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EMBEDDING THEOREMS FOR GRAPHS ESTABLISHING NEGATIVE PARTITION RELATIONS

by

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§ 0. Introduction. Notation

We started to work on this paper with the following observation. If $\mathcal{G}$ is a graph on $\aleph_1$ vertices and establishing the negative partition relation $\aleph_1 \nrightarrow ([\aleph_0, \aleph_1])^2_2$ then $\mathcal{G}$ is universal for countable graphs. This last statement means that every countable graph $\mathcal{H}$ is isomorphic to a spanned subgraph of $\mathcal{G}$.

We have stated a number of results and problems of the above type in our paper [4] written in 1971. (See Problems VIII, IX, X on pp. 285–286.) Quite a few of these problems will be solved or modified by the results of this paper.

S. Shelah has made an important remark concerning our problem. He has shown that our starting result cannot be generalized for higher cardinals.

THEOREM (S. SHELAH [8], Th. 4.1, p. 11). *Let M be a countable trasitive model of $ZFC + GCH$. Assume $M \models$ "$\varkappa, \lambda, \tau$ are cardinals, each of cofinality greater than ω, $\mathcal{G}$ is a graph establishing $\varkappa \nrightarrow ([\lambda, \tau])^2_2$". Let N be the model obtained from M by adding one Cohen real. Then $N \models$ "$\mathcal{G}$ establishes $\varkappa \nrightarrow ([\lambda, \tau])^2_2$ and there is a graph $\mathcal{H}$ on $\aleph_1$ vertices isomorphic to no spanned subgraph of $\mathcal{G}$".*

This result explains why most of this paper deals with embedding *countable* graphs into graphs of cardinality $\aleph_1$. We will discuss the problems left open by Shelah's result at the end of § 6. In what follows we work in ZFC. Set theoretic notation will be standard. In particular, ordinals are identified with the set of their predecessors, and cardinals with their initial numbers. All Greek lower case letters but φ, ψ, denote ordinals, $\varkappa$, λ always denote cardinals, i, j, n, m denote non-negative integers.

We use the well-known partition relations, the "ordinary partition relation" the "polarized partition relation" and the "square bracket partition relation" as defined e.g. in [3].

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We collect the main graph theory notation below:

DEFINITION 0.1.

(a) A *graph* $\mathcal{G}$ is an ordered pair (g, G) where $G \subset [g]^2$. g is the set of vertices of $\mathcal{G}$, G is the set of edges of $\mathcal{G}$.
(b) Where $\mathcal{G} = (g, G)$, $\mathcal{H} = (h, H)$ are graphs, $\mathcal{H}$ is a *subgraph* of $\mathcal{G}$ if $h \subseteq g$ and $H \subseteq G$. We denote this by $\mathcal{H} \subseteq \mathcal{G}$ if there is no danger of confusion.
(c) Where $\mathcal{G} = (g, G)$ is a graph, $(g, [g]^2 \setminus G)$ is the complement of $\mathcal{G}$ and is denoted by $\neg\mathcal{G}$.
(d) Where $\mathcal{G} = (g, G)$ is a graph, $h \subseteq g$, $(h, [h]^2 \cap G)$ is the *subgraph* of G *spanned by* h and will be denoted by $\mathcal{G}(h)$.
(e) A *complete* $\varkappa$*-graph* is a graph of the form $(g, [g]^2)$ where $|g| = \varkappa$. A *complete* $\varkappa, \lambda$*-bipartite graph* is a graph $\mathcal{G} = (g, G)$, where

$$g = g_0 \cup g_1,\ g_0 \cap g_1 = \emptyset,\ |g_0| = \varkappa, g_1 = \lambda,$$

$$G = [g_0, g_1]^{1,1} = \{\{x, y\} : x \in g_0 \wedge y \in g_1\}.$$

We often speak about "the" *complete* $\varkappa$*-graph* or $(\varkappa, \lambda)$*-bipartite graph* and we denote them by $[\varkappa]$, $[\varkappa, \lambda]$, respectively. Whenever we write $[\varkappa] \subseteq \mathcal{G}$ we mean that $\mathcal{G}$ contains a complete $\varkappa$-graph as a subgraph. We use this shorthand for other classes of graphs as well.

(f) Whenever $\mathcal{G} = (g, G)$ is a graph, $h \subseteq g$, $x \in g$, we put $\mathcal{G}(x, h) = \{y \in h : \{x, y\} \in G\}$. $\mathcal{G}(x, h)$ is the set of vertices of $\mathcal{G}$ lying in h and adjacent to x. $|\mathcal{G}(x, h)|$ is said to be the *valency* of *the vertex* x *in* $\mathcal{G}$ *for* h. We briefly write $\mathcal{G}(x)$ for $\mathcal{G}(x, g)$ if there is no danger of confusion.
(g) A *half* $(\varkappa, \varkappa)$*-bipartite graph* is a graph $G = (g, G)$ where $g = g_0 \cup g_1$, $g_0 \cap g_1 = \emptyset$, $|g_0| = |g_1| = \varkappa$ and there are one-to-one enumerations of g_0 and g_1, $g_0 = \{x_\alpha : \alpha < \varkappa\}$, $g_1 = \{y_\alpha : \alpha < \varkappa\}$ such that $\mathcal{G}(x_\alpha, g_1) = \{y_\beta : \alpha < \beta < \varkappa\}$ for $\alpha < \varkappa$. We denote "the half $(\varkappa, \varkappa)$-bipartite graph" by $[\varkappa/\varkappa]$.

DEFINITION 0.2. A mapping $f : [g]^2 \to \gamma$ is said to be a 2-partition of g with γ colors. There is a canonical isomorphism between the set of graphs with vertex set g and the set of 2-partitions of g with 2 colors given by the relation

$$G = \{\{x, y\} : f(\{x, y\}) = 0\}.$$

We will sometimes allow us the liberty to identify 2-partitions with 2 colors to the corresponding graph.

DEFINITION 0.3.

(a) Assume f_0, f_1 are 2-partitions of g_0 and g_1 with γ colors. We say that f_0 *embeds* into f_1 if there is a set $h \subseteq g_1$ such that f_0 is isomorphic to $f_1 \restriction h$. By this we mean that there is a one-to-one mapping φ of g_0 onto h such that

$$f_0(\{x, y\}) = f_1(\{\varphi(x), \varphi(y)\})$$

holds for all $\{x, y\} \in [g_0]^2$. If f_0 embeds into f_1 we write $f_0 \rightarrowtail f_1$.

As a special case of the above we state

(b) Assume $\mathcal{G}_0 = (g_0, G_0)$, $\mathcal{G}_1 = (g_0, G_1)$ are graphs. We say that $\mathcal{G}_0$ *embeds into* $\mathcal{G}_1$ if $\mathcal{G}_0$ is isomorphic to a spanned subgraph of $\mathcal{G}_1$. We write $\mathcal{G}_0 \rightarrowtail \mathcal{G}_1$ to denote this fact.

(c) We say that $\mathcal{G}_0$ weakly embeds into $\mathcal{G}_1$ if either $\mathcal{G}_0 \rightarrowtail \mathcal{G}_1$ or $\mathcal{G}_0 \rightarrowtail \neg\mathcal{G}_1$ We denote this fact by $\mathcal{G}_0 \overset{w}{\rightarrowtail} \mathcal{G}_1$.

