j,A_{k+j}]$ for $1 \le j \le k$.

It is easy to see that $G^3(H,n)$ is strongly $c,2$-universal.

We leave it to the reader to ponder about the restrictive effect of these examples on possible embedding theorems. We only prove one more theorem in which we can make real use of strong $c,2$-universality.

THEOREM 4. There are functions $n_4(c,\ell)$, $c_4(c,\ell) > 0$ such that for all $c > 0$, $\ell \ge 1$ and for all strongly $c,2$-universal graphs $G=\langle V,E\rangle$ with $n(G)=n > n_4(c,\ell)$ there exists a subset $X\subset V$, $|X|=\ell$ for which

$$|K(X,\varphi)| \geq c_4(c,\ell) \text{ holds for all } \varphi : X \to \{0,1\}.$$

P r o o f. Choose ℓ_o so that $c\ell_o - \ell > 0$. Put $c_4 = c_4(c,\ell) = \frac{c\ell_o - \ell}{2^{\ell_o}\ell_o}$. Let now $Y \subset V$, $|Y| = \ell_o + 1$, $Y = \{y_o, \ldots, y_{\ell_o}\}$. We claim that there exists a subset $L \subset [0,\ell_o)$, $|L| = \ell$ such that

(1) $\quad |\{x \in V : \forall j \in L(\{y_j, x\} \in E \wedge \{y_{j+1}, x\} \notin E)\}| \geq c'n.$

Let $W_x = \{j : 0 \leq j < \ell_o \wedge \{x_j, x\} \in E \wedge \{y_{j+1}, x\} \notin E\}$ for $x \in V$, and $W = \{\langle j,x\rangle : j \in W_x \wedge x \in M\}$. By the assumption, $|W| \geq \ell_o cn$. Let $V_1 = \{x \in V : |W_x| \geq \ell\}$. Since

$|W| \leq n\ell + |V_1|\ell_o$, it follows that $|V_1| \geq$ $\geq \frac{\ell_o c - \ell}{\ell_o} n$. It follows that there is an $L \subset [0,\ell]$, $|L| = \ell$ which is contained in W_x for at least $c'n$ elements x of V. This proves (1).

Let now $\Psi : [0,\ell) \to \{0,1\}$ be any function, and $Z = \{z_1, \ldots, z_t\}$ an ordering of a subset of V, $X \in [Z]^\ell$. The mapping Ψ induces a mapping $\phi : X \to \{0,1\}$ by the stipulation $\varphi(z_j) = \Psi(k)$ provided z_j is the k-th element of X. We may denote $K(X,\varphi)$ by $K(X,\Psi)$ as well.

Let now ℓ_1 be so large that

$$\ell_1 \to (\ell_o + 1)^{\ell}_{2\ell} \text{ holds.}$$

For sufficiently large n choose an ordered set.

$Z=\{z_1,\ldots,z_{\ell_1}\}\subset V$. For $\Psi:[0,\ell)\to\{0,1\}$.

Let $I_\Psi=\{X\in[Z]^\ell : |K(x,\Psi)|<c'n\}$.

It is sufficient to see, that there is an $X\in[Z]^\ell$ not contained in I_Ψ for all $\Psi:[0,\ell)\to\{0,1\}$. If this is not true, then $[Z]^\ell=\cup\{I_\Psi : \Psi:[0,\ell)\to\{0,1\}\}$ is a partition of ℓ tuples of Z into 2^ℓ classes. By the choice of ℓ_1 there are a $Y\subset Z$, $|Y|=\ell_0+1$ and a $\Psi:[0,\ell)\to\{0,1\}$ such that

$$[Y]^{\ell_0}\subset I_\Psi.$$

This contradicts (1).

Note that we only used the following consequence of strong $c,2$-universality in the proof.

For $\{u,v\}\in[V]^2$ and for $\varphi:\{u,v\}\to\{0,1\}$

$$|K(X,\varphi)|\geq cn \text{ provided } \varphi(u)\neq\varphi(v).$$

§.4. Ramsey type conditions

THEOREM 5. There is a function $n_5(c,k)$ such that for all graphs G establishing the negative partition relation $n\not\to(c\log n)^2_2$ and for all graphs H with $n(H)=k$, H is isomorphic to a spanned subgraph of G provided $n=n(G)>$ $>n_5(c,k)$.

P r o o f.

By the result of [5] already used, there is a $d>0$ such that

(1) $n\to(2c\log n,\ d\log n)^2$ holds for all sufficiently large n.

First we prove that there exists $n_6(c,k)$ such that

(2) For all graphs $G=\langle V,E\rangle$ establishing $n \nrightarrow (c \log n)^2_2$ with $n=n(G) > n_6(c,k)$ and for all H with $n(H)=k$ either H is isomorphic to a spanned subgraph of G or there are $i < 2$, $A,B\subset V$ such that $A\cap B=\emptyset$ $[A]^2\subset E_i$, $[A,B]\subset E_i$, $|A| \geq \frac{d}{2}\log n$, $|B| \geq \frac{n}{(\log n)^{2k}}$.

Let $H = \langle V',E'\rangle$ and $V'=\{x_1,\dots,x_k\}$. We are going to select $Y=\{y_1,\dots,y_k\}\subset V$ in such a way that $x_j \to y_j$ is an isomorphism between H and $G(Y)$.

