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PROBLEMS AND RESULTS IN COMBINATORIAL ANALYSIS

By

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4. Some remarks on a theorem of Stone and myself. Stone and I proved that for $n > n_o(\varepsilon, k, l)$ every $G(n; \frac{n^2}{2}(1 - \frac{1}{k-1} + \varepsilon))$ contains a $K_k^{(2)}(l)$ (for $k=2$ this is again a weaker form of the Kovari-Sos, Turan theorem). Our original proof did not give a very good dependence of n on l and ε. A very much sharper result in this divertion was just published by Bollobas and myself; a further improvement which is nearly best possible has recently been obtained by Bollobas, Simonovitz and myself.

Recently I succeeded to extend this theorem to r-graphs as follows: To every r, ε, t and l there is an $n_o = n_o(\varepsilon, r, t, l)$ so that every $G^{(r)}(n; \alpha(t, r) + \varepsilon)\binom{n}{r})$ contains a $K_t^{(r)}(l)$ where $\alpha(t, r)$ is defined by (1) of chapter 1. Here we do not yet have a good estimate of n in terms of ε, k and l (unlike for $r=2$).

The following problem is open and seems very challenging to me: Let $G^{(r)}(n_i)$, $i = 1, \ldots, n_i \to \infty$ be a sequence of r-graphs of n_i vertices. We say that the family has subgraphs of density $\geq \alpha$ if there is a sequence of subgraphs $G(m_i)$ of $G^{(r)}(n_i)$, $m_i \to \infty$, so that $G(m_i)$ has at least $(\alpha + o(1))\binom{m_i}{r}$ edges. The theorem of Stone and myself implies that every $G(n; \frac{n^2}{2}(1 - \frac{1}{l} + \varepsilon))$ contains a subgraph of density $1 - \frac{1}{l+1}$ and it is easy to see that this is best possible. Thus the possible maximal densities of subgraphs are of the form $1 - \frac{1}{l}$, $2 \leq l < \infty$. Now it may be true that for $r > 2$ there are also only a denumerable number of possible values of the maximal densities of subgraphs. As stated at the end of the previous

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 28

chapter, I proved that every r-graph of density ε contains a subgraph of density $\geq \frac{r!}{r^r}$. The simplest unsolved problem states: Is there a constant $\alpha_r > 0$ so that every r-graph of n vertices (n large) and $(\frac{r!}{r^r} + \varepsilon)n^r$ edges contains a subgraph of density $\geq \frac{r!}{r^r} + \alpha_r$. This is unsolved even for r=3. Perhaps every $G^{(3)}(3n; n^3+1)$ contains either a $G^{(3)}(4;3)$ or a $G^{(3)}(5;4)$; (1,2,3),(1,2,4),(1,2,5),(3,4,5) or a $G^{(3)}(5,5)$: (1,2,3), (1,2,4), (1,3,5),(2,4,5), (3,4,5). The same unsolved problems on the possible maximal densities arise on multigraphs and digraphs as stated in a recent paper of Brown, Simonovits and myself.

By the method of probabilistic graph theory it is easy to prove that to every ε and $0 < \alpha < 1$ there is a $C = C(\varepsilon, \alpha)$ so that for $n > n_0(C, \varepsilon, \alpha)$ there is a $G^{(r)}(n, \alpha\binom{n}{r})$ so that for every $m > C(\log n)^{\frac{1}{r-1}}$ every spanned subgraph of its m vertices has more than $(\alpha - \varepsilon)\binom{m}{r}$ and less than $(\alpha + \varepsilon)\binom{m}{r}$ edges and it follows from the results of my paper on graphs and generalized graphs that this result is best possible (Israel Journal Math. 2(1965), 183-190).

P. Erdös and A. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52(1946), 1087-1091.

B. Bollobas and P. Erdös, On the structure of edge graphs, Bull. London Math. 15(1973), 317-321. The triple paper with Simonovits will soon appear in J. London Math. Soc.

P. Erdös, On some extremal problems on r-graphs, Discrete Math. 1(1971), 1-6 .

W.G. Brown, P. Erdös and M. Simonovits, Extremal problems for directed graphs, J. Comb. Theory, ser. B. 15(1973), 77-9

5. In this chapter I discuss various combinatorial problems on subsets. First of all I call attention to my paper with Kleitman quoted in the introduction. Here I mainly discuss problems not considered in our survey paper.