DEFINITION 0.4. (Establishing negative partition relations).

(a) Assume $\mathcal{G} = (g, G)$ is a graph, and Δ_0, Δ_1 are either $[\varkappa]$ or $[\varkappa, \lambda]$ or $[\varkappa/\varkappa]$ for suitable cardinals $\varkappa, \lambda$. We say that $\mathcal{G}$ *establishes the negative partition relation* $\varkappa \nrightarrow (\Delta_0, \Delta_1)^2$ if $\Delta_0 \not\subseteq \mathcal{G}$, $\Delta_1 \not\subseteq \neg\mathcal{G}$ and $|g| = \varkappa$.

The reader at first may skip the following more general definition which we will only use to give hints to possible generalizations of the theorems we will state.

(b) Assume f is a 2-partition of g with γ colors. Assume further that Δ_ν: $\nu < \gamma$ are symbols like Δ_0, Δ_1 appearing in definition (a). We say that f *establishes the negative square bracket relation*

$$\varkappa \nrightarrow [\Delta_\nu]^2_{\nu<\gamma}$$

if $|g| = \varkappa$ and $\Delta_\nu \not\subseteq \mathcal{G}_\nu$ for $\nu < \gamma$ where $g_\nu = g$ and

$$G_\nu = \{\{x, y\} \in [g]^2 : f(\{x, y\}) \neq \nu\}.$$

Assume that Δ_0, Δ_1 are one of the following symbols:

$$[\aleph_1],\ [\aleph_1, \aleph_1],\ [\aleph_1/\aleph_1],\ [\aleph_0, \aleph_1].$$

Our main aim in this paper is to characterize the class of all countable graphs which embed in to all graphs $\mathcal{G}$ establishing $\aleph_1 \nrightarrow (\Delta_0, \Delta_1)^2$.

To have short notation we make the following definitions.

DEFINITION 0.5.

(a) $C_\alpha(\Delta_0, \Delta_1, \varkappa) = \{\mathcal{H}$: $\mathcal{H}$ is a graph of cardinality less than $\aleph_\alpha$ and $\mathcal{H} \rightarrowtail G$ for all graphs G establishing $\varkappa \nrightarrow (\Delta_0, \Delta_1)^2\}$.

(b) $Cw_\alpha(\Delta_0, \Delta_1, \varkappa) = \{\mathcal{H}: \mathcal{H}$ is a graph of cardinality less than $\aleph_\alpha$ and $\mathcal{H} \overset{w}{\rightarrowtail} \mathcal{G}$ for all graphs $\mathcal{G}$ establishing $\varkappa \nrightarrow (\Delta_0, \Delta_1)^2\}$.

(c) $\mathrm{Sub}_\alpha(\mathcal{G}) = \{\mathcal{H}: \mathcal{H}$ is a graph of cardinality less than $\aleph_\alpha$ and $\mathcal{H} \rightarrowtail \mathcal{G}\}$.

Since we have a special interest in the case $\alpha = 1$ we briefly write $C_\alpha(\Delta_0, \Delta_1, \varkappa) = C(\Delta_0, \Delta_1, \varkappa)$, $Cw_\alpha(\Delta_0, \Delta_1, \varkappa) = Cw(\Delta_0, \Delta_1, \varkappa)$ and $\mathrm{Sub}_\alpha(\mathcal{G}) = \mathrm{Sub}(\mathcal{G})$.

Clearly

$$C_\alpha(\Delta_0, \Delta_1, \varkappa) = \bigcap \{\mathrm{Sub}_\alpha(\mathcal{G}):\ \mathcal{G} \text{ establishes } \varkappa \nrightarrow (\Delta_0, \Delta_1)^2\},$$

$$Cw_\alpha(\Delta_0, \Delta_1, \varkappa) = \bigcap \{\mathrm{Sub}_\alpha(\mathcal{G}) \cup \mathrm{Sub}_\alpha(\neg\mathcal{G}):\ \mathcal{G} \text{ establishes } \varkappa \nrightarrow (\Delta_0, \Delta_1)^2\} \text{ and}$$

$$C_\alpha(\Delta_0, \Delta_1, \varkappa) \cup C_\alpha(\Delta_1, \Delta_0, \varkappa) \subseteq Cw_\alpha(\Delta_0, \Delta_1, \varkappa).$$

We will see that in the last line proper inclusion holds in some cases. We will concentrate our main efforts on the investigation of the classes $C_\alpha(\Delta_0, \Delta_1, \varkappa)$ since the classes Cw_α do not seem to lead to genuinely new problems.

To give some information we describe a typical situation in advance.

DEFINITION 0.6.

(a) We say that a graph $\mathcal{G} = (g, G)$ is a *Sierpiński graph* if there exist a well-ordering $\prec_0$ and an ordering $\prec_1$ of g such that

$$G = \{\{x, y\} : x \prec_0 y \Leftrightarrow x \prec_1 y\}.$$

(b) $\mathcal{G}$ is said to be an ω_1 *real-Sierpiński graph* if g consists of reals, $\prec_0$ is an ω_1-type well-ordering of g and $\prec_1$ is the natural ordering.

It is a well-known fact that all $\omega_1$real-Sierpinski graphs establish $\aleph_1 \nrightarrow ([\aleph_1, \aleph_1])^2$. This is the strongest relation of type $\aleph_1 \nrightarrow (\Delta_0, \Delta_1)^2$ known to be true in *ZFC*. In § 2, Theorem 2.1 we are going to prove that

$$C([\aleph_1, \aleph_1], [\aleph_1, \aleph_1], \aleph_1) = \text{The class of countable Sierpiński graphs} = \mathrm{Sub}(\mathcal{G}), \text{ for all } \omega_1 \text{ real-Sierpiński graphs } \mathcal{G}.$$

Since $\mathrm{Sub}(\mathcal{G}) = \mathrm{Sub}(\neg\mathcal{G})$ holds for all ω_1 real-Sierpiński graphs, the above class is equal to $Cw([\aleph_1, \aleph_1], [\aleph_1, \aleph_1], \aleph_1)$ as well. The theorem mentioned gives a positive solution of Problem X. (1) of [4].

In fact there are only two more graphs $\mathcal{G}$ known to establish a partition relation of type $\aleph_1 \nrightarrow (\Delta_0, \Delta_1)^2$ and nice enough to restrict $\mathrm{Sub}(\mathcal{G})$.

One of them is "the Souslin tree" (see 3.1) which exists under the assumption that Souslin's hypothesis is false, and the other "the Shelah graph" constructed by Shelah in [8] under the assumption that CH holds (see Def. 5.4).

In the main line of the paper we will prove embedding theorems in *ZFC*, and we will try to show them to be "best possible" assuming CH or the existence of a Souslin tree or both. However we will prove embedding theorems assuming Souslin's hypothesis and Martin's axiom as well.

In §§ 1–6 we treat the cases $C(\Delta_0, \Delta_1, \aleph_1)$. We will summarize the results and problems concerning to the different special cases in the respective chapters. In § 7 we summarize all the results concerning the embedding of finite graphs. In § 8 we give some results concerning weak embeddings and correct the result (4) (a) on p. 286 of [4] which was incorrectly stated there.

At the end of the paper we give a list of special notation we use for graphs and classes of graphs.