Assume that for some $1 \leq j \leq k$ the elements $Y_j = \{y_\ell : 1 \leq \ell < j\}$ have already been chosen in such a way that for each $\varphi : Y_j \to \{0,1\}$

$$|K(Y_j,\varphi)| \geq \frac{n}{(\log n)^{2(j-1)}} \text{ and}$$

$\{y_\ell,y_s\}\in E_i$ iff $\{x_\ell,x_s\}\in E'_i$ for $1 \leq \ell,s < j$ and $i < 2$.

Let $\varphi_o(y_\ell)=i$ iff $\{x_\ell,x_j\}\in E'_i$ for $\ell < j$.

If there is an element $y_j\in K(Y_j,\varphi_o)$ and such that

(3) $d_i(y_j,K(Y_j,\varphi)) \geq \frac{n}{(\log n)^{2j}}$ for $i < 2$ and for all $\varphi : Y_j \to \{0,1\}$ we can continue the induction. Otherwise all elements y of $K(Y_j,\varphi_o)$ fail to satisfy (3) for some $i < 2$ and $\varphi : Y_j \to \{0,1\}$.

It follows that there are $A_1\subset K(Y_j,\varphi_o)$, $i < 2$ and a $\varphi_1 : Y_j \to \{0,1\}$ such that

(4) $|A_1| \geq \frac{n}{2^j\cdot \log n^{2(j-1)}}$, and $d_i(x,B_1) < \frac{n}{\log n^{2j}}$ holds

for $x \in A$ and $B_1 = K(Y_j, \varphi_1)$

Now, if n is large enough, $\frac{n}{2^j \log n^{2(j-1)}} \geq n^{1/2}$.

By (1), $n^{1/2} \to (c \log n, \frac{d}{2} \log n)^2$. Considering that G_i does not contain a complete $c \log n$, it follows that there is an $A \subset A_1$, $|A| = [\frac{d}{2} \log n] + 1$, $[A]^2 \subset E_{1-i}$. Now put $B = B_1 \setminus (A \cup \bigcup \{D_i(x,B) : x \in A\})$

$$|B| \geq \frac{n}{(\log n)^{2(j-1)}} - ([\frac{d}{2} \log n] + 1)(\frac{n}{(\log n)^{2j}} + 1) \geq$$

$$\geq \frac{n}{2 \log n^{2(j-1)}} \geq \frac{n}{\log n^{2k}}$$

provided n is large enough. Clearly $[A,B] \subset E_{1-i}$ and this proves (2). In what follows we assume indirectly that the second part of (2) is true for all large enough spanned subgraphs G' of G.

We now "iterate" (2). Let t be such that $\frac{1}{8} t\, d > c$. We define a sequence $A_1, \dots, A_t$ of pairwise disjoint subsets of V by induction on j. Assume that $1 \leq j \leq t$, and the sets A_ℓ, $1 \leq \ell < j$ have already been defined and

a set B_{j-1}, $|B_{j-1}| \geq \frac{n}{(\log n)^{k(j-1)}}$ is defined as well.

(Set $B_0 = V$). Assume further that $A_1, \dots, A_{j-1}, B_{j-1}$ are pairwise disjoint.

Put $m = \frac{n}{(\log n)^{k(j-1)}}$. We choose sets $A_j, B_j \subset B_{j-1}$ and an $i(j) < 2$ in such a way that

$|A_j| = [\frac{d}{2} \log m] + 1$, $|B_j| \geq \frac{m}{(\log m)^k}$, $A_j \cap B_j = \emptyset$ and

$[A_j]^2 \subset E_{i(j)}$, $[A_j, B_j] \subset E_{i(j)}$. Then $\log n \geq \log m \geq$
$\geq \log n - \log(\log n)^{k(j-1)} \geq \frac{\log n}{2}$ provided n is large enough. It follows that $|B_j| \geq \frac{n}{(\log n)^{kj}}$ and $|A_j| \geq$
$\geq \frac{d}{2} \log m \geq \frac{d}{4} \log n$ for $1 \leq j \leq t$. This defines the sets $A_1, \ldots, A_t$. Moreover we know that for each $1 \leq j \leq t$ there is an $i(j) < 2$ such that $[A_j, A_\ell]^2 \subset E_{i(j)}$, for $j \leq \ell \leq t$.
There is an $i < 2$ such that $L_i = \{1 \leq j \leq t : i(j) = i\}$ has $\geq \frac{t}{2}$ elements.

Put $A = \cup\{A_j : j \in L_i\}$. Then $|A| \geq \frac{1}{8} td \log n > c \log n$ and $[A]^2 \subset E_i$. This is a contradiction.

A refirement of this proof gives the following

There is a function $n_7(k)$ such that for all graphs G establishing the negative partition relation

$$n \nrightarrow (2^{\frac{1}{2}(\frac{1}{2k+1} \log n)^{1/2}})^2_2,$$

and for all graphs H with $n(H) = k$, H is isomorphic to a spanned subgraph of G provided $n > n_7(k)$. We do not discuss this since we do not know how far this is from being best possible. As far as we know the theorem could be true with n^{ε_k} in place of $2^{\frac{1}{2}(\frac{1}{2k+1} \log n)^{1/2}}$.

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