

COMBINATORIAL PROBLEMS ON SUBSETS  
AND THEIR INTERSECTIONS

by

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ABSTRACT

Let  $|S| = n$ ,  $m(n; \ell_1, \ell_2, k)$  respectively  $m'(n, \ell_1, \ell, k)$  denote the cardinality of the largest family of subsets  $A_i \subset S$  satisfying  $|A_i| = k$  (respectively  $|A_i| \leq k$ ) and  $|A_{i_1} \cap A_{i_2}| = \ell_1$  or  $\ell_2$ . In this paper we prove

a)  $m(n, 0, \ell_2, k) \leq \binom{n}{2}$ ,  $m'(n, 0, \ell_2, k) \leq \binom{n}{2} + n + 1$ ; equality, iff  $k = 2$ ;

b)  $m(n, 0, \ell_2, k) \leq n$ , if  $\ell_2 \neq k$ , with equality for an infinity of  $n$ .

For  $n \geq n_0(k)$  we show that:

a)  $m(n_1, \ell_1, \ell_2, k) \leq \binom{n-\ell_1}{2}$ ,  $m'(n, \ell_1, \ell_2, k) \leq \binom{n-\ell_1}{2} + (n-\ell_1) + 1$ ;

b) more exactly,  $m(n, \ell_1, \ell_2, k) \leq \left[ \frac{n-\ell_1}{k-\ell_1} \left[ \frac{n-\ell_2}{k-\ell_2} \right] \right]$  with equality for an infinity of  $n$ .

Let integers  $0 \leq \ell_1 \leq \ell_2 < k < n$  be given. Denote by  $M(n, \ell_1, \ell_2, k)$  any maximal system  $\alpha = \{A_i\}$  of different sets such that

$$\left| \bigcup_{A_i \in \alpha} A_i \right| \leq n, \quad |A_i| = k (A_i \in \alpha), \quad |A_i \cap A_j| = \ell_1, \ell_2 (A_i, A_j \in \alpha, i \neq j), \quad (1)$$

$$\text{by } m(n, \ell_1, \ell_2, k) = |M(n, \ell_1, \ell_2, k)|, \quad (2)$$

by  $M'(n, \ell_1, \ell_2, k)$  any maximal system  $\alpha = \{A_i\}$  such that,

$$\left| \bigcup_{A_i \in \alpha} A_i \right| \leq n, \quad |A_i| \leq k (A_i \in \alpha), \quad |A_i \cap A_j| = \ell_1, \ell_2 (A_i, A_j \in \alpha, i \neq j), \quad (1')$$

$$\text{and by } m'(n, \ell_1, \ell_2, k) = |M'(n, \ell_1, \ell_2, k)|. \quad (2')$$

Let  $\ell > 0$  be a given integer. The *kernal* of the system  $\alpha = \{A_i\}$  is the intersection  $K(\alpha) = \bigcap_{A_i \in \alpha} A_i$ .  $|\alpha| > 2$ . (3)

System  $\alpha$  is an  $\ell$ -*star*, if

$$|K(\alpha)| \geq \ell. \quad (4)$$

System  $\alpha$  is a  $\Lambda$ -*system*, if

$$\text{all sets } A_i \setminus K(\alpha) \text{ are disjoint}. \quad (5)$$

Assume first  $\ell_1 = \ell_2 = \ell$ . Then Ryser proved the following (in other terms) Theorem 1 ([8])

$$m(n, \ell, \ell, k) \leq n, \quad (6)$$

$$m'(n, \ell, \ell, k) \leq n + 1, \quad (6')$$

equality holds, if there exist an  $(n, k, \ell)$ -design.

In fact, it was also shown in Theorem 1 of [8], that if  $\alpha = \{A_1, A_2, \dots, A_n\}$  satisfies  $|\bigcup_{i=1}^n A_i| = n$ ,  $|A_i \cap A_j| = \ell (\forall 1 \leq i < j \leq n)$  then it is either  $(n, k, \ell)$ -design or a  $\lambda$ -design,  $\lambda = \ell$ . Theorem 1 is a generalization of the <sup>de</sup>Bruijn-Erdos's Theorem (case  $\ell = 1$ ), which in turn is a generalization of Fisher's inequality for  $(b, v, r, k, \lambda)$ -design. Deza proved (in other terms)

Theorem 2 ([2])