§ 1. The case $\Delta_0 = [\aleph_1/\aleph_1]$, $\Delta_1 = [\aleph_1, \aleph_1]$

THEOREM 1.1. *Assume $\varkappa > \omega$ is a regular cardinal. Then $C([\varkappa/\varkappa], [\varkappa, \varkappa]; \varkappa)$ is the class of all countable graphs, hence*

$$C([\varkappa/\varkappa], [\varkappa, \varkappa], \varkappa) = C([\varkappa, \varkappa], [\varkappa/\varkappa], \varkappa)$$

holds as well.

COROLLARY 1.2. $C([\aleph_1/\aleph_1], [\aleph_1, \aleph_1], \aleph_1]$ *is the class of all countable graphs; hence*

$$C([\aleph_1/\aleph_1], [\aleph_1, \aleph_1], \aleph_1) = C([\aleph_1, \aleph_1], [\aleph_1/\aleph_1], \aleph_1)$$

holds as well.

This is a stronger result then the one we claimed in [4] (p. 285).

First we should mention that if CH is true then Corollary 1.2 does not hold vacuously because of

PROPOSITION 1.3 ([5], Theorem 17/A). *Assume $\varkappa > \omega$ and $2^\varkappa = \varkappa^+$. Then $\varkappa^+ \nrightarrow [[\varkappa, \varkappa^+]]^2_{\varkappa^+}$.*

On the other hand to the best of our knowledge it is not known if $\aleph_1 \to ([\aleph_1/\aleph_1], [\aleph_1, \aleph_1])^2$ is consistent with *ZFC*. Nor doe we think that $\varkappa \nrightarrow ([\varkappa/\varkappa], [\varkappa, \varkappa])^2$ has been proved in *ZFC* for any $\varkappa > \omega$.

The following generalization of Theorem 1 is true as well.

Assume $\varkappa > \omega$ is a regular cardinal, $2 \leq n < \omega$. Assume further that $f\colon [\varkappa]^2 \to n$ establishes

$$\varkappa \nrightarrow [[\varkappa, \varkappa], [\varkappa/\varkappa]_{n-1}]^2.$$

Then $h \rightarrowtail f$ holds for all 2-partitions h with n colors of a countable set.

We omit the proof of this.

For the proof of Theorem 1.1 we will need a sequence of easy lemmas.

DEFINITION 1.1. Assume $\mathcal{G} = (g, G)$ is a graph, $\varkappa$ a cardinal, $A, B \subseteq g$. The pair (A, B) is said to be $\varkappa$-good for $\mathcal{G}$ if for all $A' \in [A]^\varkappa$, $B' \in [B]^\varkappa$ there is an $x \in A'$ such that $|\mathcal{G}(x, B')| = \varkappa$.

LEMMA 1.3. *Assume $\mathcal{G} = (g, G)$ is a graph, $cf(\varkappa) > \omega$, $|g| = \varkappa$. Assume further that*

$$\forall C \in [g]^\varkappa \, \exists A, B \in [C]^\varkappa \, ((A, B) \text{ is } \varkappa\text{-good for both } \mathcal{G} \text{ and } \neg\mathcal{G}).$$

Then $\mathcal{H} \rightarrowtail \mathcal{G}$ holds for all countable graphs $\mathcal{H}$.

PROOF. Let us first remark that if (A, B) is $\varkappa$-good for $\mathcal{G}$, $A' \in [A]^\varkappa$ and $B' \in [B]^\varkappa$ then (A', B') is $\varkappa$-good for $\mathcal{G}$ as well.

Define sequences A_n, B_n of subsets of g by induction on $n \in \omega$ as follows. $A_0 = g$. Assume $A_n \in [g]^\varkappa$ has already been defined. Pick disjoint subsets $B_n, A_{n+1} \in [A_n]^\varkappa$ such that (B_n, A_{n+1}) is $\varkappa$-good for both $\mathcal{G}$ and $\neg\mathcal{G}$. Then $B_n \in [g]^\varkappa$, the B_n are pairwise disjoint and (B_n, B_m) is $\varkappa$-good for $\mathcal{G}$ and $\neg\mathcal{G}$ for all $n < m \, (< \omega)$. We may now assume that $\mathcal{H} = (\omega, H)$. Claim: There is an $x_0 (\in B_0)$ and a sequence $B_n' (\in [B_n]^\varkappa)$ for $1 \leq n < \omega$ such that $B_n' \subseteq \mathcal{G}(x_0, B_n)$ if $\{0, n\} \in H$ and $B_n' \subseteq \neg G(x_0, B_n)$ if $\{0, n\} \notin H$.

If this is not true then, because of $cf(\varkappa) > \omega$, there are an n $(1 \leq n < \omega)$ and a subset $A' \in [B_0]^\varkappa$ such that either $|\mathcal{G}(x, B_n)| < \varkappa$ for all $x \in A'$, or $|\mathcal{G}(x, B_n)| < \varkappa$ for all $x \in A'$. This contradicts the fact that (B_0, B_n) is $\varkappa$-good for $\mathcal{G}$ and $\neg\mathcal{G}$.

By repeating this procedure we can obtain a sequence $x_n \in B_n$ such that $\{x_n, x_m\} \in G$ iff $\{n, m\} \in H$.

Hence $\mathcal{H} \rightarrowtail \mathcal{G}$.

LEMMA 1.4. *Assume $\mathcal{G} = (g, G)$ is a graph, $\varkappa > \omega$, $\varkappa$ is regular. Assume further that $[\varkappa, \varkappa] \nsubseteq \neg\mathcal{G}$. Then*

$$\forall A_0 B_0 \in [g]^\varkappa \, \exists A \in [A_0]^\varkappa \, \exists B \in [B_0]^\varkappa$$

either (A, B) is $\varkappa$-good for G or (B, A) is $\varkappa$-good for $\mathcal{G}$.

PROOF. If (A_0, B_0) is $\varkappa$-good for $\mathcal{G}$ then we are home. We may assume that the are $A_1 \in [A_0]^\varkappa$, $B_1 \in [B_0]^\varkappa$ such that $|\mathcal{G}(x, B_1)| < \varkappa$ for all $x \in A_1$.

If (B_1, A_1) is $\varkappa$-good for G then we are again done. We may assume that there are $A_2 \in [A_1]^\varkappa$ and $B_2 \in [B_1]^\varkappa$ such that $|\mathcal{G}(y, A_2)| < \varkappa$ for $y \in B_2$. Using the fact that if $\varkappa > \omega$ is regular and $\mathcal{H}$ is a graph all whose vertices have valency $< \varkappa$ then all components of $\mathcal{H}$ have cardinality $< \varkappa$, this would imply that there are $A_3 \in [A_2]^\varkappa$, $B_3 \in [B_2]^\varkappa$ such that $[A_3, B_3] \subseteq \neg\mathcal{G}$; a contradiction.