First we consider some problems related to a result of

Ko, Rado and myself. Let $|S|=n, A_i \subset S, |S|=k$. Denote by $t(n; k, r, \alpha)$ the size of the largest family A_j, $1 \leq j \leq t(n;k, r, \alpha)$ satifying $|A_{j_1} \cap A_{j_2}| \leq r$ and every element is contained in at most $\alpha t(n; k, r, \alpha)$ of the A's. $t(n;k,r,<\alpha)$ is the size of the largest subfamily with the same properties but now every element is contained in fewer than $\alpha t(n; k, r, <\alpha)$ of the A's. Ko, Rado and I proved that for $n \geq 2k$:

(1) $$t(n; k, 1, 1) = \binom{n-1}{k-1}$$

For $n > 2k$ equality holds iff all the A's have a common element. For $n > n_0(k, r)$ we further proved

(2) $$t(n; k, r, 1) = \binom{n-r}{k-r} .$$

Our estimation for $n_0(k, r)$ is probably very poor, but Min observed (2) does not hold for all $n \geq 2k$. We conjectured that

(3) $$t(4l; 2l, 2, 1) = \frac{1}{2}\left\{\binom{4l}{2l} - \binom{2l}{l}^2\right\}$$

(3) if true is best possible. We state in our paper several other problems most of which has been settled since then, but as far as I know (3) has not been settled as yet.

Hilton and Milner proved that for $n \geq 2k$

(4) $$t(n; k, 1, <1) = 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$$

Equality in (4) occurs if (and no doubt only if $n > n_0(k, r)$), A_1 is an arbitrary k-tuple, x_1 is not in A_1. All the other A's contain x_1 and have a non-empty intersection with A_1.

Observe that for fixed k

$$t(n; k, 1, 1) = (1 + o(1))n^{-2}\binom{n}{k} .$$

Now Rothschild, Szemeredi and I took up this investiga-
13. tion. We first of all showed that for $\alpha = \frac{2}{3}$

(5) $$t(n;k, 1, \tfrac{2}{3}) = 3\binom{n-2}{k-2} - 2\binom{n-3}{k-3}$$

Equality iff (until further notice n is supposed to be large), there are three elements and the A's contain at least two of them.

We further proved: $t(n;k, t, < \frac{2}{3}) = cn^{-3}\binom{n}{k}$.

The extremal family is obtained as follows: given three ele ments x_1, x_2, x_3 and a set A_1 not containing any of the All the other A 's meet A_1 and contain at least two of the x 's.

Let now $\varepsilon > 0$ be sufficiently small. We are fairly sur that a family of size $t(n; k, 1, \frac{2}{3} - \varepsilon)$ is obtained as fo lows: Let $x_1, \dots, x_5$ be five elements, the A 's contai three or more of them and $t(n; k, 1, \alpha)$ is constant between $\frac{1}{2}$ and $\frac{3}{5}$. There seem to be only a finite number of values of $t(n; k, 1, \alpha)$ for $\frac{3}{7} < \alpha < \frac{2}{3}$. $t(n;k, 1, \frac{3}{7})$ is probably obtained as follows: Consider a set $B <$ $|B| = 7$ and the 7 Steiner triples of B. The A 's are all the sets which meet B in a set which contains at least one of these triples. We also are fairly sure that

$$t(n; k, 1, < \tfrac{3}{7}) < \frac{c}{n^2}\binom{n}{k} .$$

More generally we conjecture that

$$t(n; k, 1, < \frac{l}{l^2-l+1}) - \frac{c}{n^{l+1}}\binom{n}{k} .$$

If there is a finite geometry on $l^2 - l + 1$ elements, then it is easy to see that

$$t(n; k, 1, \frac{l}{l^2-l+1}) = \frac{c}{n^l}\binom{n}{k} ,$$

but if there is no such finite geometry we conjecture that

$$t(n; k, 1, \frac{l}{l^2-l+1}) < \frac{c}{n^{l+1}}\binom{n}{k} .$$

Needless to say these last two conjectures are very specul tive.

Kneser made the following pretty conjecture: - Let $|S| = 2n + k$ and define a graph $G_{n,k}$ as follows: Its ve tices are the $\binom{2n+k}{n}$ n-tuples of S. Two vertices are j ned if the corresponding n-sets are disjoint. Denote by $K(G)$ the chromatic number of G. Kneser conjectured $K(G_{n,k}$ $= k+2$. $K(G_{n,k}) \leq k+2$ is immediate but the opposite in- equality seems to present great and unexpected difficulti Szemeredi proved (unpublished) that $K(G_{n,k})$ tends to inf nity uniformly in k. Hajnal and I and no doubt many other tried to attack this problem by the following ....

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