There is an  $r(k, \ell)$  such that

$$r(k, \ell) \leq k^2 - k + 1, \quad (7)$$

$$n > \ell + r(k, \ell)(k - \ell) \Rightarrow m(n, \ell, \ell, k) > r(k, \ell) \Rightarrow$$

$$\Rightarrow \text{any } M(n, \ell, \ell, k) \text{ is a } \Delta\text{-system} \Rightarrow m(n, \ell, \ell, k) = \left\lceil \frac{n - \ell}{k - \ell} \right\rceil, \quad (8)$$

$$n > \ell + r(k, \ell) - 1 \Rightarrow m'(n, \ell, \ell, k) > r(k, \ell) \Rightarrow$$

$$\Rightarrow \text{any } M'(n, \ell, \ell, k) \text{ is a } \Delta\text{-system} \Rightarrow m'(n, \ell, \ell, k) = n - \ell + 1. \quad (8')$$

For  $\ell = 1$  and infinitely many  $k$  (7) is best possible. We obtain from [1], [2] and [7] that

$$k^2 - k + 1 \geq \max(\ell + 2, (k - \ell)^2 + k - \ell + 1) \geq r(k, \ell) \geq \max(\ell + 2, q^2 + q + 1), \quad (9)$$

where  $q = \max q^*$ , such that  $q^* \leq k - \ell$  and  $PG(2, q^*)$  exists. The function  $r(k, \ell)$  and several generalizations of it were considered in detail in [3].

In this paper we consider the case  $\ell_1 < \ell_2$ . From now on we assume  $\ell_1 < \ell_2$ . It is evident that

$$m(n, \ell_1, \ell_2, k) \geq m(n - \ell_1, 0, \ell_2 - \ell_1, k - \ell_1), \quad (10)$$

$$m'(n, \ell_1, \ell_2, k) \geq m(n - \ell_1, 0, \ell_2 - \ell_1, k - \ell_1), \quad (10')$$

since for example if  $\alpha = \{A_i\} = M(n - \ell_1, 0, \ell_2 - \ell_1, k - \ell_1)$

and  $|\Lambda| = \ell_1$ ,  $\Lambda \cap (\cup_{A_i \in \alpha} A_i) = \emptyset$  then

$$|\{A_i \cup \Lambda\}| \leq m(n, \ell_1, \ell_2, k).$$

Deza and Erdos proved the following ~~Theorem~~ <sup>Theorem</sup> (this is inversion of (10), (10') and generalization of Theorem 2).

Theorem 3 ([4])

Let  $0 < \ell_1 < \ell_2 < k < n$ . There are  $s(k)$  and  $s'(k)$ , such that

$$m(n, \ell_1, \ell_2, k) > \frac{\ell_2^2 - \ell_2 + 1}{k} n + s(k) \Rightarrow \text{any } M(n, \ell_1, \ell_2, k) \text{ is an } \ell_1\text{-star} \Rightarrow$$

$$\Rightarrow m(n, \ell_1, \ell_2, k) = \max \left( \frac{\ell_2^2 - \ell_2 + 1}{k} n + s(k), m(n - \ell_1, 0, \ell_2 - \ell_1, k - \ell_1) \right), \quad (11)$$

$$m'(n, \ell_1, \ell_2, k) > (\ell_2^2 - \ell_2 + 1)n + s'(k) \Rightarrow \text{any } M'(n, \ell_1, \ell_2, k) \text{ is an } \ell\text{-star} \Rightarrow$$

$$\Rightarrow m'(n, \ell_1, \ell_2, k) = \max \left( (\ell_2^2 - \ell_2 + 1)n + s'(k), m'(n - \ell_1, 0, \ell_2 - \ell_1, k - \ell_1) \right). \quad (11')$$

Assume now  $\ell_1 = 0, \ell_2 = \ell > 0$ .

Theorem 4. Let  $0 < \ell < k < n$ . Then

$$m(n, 0, \ell, k) = \binom{n}{2} \quad \text{for } k = 2, \quad (12)$$

$$m(n, 0, \ell, k) \leq \left\lceil \frac{n^2}{k} \right\rceil \quad \text{for } k > 2,$$

$$m(n, 0, \ell, k) \leq \left\lceil \frac{n}{k} \left\lceil \frac{n - \ell}{k - \ell} \right\rceil \right\rceil \quad \text{for } n > \ell + r(k, \ell)(k - \ell), \quad (13)$$

$$m(n, 0, \ell, k) = \frac{n(n - \ell)}{k(k - \ell)} \quad \text{for the case } \ell | k \text{ and}$$

$$n > f_0(k, \ell), \quad \ell | n, \quad \frac{k}{\ell} - 1 \mid \frac{n}{\ell} - 1, \quad \frac{k}{\ell} \left( \frac{k}{\ell} - 1 \right) \mid \frac{n}{\ell} \left( \frac{n}{\ell} - 1 \right);$$

$$m(n, 0, \ell, k) \leq n \quad \text{if } \ell \nmid k, \quad (14)$$

$m(n, 0, \ell, k) = n$  ~~for~~  $v \mid n$  where  $v$  is an integer, such that ~~there exists a~~ <sup>and</sup>  $(v, k, \ell)$ -design.