LEMMA 1.5. *Let* $\mathcal{G} = (g, G)$ *be a graph,* $\varkappa > \omega$ *a regular cardinal. Assume that* $[\varkappa/\varkappa] \nsubseteq \mathcal{G}$. *Then*

$$\forall A, B \in [g]^{\varkappa} \ ((A, B) \text{ is } \varkappa\text{-good for } \neg\mathcal{G}).$$

PROOF. Assume there are $A', B' \in [g]^{\varkappa}$ such that $|\neg\mathcal{G}(x, B')| < \varkappa$ for all $x \in A'$. Using the regularity of $\varkappa$ we can pick sequences $x_\alpha \in A'$, $y_\alpha \in B'$ so that x_α, y_α are all different and $y_\alpha \notin \bigcup \{\neg\mathcal{G}(x_\beta): \beta < \alpha\}$ holds for all $\alpha < \varkappa$. This in view of Definition 0.1 (g) would mean $[\varkappa/\varkappa] \subseteq \mathcal{G}$, a contradiction.

PROOF of Theorem 1.1. Assume $\mathcal{G} = (g, G)$ is a graph establishing $\varkappa \nrightarrow ([\varkappa/\varkappa], [\varkappa, \varkappa])^2$. Assume $C \in [g]^{\varkappa}$. By Lemma 1.4, there are $A, B (\in [C]^{\varkappa})$ such that (A, B) is $\varkappa$-good for $\mathcal{G}$. By Lemma 1.5, (A, B) is $\varkappa$-good for $\neg\mathcal{G}$ as well. Then, by Lemma 1.3, all countable graphs embed into $\mathcal{G}$.

§ 2. The case $\Delta_0 = \Delta_1 = [\aleph_1, \aleph_1]$

DEFINITION 2.1. Let S denote the class of all countable Sierpiński graphs.

THEOREM 2.1. *Assume* $\varkappa > \omega$ *is a regular cardinal. Then*

$$S \subseteq C([\varkappa, \varkappa], [\varkappa, \varkappa], \varkappa).$$

If in addition $\varkappa$ *is not weakly compact, then* $S = C([\varkappa, \varkappa], [\varkappa, \varkappa], \varkappa)$.

COROLLARY 2.2. $C([\aleph_1, \aleph_1], [\aleph_1, \aleph_1], \aleph_1) = Cw[\aleph_1, \aleph_1], [\aleph_1, \aleph_1], \aleph_1) = S =$ *= the class of all countable Sierpiński graph.*

The second part of Theorem 2 follows from the following observations. If $\varkappa \nrightarrow (\varkappa)_2^2$ then, by a theorem of HANF [7], there is an ordered set $(\varkappa, \prec_1)$ not containing well-ordered and reversely well-ordered subsets of cardinality $\varkappa$ and from

PROPOSITION 2.3. *Assume* $(\varkappa, \prec_1)$ *is an ordered set satisfying the above condition and let* $\mathcal{G} = (\varkappa, G)$ *be the Sierpiński graph obtained by choosing* $\prec_0$ *as the natural well-ordering of* $\varkappa$. *Then* $\mathcal{G}$ *establishes* $\varkappa \nrightarrow ([\varkappa, \varkappa])_2^2$.

PROOF. Assume $A, B \in [\varkappa]^{\varkappa}$. It is easy to see that because of the assumption imposed on $(\varkappa, \prec_1)$ there is an $x \in \varkappa$ such that both $A|\prec_1 x$ and $A|_1\succ x$ are of cardinality $\varkappa$.

Then we may assume that $B|_1\succ x$ has cardinality $\varkappa$. Obviously $A|\prec_1 x \prec_1 B|\succ_1 x$ and there are $y \in A|\prec_1 x$, $z \in B|_1\succ x$ satisfying either of the conditions $y \prec_0 z$, $z \prec_0 y$.

As to the generalization of the result for 2-partitions with more than two colors we could not even formulate a conjecture.

PROBLEM 1. Characterize the class of those 2-partitions h of countable sets with 3-colors for which $h \mapsto f$ for all f establishing $\aleph_1 \nrightarrow ([\aleph_1, \aleph_1])_3^2$.

We mention that if $g = 3$, $h(\{0, 1\}) = 0$, $h(\{0, 2\}) = 1$ $h(\{1, 2\}) = 2$, then h is a member of the above class.

Finally we remark that "most" countable graphs are *not* Sierpiński graphs. E.g., a pentagon without a diagonal is not a Sierpiński graph.

PROOF of Theorem 2.1. Let $\mathcal{G} = (g, G)$ be a graph establishing $\varkappa \nrightarrow ([\varkappa, \varkappa])_2^2$. We may assume that there is a countable graph $\mathcal{H}$ which does not embed into $\mathcal{G}$. Then, by Lemma 1.3, there is a $C \in [g]^\varkappa$ such that no pair (A, B) with $A, B \in [C]^\varkappa$ is $\varkappa$-good for both $\mathcal{G}$ and $\neg\mathcal{G}$. We may as well assume that $C = g$.

As a corollary of this, and Lemma 1.4 we may assume that for all $C \in [g]^\varkappa$ there are $A, B \in [C]^\varkappa$ such that

$$\text{(1)} \qquad A \cap B = \emptyset,\ \forall x \in A(|\neg\mathcal{G}(x, B)| < \varkappa) \text{ and } \forall y \in B(|\mathcal{G}(y, A)| < \varkappa).$$

Call such a pair $A, B(\in [g]^\varkappa)$ a *convenient* pair. Notice that if $A' \in [A]^\varkappa$, $B' \in [B]^\varkappa$ and (A, B) is convenient then so is (A', B'). Let now Q be the set of diadically rational numbers $r(\in(0, 1))$. We claim that for $r \in Q$ we can select a set $A_r \in [g]^\varkappa$ in such a way that for all $r, s \in Q$, $r < s$, the pair (A_r, A_s) is convenient.

We outline the construction. Applying (1) twice we can select sets $B_0, B_1, B_2 \in [g]^\varkappa$ such that (B_i, B_j) is convenient for $i < j < 3$. Call B_1 $A_{1/2}$. Repeat this procedure in both B_0 and B_2. Call the sets in the middle $A_{1/4}$, $A_{3/4}$ respectively and repeat the procedure in the remaining four sets, and so on. This procedure obviously leads to the required system.

Using that $\varkappa$ is regular, by an easy transfinite recursion we can define a subset $D(\in [g]^\varkappa)$ and a well ordering $\prec$ of type $\varkappa$ of D satisfying the following conditions:

$$A'_r = A_r \cap D \in [D]^\varkappa \quad \text{for} \quad r \in Q \quad \text{and}$$

for all $x \in A'_r$, $y \in A'_s$, $r < s$; $r, s \in Q$ we have

$$x \prec y \Leftrightarrow \{x, y\} \in G.$$

Let now $\mathcal{H}$ be a countable Sierpiński graph. We may assume that $\mathcal{H} = (\alpha, H)$ for some $\alpha < \omega_1$ and there is a one-to-one mapping φ of α into Q such that for all $\beta < \gamma < \alpha$

$$\{\beta, \gamma\} \in H \Leftrightarrow \varphi(\beta) < \varphi(\gamma).$$

Using that $cf(\varkappa) > \omega$, we can select a sequence x_β $(\beta < \varkappa)$ in such a way that $x_\beta \in A'_{\varphi(\beta)}$ and $x_\gamma \prec x_\beta$ holds for all $\gamma < \beta < \alpha$.

Then $\{x_\gamma, x_\beta\} \in \mathcal{G} \Leftrightarrow \varphi(\beta) < \varphi(\gamma) \Leftrightarrow \{\beta, \gamma\} \in H$ for all β, γ $(< \alpha)$. Hence $\mathcal{H} \rightarrowtail \mathcal{G}$.