In fact, equality (12) is trivial, because  $m(n,0,\ell, 2) = m(n,0,1,2) \leq |\{A_i: |A_i| = 2\}| = \binom{n}{2}$ . It is easy to see that  $M(n^*,0,1,k^*)$  is a *pairwise balanced design* PBD $[k^*, n^*]$ . R.M. Wilson proved in [9] that a PBD  $[k^*,n^*]$  exists if  $n^* > f_0(k^*)$ ,  $k^*|n^*$ ,  $k^*(k^*-1) | n^*(n^*-1)$ . In this case, we have  $m(n^*, 0, 1, k^*) = \frac{n^*(n^*-1)}{k^*(k^*-1)}$ . Now we take a  $\ell$ -multiple of PBD  $[k^*, n^*]$  and put  $n = \ell n^*$ ,  $k = \ell k^*$ . We obtain

$$m(n,0,\ell,k) \geq m(n^*,0,1,k^*) = \frac{\frac{n}{\ell}(\frac{n}{\ell} - 1)}{\frac{k}{\ell}(\frac{k}{\ell} - 1)} = \frac{n(n-\ell)}{k(k-\ell)}$$

for  $n^* = \frac{n}{\ell} > f_0(k^*)$ , i.e.  $n > \ell f_0(k/\ell)$ . If also  $n > \ell + r(k,\ell)(k-\ell)$  holds then we have equality in (13). We obtain second inequality (14) by taking  $n/v$   $(v,k,\ell)$ -designs  $\alpha_j = \{A_{ij}\}$ ,  $1 \leq j \leq n/v$ , such that

$$(\cup_{A_{ij_1} \in \alpha_{j_1}} A_{ij_1}) \cap (\cup_{A_{ij_2} \in \alpha_{j_2}} A_{ij_2}) = \emptyset \text{ for } 1 \leq j_1 < j_2 \leq n/v ;$$

It is evident that  $m(n,0,\ell,k) \geq |\alpha_1| n/v = n$ . Now we will prove <sup>the</sup> upper bounds (12), (13), (14). Let any  $M(n,0,\ell,k) = \alpha = \{A_i\}$  be given. We have

$$|\alpha|k \leq n m(n,\ell,\ell,k) \text{ and so } |\alpha| \leq \left\lceil \frac{m(n,\ell,\ell,k)n}{k} \right\rceil . \tag{15}$$

Now inequality (12) follows from (15) and (6) of Theorem 1; inequality (13) follows from (15) and (8) of Theorem 2.

To prove (14), assume that there exists  $M(n,0,\ell,k) = \{A_1 A_2, \dots, A_b\}$ ,  $b > n$ . Let  $\cup_{i=1}^b A_i = \{x_1, x_2, \dots, x_n\}$ .



Define  $n \times b$  incidence matrix  $N$  as follows:

$$N = (n_{ij}) \text{ where } b_{ij} = \begin{cases} 1 & \text{if } x_i \in A_j \\ 0 & \text{if } x_i \notin A_j \end{cases} .$$

Clearly,  $N^T N = (b_{ij})$ , where

$$b_{ij} = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } |A_i \cap A_j| = 0 \\ \ell & \text{if } |A_i \cap A_j| = \ell \end{cases} .$$

Since  $N$  is  $n \times b$  matrix and  $b > n$ ,  $N^T N$  is singular. Hence there exists a rational vector  $(y_1, y_2, \dots, y_b)^T$  such that

$$N^T N (y_1, y_2, \dots, y_b)^T = 0 . \quad (16)$$

Now by choosing  $(y_1, y_2, \dots, y_b)$  suitably we can assume that  $y_1, y_2, \dots, y_b$  are integers and if  $y_{i_1}, y_{i_2}, \dots, y_{i_r}$  are the nonzero integers among these, then

$$\text{g.c.d. } (y_{i_1}, y_{i_2}, \dots, y_{i_r}) = 1 .$$

Now from (16) we have  $ky_i + \ell(\sum y_j) = 0$ ,  $i = 1, 2, \dots, b$  (17)  
 where terms in the sum  $\sum y_j$  are those for which  $b_{ij} = \ell$ .

Hence from (17),  $\ell | ky_i$  for each  $i$ , in particular,  
 $\ell | k y_j$ ,  $j = 1, 2, \dots, r$ . Since  $\text{g.c.d. } (y_{i_1}, y_{i_2}, \dots, y_{i_r}) = 1$  we have  
 a contradiction and so  $\ell | k$ .