§ 3. Preliminaries. A characterization of Sierpiński trees

First we recall some well-known definitions.

Definition 3.1. A poset $(T, \prec)$ is a tree if the set $T|\prec x = \{y \in : y \prec x\}$ is well-ordered by $\prec$ for all $x \in T$. tp $x(\prec)$ is called the height of x in T. It T is a fixed tree, we will denote $T|\prec x$ by $\hat{x}$, $T|\succeq x$ by $\check{x}$ and tp $x(\prec)$ by $\mathrm{ht}(x)$. The α-th level of T is the set $T_\alpha = \{x \in T\colon \mathrm{ht}(x) = \alpha\}$. The length of T is the smallest ordinal α for which $T_\alpha = \emptyset$.

A chain of a tree is a subset $X(\subseteq T)$ ordered by $\prec$. A branch of a tree is a chain of A such that $y \prec x \in A$ implies $y \in A$. An antichain of a tree is a subset $X(\subseteq T)$ such that no two different elements of X are comparable in T.

A Souslin tree is a tree of length ω_1 in which every chain and antichain is countable. Souslin's hypothesis = SH says that there are no Souslin trees. Results of Jensen say that $\neg$SH holds in L and that SH is consistent with ZFC ZFC + $2^{\aleph_0} = \aleph_1$. (See [1].)

Definition 3.2. Given a tree $(T, \prec)$, we denote by $\mathcal{G}_T$ the graph with vertex set T, and whose edges are the pairs $\{x, y\}$ for which x and y are comparable in T. We call $\mathcal{G}_T$ the projection of $(T, \prec)$. We say that $\mathcal{G} = (g, G)$ is a *p-tree* if it is the projection of a tree. We use the expression *p-tree* because the word "tree" is used for graphs in a different sense.

The reader will easily verify the following well known fact:

Proposition 3.1. *If $(T, \prec)$ is a Souslin-tree, then $\mathcal{G}_T$ establishes $\aleph_1 \nrightarrow ([\aleph_1/\aleph_1], \aleph_1)^2$.*

This is best possible since if $(T, \prec)$ is a Souslin tree then $[\aleph_0, \aleph_1] \subseteq \mathcal{G}_T$ and $[\aleph_1, \aleph_1] \subseteq \neg\mathcal{G}_T$.

Definition 3.3. From now on we denote by $\mathfrak{T}$ the class of countable p-trees.

For every ordinal α, $\mathfrak{T}_\alpha$ is the class of graphs which are projections of trees of length $\leq \alpha$.

The elements of $S \cap \mathfrak{T}$ will be called countable *Sierpiński trees*. As a corollary of 3.1 we have

PROPOSITION 3.2. *Assume* $\neg$SH. *Then*

(a) $C([\aleph_1/\aleph_1], [\aleph_1], \aleph_1) \subseteq \mathfrak{S}$

(b) $C([\aleph_1, \aleph_1], [\aleph_1], \aleph_1) \subseteq \mathfrak{s} \cap \mathfrak{S}$.

We do not know if equality holds in any of the above cases. We will prove partial results in both cases.

First we give a characterization of Sierpiński trees, which is of some independent interest.

DEFINITION 3.4.

(a) Assume Z is a class of trees. Let Z' be the class of all trees $(T, \prec)$ which contain a countable branch B such that, for all $y \in T \setminus B$, $\dot{y} = \{x \in T : y \preceq x\}$ is a subtree of T which belongs to Z.

(b) Define a transfinite sequence of classes of trees as follows. Z_0 consists of the empty tree and of the one point tree. Assume $\varrho > 0$. Let $Z_\varrho = Z'_\tau$ for $\tau + 1 = \varrho$ and let $Z_\varrho = \bigcup\limits_{\tau<\varrho} Z_\tau$ for limit ϱ.

(c) We say that the tree $(T, \prec)$ embeds into the Sierpiński graph $G = (g, G)$ determined by $\prec_0$ and $\prec_1$ if

$$x \prec y \Leftrightarrow (x \prec_0 y \wedge x \prec_1 y)$$

holds for all $x, y \in g$.

(d) We denote by S_ϱ the class of graphs which are projections of elements of Z_ϱ.

(e) We denote by T_0 the complete binary ω-tree. T_0 is a tree of length ω with a smallest element called root and such that each element of T_0 has exactly two immediate successors.

THEOREM 3.3.

(a) *Assume* $\varrho < \omega_1$ *and* $(T, \prec) \in Z_\varrho$. *Then* $(T, \prec)$ *embeds into a Sierpiński graph.*

(b) *Assume that the projection of the countable tree* $(T, \prec)$ *is a Sierpiński graph. Then* $(T, \prec) \in Z_\varrho$ *for some* $\varrho < \omega_1$. *As a corollary of these* $\mathfrak{T} \cap S = \bigcup\limits_{\varrho<\omega_1} S_\varrho$.

First we prove

LEMMA 3.4. *The projection of* T_0 *is not a Sierpiński graph.*

PROOF. Considering that a confinal subset of T_0 contains a tree isomorphic to T_0, if the statement is not true then the tree T_0 embeds into a Sierpiński graph. Let G_0 be such a Sierpiński graph with the smallest well-ordering. Let $\mathcal{G}_0 = (\alpha, G_0)$, $\prec_1$ is an ordering of α, $\prec_2$ a tree ordering on α isomorphic to T_0, and such that

$$\beta \prec_2 \gamma \Leftrightarrow (\beta \prec \gamma \wedge \beta \prec_1 \gamma) \qquad \text{for} \quad \beta, \gamma < \alpha.$$

Assume β, γ are two incomparable elements of α in the tree. $\check{\beta} = \{\delta \in \alpha : \beta \preceq_2 \delta\}$, $\check{\gamma} = \{\delta \in \alpha : \gamma \preceq_2 \delta\}$. Then $\check{\beta}, \check{\gamma}$ are isomorphic to T_0 as well. We may assume e.g. $\beta \prec_1 \gamma$.

Then $\beta \prec_1 \gamma \preceq_1 \delta$ for $\delta \in \check{\gamma}$, hence $\prec_2$ being a tree, $\delta < \beta$ holds for $\delta \in \gamma$. Then $\check{\gamma} \subseteq \beta < \alpha$ a contradiction to the minimality of α.

PROOF of Theorem 3.3.

(a) Let $\mathfrak{T}_S$ denote the class of countable trees which embed into Sierpiński graphs. It is obviously sufficient to see that $Z_\varrho \subseteq \mathfrak{T}_S$ for $\varrho < \omega_1$. Assume $Z \subseteq \mathfrak{T}_S$. Let $(T, \prec) \in Z'$. Let $B \subseteq T$ be a countable branch such that $(\check{y}, \prec) \in Z$ for $y \in T \setminus B$. Let $C = \{y \in T\colon y$ is a minimal element of $T \setminus B\}$. Let Obviously $T = B \cup \bigcup \{\check{y} : y \in C\}$ where the summands are disjoint. By the assumption, there are well-orderings $\prec_y^0$ and orderings $\prec_y^1$ for $y (\in C)$ such that

$$u \prec v \Leftrightarrow (u \prec_y^0 v \wedge u \prec_y^1 v) \quad \text{for} \quad u, v \in \check{y},\ \check{y} \in C.$$

For $y \in C$ there is a section B_y of B such that $B_y = \{u \in B : u \prec y\}$.

Define the orderings $\prec_0, \prec_1$ of T by the following stipulations:

$$u \prec_0 v \Leftrightarrow u \prec_y^0 v \quad \text{for} \quad u, v \in \check{y};\ \check{y} \in C$$

$$u \prec_0 v \Leftrightarrow u \in B_y \quad \text{for} \quad u \in B \wedge u \in \check{y};\ y \in C.$$

Choose an auxiliary well-ordering $\prec_2$ of C and put

$$u \prec_0 v \Leftrightarrow (B_y \subset B_z \wedge (B_y = B_z \wedge y \prec_2 z)) \quad \text{for} \quad u \in \check{y},\ v \in \check{z};\ y \neq z \in C.$$

Clearly $\prec_0$ is a well-ordering.

$$u \prec_1 v \Leftrightarrow u \prec_y^1 c \quad \text{for} \quad u, v \in \check{y},\ \check{y} \in C$$

$$u \prec_1 v \quad \text{for} \quad u \in B \wedge v \in \check{y};\ y \in C$$

$$u \prec_1 v \Leftrightarrow v \prec_0 u \quad \text{for} \quad u \in \check{y},\ v \in z,\ y \neq z \in C.$$

It is easy to check that $\prec_1$ is an ordering of T and

$$u \prec v \Leftrightarrow (u \prec_0 v \wedge u \prec_1 v) \quad \text{holds for} \quad u, v \in T.$$

Hence $Z' \subseteq \mathfrak{T}$. $Z_\varrho \subseteq \mathfrak{T}_S$ follows by transfinite recursion on $\varrho < \omega_1$.

(b) By (a) $\bigcup_{\varrho<\omega_1} Z_\varrho \subseteq \mathfrak{T}_S$, and so $\bigcup_{\varrho<\omega_1} S_\varrho \subseteq S \cap \mathfrak{T}$.

We claim that if $(T, \prec)$ is a countable tree and $(T \prec) \notin \bigcup_{\varrho<\omega_1} Z_\varrho$ then $(T, \prec)$ contains T_0. First we prove that every such tree contains two incomparable elements x, y with $\check{x}, \check{y} \notin \bigcup_{\varrho<\omega_1} Z_\varrho$. If this is not the case then let $B = \{x \in T\colon$

$(\check{x}, \prec) \notin \bigcup_{\varrho<\omega_1} Z_\varrho\}$. By the indirect assumption, B is a countable branch of T, and $(\check{y}, \prec) \in \bigcup_{\varrho<\omega_1} Z_\varrho$ for $y(\in T\setminus B)$. But then $(T, \prec) \in \bigcup_{\varrho<\omega_1} Z_\varrho$, a contradiction.

This proves our last claim. By a repeated application of this it follows that all such trees contain T_0 indeed.

Assume now that the projection of $(T, \prec)$ is a Sierpiński tree. By the claim just proved and by Lemma 3.4 $(T, \prec) \in \bigcup_{\varrho<\omega_1} Z_\varrho$ This implies $\mathfrak{S} \cap S = \bigcup_{\varrho<\omega_1} S_\varrho$ as well.

§ 4. The cases $\Delta_0 = [\aleph_1/\aleph_1]$ or $[\aleph_1, \aleph_1]$, $\Delta_1 = [\aleph_1]$. A consequence of SH

THEOREM 4.1.

(a) *Assume* $\mathcal{G}$ *establishes* $\aleph_1 \nrightarrow ([\aleph_1, \aleph_1], \aleph_1)^2$. *Then either* $S \subseteq \mathrm{Sub}(\mathcal{G})$ *or* $\mathcal{G}_{T_0} \in \mathrm{Sub}\,(\mathcal{G})$.

(b) *Assume* $\mathcal{G}$ *establishes,* $\aleph_1 \nrightarrow ([\aleph_1/\aleph_1], \aleph_1)^2$. *Then either* $\mathrm{Sub}\,(\mathcal{G})$ *is the class of all countable graphs or* $\mathcal{G}_{T_0} \in \mathrm{Sub}\,(\mathcal{G})$.

COROLLARY 4.2.

(a) $$\mathfrak{S}_\omega \cap S \subseteq C([\aleph_1, \aleph_1], \aleph_1], \aleph_1)$$

(b) $$\mathfrak{S}_\omega \subseteq C([\aleph_1/\aleph_1], [\aleph_1], \aleph_1)$$

PROBLEM 2. Is it true (in ZFC) that

(a) $$\mathfrak{S}_{\omega+1} \cap S \subseteq C\,([\aleph_1, \aleph_1], [\aleph_1], \aleph_1)\,?$$

(a) $$\mathfrak{S}_{\omega+1} \subseteq C([\aleph_1/\aleph_1], [\aleph_1], \aleph_1)\,?$$

By 3.2, if $\neg SH$ holds, then $\mathfrak{S} \cap S$ and $\mathfrak{S}$ include the respective classes. We will see in Theorem 4.5 that $\mathrm{SH} \Rightarrow C\,([\aleph_1], [\aleph_1], \aleph_1) \subsetneq \mathfrak{S}$. First we prove

LEMMA 4.3. *Assume* $\mathcal{G} = (\omega_1, G)$ *is a graph such that* $[\aleph_1] \subsetneq \neg\,\mathcal{G}$ *and*

$$\forall C \in [\omega_1]^{\aleph_1}\ \exists A, B \in [C]^{\aleph_1} \text{ with } [A, B]^{1,1} \cap G = \emptyset.$$

Then $\mathcal{G}_{T_0} \mapsto \mathcal{G}$.

PROOF. To establish that the projection of T_0 embeds into G it is obviously sufficient to see that for all $D \in [\omega_1]^{\aleph_1}$ we can find $x \in D$ and two disjoin subsets $A, B \subset [D]^{\aleph_1}$ such that $A \cup B \subseteq \mathcal{G}(x)$ and $[A, B]^{1,1} \cap G = \emptyset$.

Assume now $D \in [\omega_1]^{\aleph_1}$. Then, because of $[\aleph_1] \subsetneq \neg\,\mathcal{G}$ there is an $x \in D$ with $|\mathcal{G}(x, D)| = \aleph_1$. Then, by the assumption on $\mathcal{G}$, there are $A, B(\in [\mathcal{G}(x, D)]^{\aleph_1})$ such that $A \cap B = \emptyset$ and $[A, B]^{1,1} \cap G = \emptyset$.

PROOF of Theorem 4.1. Let $\mathcal{G} = (\omega_1 G)$ be a graph establishing $\aleph_1 \nrightarrow ([\aleph_1, \aleph_1], \aleph_1)^2$ or $\aleph_1 \nrightarrow ([\aleph_1/\aleph_1], \aleph_1)^2$. It follows from Theorems 2.1 and 1.1 respectively that either $S \subseteq \mathrm{Sub}(G)$ or $\mathrm{Sub}(G)$ contains all countable graphs or

$$\forall C \in [\omega_1]^{\aleph_1}\ \exists A, B \in [C]^{\aleph_1}\ [A, B]^{1,1} \cap G = \emptyset.$$

Since $[\aleph_1] \not\supseteq \neg\mathcal{G}$ holds in both cases, it follows from Lemma 4.3 that $\mathcal{G}_{T_\kappa} \mapsto \mathcal{G}$.