Theorem 5. Let  $0 < \ell < k < n$ . Then

$$m'(n, 0, \ell, k) = \binom{n}{2} + n + 1 \quad \text{for } \ell = 1, \quad (18)$$

$$m'(n, 0, \ell, k) = 9 < \binom{n}{2} + n + 1 \quad \text{for } n = 4, k = 3, \ell = 2,$$

and  $m'(n, 0, \ell, k) \leq \left\lfloor \frac{n(n+1)}{\ell+1} \right\rfloor + n + 1 < \binom{n}{2} + n + 1$  otherwise;

$$m'(n, 0, \ell, k) \leq \left\lfloor \frac{n(n-\ell+1)}{\ell+1} \right\rfloor + n + 1 \quad \text{for } n > \ell + r(k, \ell) - 1. \quad (19)$$

In fact, the proof is analogous to the proof of Theorem 4. But instead of (15) we have  $|\alpha| \leq \left\lfloor \frac{nm'(n, \ell, \ell, k)}{\ell+1} \right\rfloor + n + 1$  for (15')  
 $M'(n, 0, \ell, k) = \alpha = \{A_i\}$  because denoting  $\alpha^* = \{A_i \in \alpha : |A_i| \geq \ell + 1\}$ , we obtain

$$|\alpha^*| (\ell+1) \leq nm'(n, \ell, \ell, k),$$

$$|\alpha^*| \geq |\alpha| - m'(n, 0, 0, \ell).$$

Now we return to the general case.

Theorem 6. Let  $0 \leq \ell_1 < \ell_2 < k \leq n$ . Then

$$m(n, \ell_1, \ell_2, k) \leq \binom{n-\ell_1}{2} \text{ for } n \leq k + \sqrt{k^2 + 2s(k, \ell)} , \quad (20)$$

$$m'(n, \ell_1, \ell_2, k) \leq \binom{n-\ell_1}{2} + (n-\ell_1) + 1 \text{ for } n \leq (\ell_2^2 - \ell_2 + 1) + \sqrt{(\ell_2^2 - \ell_2 + 1)^2 2s'(k, \ell)} \quad (20')$$

$$m(n, \ell_1, \ell_2, k) \leq \left[ \frac{\binom{n-\ell_1}{k-\ell_1}}{\binom{n-\ell_2}{k-\ell_2}} \right] \text{ for } n \geq n_0(k, \ell) , \quad (21)$$

$$m'(n, \ell_1, \ell_2, k) \leq \left[ \frac{\binom{n-\ell_1}{\ell_2 - \ell_1 + 1}}{\binom{n-\ell_2}{\ell_2 - \ell_1 + 1}} \right] + (n-\ell_1) + 1 \text{ for } n \geq n_0(k, \ell) ; \quad (21')$$

$$m(n, \ell_1, \ell_2, k) \leq n \text{ for } \ell_2 - \ell_1 \mid k - \ell_1, \quad n \geq n_0(k, \ell) . \quad (22)$$

In fact, (20), (21), (22) follow from Theorem 3 and Theorem 4, applied to the case  $m(n-\ell_1, \ell_1-\ell_1, \ell_2-\ell_1, k-\ell_1)$ . Similarly, we obtain (20'), (21').

This paper was initiated by the following problem of R.Lemmon communicated to P.Erdos by A.Stone:

Estimate  $f(m, \ell, k) = \min \left| \bigcup_{i=1}^m A_i \right|$  if there exists a family  $A_1, A_2, \dots, A_m$  such that  $|A_i| = k (1 \leq i \leq m)$ ,  $|A_i \cap A_j| = \ell (1 \leq i < j \leq m)$ . A.Stone and R.Lemmon considered  $f(m, \ell, k)$  for small  $n$ ; it is easy to show that  $f(m, \ell, k) \geq mk - \ell \binom{m}{2}$  with equality for  $m = k/\ell + 1$ , if  $\ell \mid k$ .

The following problems are still open:

1) Does  $m(n, \ell_1, \ell_2, k) \leq \binom{n}{2}$  hold for  $\ell_1 > 0$  and all  $n$  (not only for the case  $n \geq n_0(k)$  as in Theorem 6) ? This is a conjecture of Erdos and Lovasz ;

2) Does a maximal system  $\alpha = \{A_i\}$  of subsets of an  $n$ -set such that

$$|A_i| = k \quad (\forall A_i \in \alpha), \quad (A_i \cap A_j) = \emptyset, \quad \ell_2, \ell_3 \quad (\forall A_i, A_j \in \alpha, \quad i \neq j)$$

contain at most  $\binom{n}{3}$  sets? Also, it would be interesting to find analog of equality (13) for this case.

3) Find an analog of (14) for  $m'(n, 0, \ell, k)$  ; we proved only  $m'(n, 0, \ell, k) \leq n$  for  $\ell > k/2$  .

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