To see the effect of S.H. on our problems we prove first

LEMMA 4.4. *Assume $\mathcal{G} = (\omega_1, G)$ establishes $\aleph_1 \nrightarrow (\aleph_1)^2_2$. Then one of the following assertions* (1) (2) *holds:*

(1) *There are $\alpha \neq \beta < \omega_1$ with $\{\alpha, \beta\} \notin G$ and $|\mathcal{G}(\alpha, \omega_1) \cap \mathcal{G}(\beta, \omega_1)| = \aleph_1$.*

(2) *Putting $\alpha \prec \beta \Leftrightarrow (\alpha < \beta \wedge \{\alpha, \beta\} \in G)$, there is a $T \in [\omega_1]^{\aleph_1}$ such that $(T, \prec)$ is a Souslin tree.*

PROOF. Assume (1) is false. For $X \subseteq \omega_1$ let

$$N(X) = \{\mathcal{G}(\alpha, \omega_1) \cap \mathcal{G}(\beta, \omega_1) \colon \{\alpha, \beta\} \in [X]^2 \setminus G\}.$$

Then $|N(X)| \leq \aleph_0$ for all countable subsets X of ω_1. We define the sequence $T_\alpha\colon \alpha < \omega_1$ by recursion on α so that T_α is a maximal independent subset of $\omega_1 \setminus \sup N(\bigcup_{\beta<\alpha} T_\beta)$. By the assumption $[\aleph_1] \not\subseteq \neg\mathcal{G}$, $0 < |T_\alpha| < \aleph_1$ holds for all $\alpha < \omega_1$. We claim that $T = \bigcup_{\alpha<\omega_1} T_\alpha$ satisfies the requirement, and in fact T_α will be the α-th level of the tree.

First we see that $\xi, \eta \prec \zeta \Rightarrow \xi \prec \eta \vee \eta \prec \xi$. Indeed, $\xi, \eta \prec \zeta$ implies $\xi, \eta < \zeta$. Hence if $\zeta \in T_\alpha$ then T_α being independent, $\xi, \eta \in \bigcup_{\beta<\alpha} T_\beta$, and $\{\xi, \eta\} \notin G$ would imply $\zeta \in N(\bigcup_{\beta<\alpha} T_\beta)$, a contradiction. Next we see that for all $\beta < \alpha$ and $\xi \in T_\alpha$ there is a unique $\eta \in T_\beta$ such that $\eta \prec \alpha$. Indeed, by the maximality of T_β, there is an $\eta \in T_\beta$ such that $\{\eta, \xi\} \in G$. Then $\eta \prec \xi$ since $\eta < \xi$. The uniqueness of η follows from the statement proved previously and from the fact that T_β is independent.

To see that $(T, \prec)$ is a tree we only have to show that $\prec$ is a partial order. Assume $\xi \in T_\alpha, \eta \in T_\beta, \zeta \in T_\gamma$ and $\xi \prec \eta \prec \zeta$. Then $\alpha < \beta < \gamma$. There is a $\xi' \in T_\alpha$ such that $\xi' \prec \zeta$. Then $\xi' \prec \eta$. Hence $\xi' = \xi$ and $\xi \prec \zeta$.

To conclude the proof we remark that T is a Souslin tree indeed because G establishes $\aleph_1 \nrightarrow (\aleph_1)^2$.

DEFINITION 4.1. Let R_0 be the class of countable complete graphs and independent graphs (containi ng no edges). For $n \in \omega$ let R_{n+1} be the class of graphs $\mathcal{G}$ of the form $g = \{a, b\} \cup h;, a \neq b \notin h, \{a, b\} \notin G, [\{a, b\}, h]^{1,1} \subseteq G$ and $\mathcal{G}(h) \in R_n$, i.e., R_{n+1} is the class of graphs which can be obtained from a member G' of R_n by adding two new vertices which are not adjacent to each other but are adjacent to all vertices of G'.

Finally put $R = \bigcup_{n<\omega} R_n$.

Clearly $\mathcal{H}_0 \in R$ where $\mathcal{H}_0$ is the "quadrilateral without diagonals", i.e., $h_0 = 4$, $H_0 = \{\{0,1\}, \{1,2\}, \{2,3\}, \{0,3\}\}$.

THEOREM 4.5. *SH implies that*

$$R \subseteq C([\aleph_1], [\aleph_1], \aleph_1).$$

In § 6 we are going to prove a stronger embedding theorem assuming Martin's axiom. We do not know if Theorem 4.5 can be improved using *SH* alone. The following is the simplest case:

PROBLEM 3. Assume SH Let $\mathcal{G}$ establish $\aleph_1 \nrightarrow (\aleph_1)^2_2$.

Does then there exist three vertices x_i: $i < 2$, $\{x_i, x_j\} \notin \mathcal{G}$ for $i < j < 3$ and such that $|\mathcal{G}(x_0, g) \cap \mathcal{G}(x_1, g) \cap \mathcal{G}(x_2, g)| = \aleph_1$?

PROOF of Theorem 4.5. We prove $R_n \subseteq C([\aleph_1], [\aleph_1], \aleph_1)$ by induction on $n \in \omega$. For $n = 0$ this is true because of $\aleph_1 \to (\aleph_1, \aleph_0)^2$. Assume the statement is true for some $n (\in \omega)$, and $\mathcal{G} = (\omega_1, G)$ establishes $\aleph_1 \nrightarrow (\aleph_1)^2$. Then, by SH and Lemma 4.4, there are $\alpha, \beta \in \omega_1$ and $C \in [\omega_1]^{\aleph_1}$ such that $\{\alpha, \beta\} \notin G$ and $C \subset \mathcal{G}(\alpha, \omega) \wedge \mathcal{G}(\beta_1, \omega_1)$. By the induction hypothesis each member of R_n embeds into $\mathcal{G}(C)$, hence each member of R_{n+1} embeds into $\mathcal{G}$.

Since $\mathcal{H}_0$ is not a p-tree, Theorem 4.5 shows that even for finite graphs one cannot prove in ZFC that Theorem 4.1 is best possible.

Nothing we have said up to now prevents the following to be true

PROBLEM 4. Is there a nice consistent extension of ZFC in which

$$C([\aleph_0, \aleph_1], [\aleph_1], \aleph_1) = C([\aleph_1/\aleph_1], [\aleph_1], \aleph_1) = \text{the class of countable graphs}$$

and

$$C([\aleph_1, \aleph_1], [\aleph_1], \aleph_1) = C([\aleph_1], \aleph_1) = C([\aleph_1], [\aleph_1], \aleph_1) = S?$$

§ 5. The case $\Delta_0 = \Delta_1 = [\varkappa]$. Preliminaries

In this case we will be able to prove genuine embedding results for uncountable graphs as well. First we need some preliminaries concerning the classes of graphs which will appear in the main result. We start with the simpler one.

DEFINITION 5.1.

(a) Let P be the class of graphs $\mathcal{G}$ such that both $\mathcal{G}$ and $\neg\mathcal{G}$ are p-trees.

(b) We briefly denote by $C^*(\varkappa)$ the class of graphs $\mathcal{H}$, $|\mathcal{H}| < \varkappa$, which embed into all $\mathcal{G}$ establishing $\varkappa \nrightarrow (\varkappa)_2^2$.

Note that $C^*(\varkappa) = C_\alpha([\varkappa], [\varkappa], \varkappa)$ where $\aleph_\alpha = \varkappa$.

First we need an easy characterization of the elements of P.

LEMMA 5.1. *Assume that for an element $\mathcal{G} \in P$, $\mathcal{G}$ and $\neg\mathcal{G}$ are the projections of the trees $(g, \prec_0)$ $(g, \prec_1)$, respectively. Then there is a unique well-ordering $\prec$ of g and a partition $g = g_0 \cup g_1$, $g_0 \cap g_1 = \emptyset$ such that*

$$\alpha \prec \beta \Leftrightarrow \alpha \prec_i \beta$$

holds for all $\alpha, \beta \in g$, $\alpha \in g_i$, $i < 2$.

PROOF. For all pairs $\alpha \neq \beta \in g$ exactly one of the relations $\alpha \prec_0 \beta$, $\alpha \prec_1 \beta$ holds. An easy discussion shows that

(1) $$(\alpha \prec_i \beta \wedge \beta \prec_{1-i} \gamma) \Rightarrow \alpha \prec_i \gamma$$

since all the cases

$$\gamma \prec_i \alpha, \ \alpha \prec_{1-i} \gamma, \quad \gamma <_{1-i} \alpha$$

are excluded.

Define $\alpha \prec \beta \Leftrightarrow \exists i \in 2(\alpha \prec_i \beta)$. Then by (1), $\prec$ is transitive, hence $\prec$ is a well-ordering of g. (1) also tells us that for all $\alpha \prec \beta \prec \gamma$, $\alpha \prec_i \beta \Leftrightarrow \alpha \prec_i \gamma$, and this yields the required partition of g.

DEFINITION 5.2. We denote by P_ξ the class of graphs $\mathcal{G}(\in P)$ for which there is a well-ordering $\prec$ of g of type at most ξ satisfying the requirements of Lemma 5.1.

In [8] answering a problem of us SHELAH proved that if G.C.H. holds and $\varkappa = \lambda^+ = 2^\lambda$ for some regular $\lambda(\geq \omega)$ then there is a 2-partition f of $\varkappa$ with λ colors establishing $\varkappa \nrightarrow [\varkappa]_\lambda^2$ and such that all three vertices span a two. colored subgraph, i.e., the function h mentioned on p. 212 does not embed into f. This result certainly shows that only very weak embedding results can be proved for 2-partitions with at least 3 colors. However we will make further use of it since the 2-partition f constructed by Shelah has some very strong properties even in case of 2-colors. For the convenience of the reader we will restate Shelah's result for this special case with detailed proofs.

DEFINITION 5.3. For $\lambda(\geq \omega)$ we denote by R_λ the set $\{f \in {}^\lambda 2 : f$ is not eventually 1$\}$, and by Q_λ the set $\{f \in R_\lambda :$ there is a largest $\alpha(< \lambda)$ with $f(\alpha) = 1\}$. If there is no danger of misunderstanding we will denote the usual lexicographical order by $<$.

Clearly R_ω, Q_ω are the set of reals and rationals, respectively. It is well known that quite a few properties of reals and rationals remain true for these general sets. We need the following well-known results.

PROPOSITION 5.2. *Assume $\lambda(\geq \omega)$ is regular. Then*

(a) $|R_\lambda| = 2^\lambda$, $|Q_\lambda| = 2^{\underline{\lambda}} = \sum_{\tau<\lambda} 2^\tau$

(b) *Q_λ is a dense subset of R_λ.*

(c) *For all subsets $A, B(\in [Q_\lambda]^{<\lambda})$ $A < B$ implies that there is an $r \in Q_\lambda$ with $A < \{r\} < B$.*

(d) *Assuming $2^\lambda = \lambda^+ = \varkappa$ and $2^{\underline{\lambda}} = \lambda$, there is a "Luzin set" $\mathfrak{L}_\lambda \subseteq R_\lambda$ such that $|\mathfrak{L}_\lambda| = \varkappa$ and for all subsets $A(\in [\mathfrak{L}_\lambda]^\varkappa)$ A is dense in some non empty intervall of R_λ. We may assume $\mathfrak{L}_\lambda \cap Q_\lambda = \emptyset$.*

DEFINITION 5.4. Assume $\lambda \geq \omega$ is regular. $2^{\underline{\lambda}} = \lambda$, $2^\lambda = \lambda^+ = \varkappa$.

Define a graph $\mathcal{G}_\lambda = (\mathfrak{L}_\lambda, G_\lambda)$ as follows.

Let $Q_\lambda = \{r_\alpha : \alpha < \lambda\}$ be a one-to-one enumeration of Q_λ. Let $Q_\lambda^0 \cup Q_\lambda^1 = Q_\lambda$ be two disjoint subsets of Q_λ both dense in R_λ. For $x, y \in R_\lambda$ let $\alpha(x, y, \lambda) = \alpha(x, y) = \min \{\alpha : r_\alpha \in (x, y)\}$ and put $\{x, y\} \in G_\lambda$ if $r_\alpha \in Q_\lambda^0$ for $\alpha = \alpha(x, y)$.

THEOREM 5.3. (SHELAH's theorem). *Under the conditions of 5.4, $\mathcal{G}_\lambda$ establishes $\varkappa \nrightarrow (\varkappa)_2^2$.*

PROOF. Assume $A \in [\mathfrak{L}_\lambda]^\varkappa$. By 5.2(d), there are $x < y$, $x, y \in R_\lambda$ such that A is dense in (x, y). For $i < 2$ there are $\alpha < \lambda$ and $r_\alpha \in Q_\lambda^i \cap (x, y)$. By 5.2(c), there are $r, s \in (x, y) \cap Q_\lambda$ such that $r < r_\alpha < s$ and $r_\beta \notin (r, s)$ for $\beta < \alpha$. Then A being dense in (x, y) there are $r < u < r_\alpha < v < s$; $u, v \in A$. Then $\alpha(u, v) = \alpha$, hence $\{u, v\} \in G$ for $i = 0$ and $\{u, v\} \notin G$ for $i = 1$.

Note that if we split Q_λ into the union of λ pairwise disjoint sets and define a 2-partition with λ colors similarly as in Proposition 5.4, we get Shelah's result for colorings with λ colors.

DEFINITION 5.5. We call a graph a *λ-Shelah-graph* if it can be embedded into some $\mathcal{G}_\lambda$.

We will denote by $\mathcal{S}he_\lambda$ the class of subgraphs of cardinality $\leq \lambda$ of $\mathcal{G}_\lambda$ for all $\lambda(\geq \aleph_0)$ satisfying the requirements. $\mathcal{S}he_{\aleph_1}$ will be denoted by $\mathcal{S}he$.

The following result contains some useful information about Shelah graphs.

PROPOSITION 5.4.

(a) *For all λ, and for all λ-Shelah graphs $\mathcal{G} = (g, G)$ there is a partition $g_0 \cup g_1$, $g_0 \cap g_1 = \emptyset$ $(g_0, g_1 \neq \emptyset)$ such that either* $[g_0, g_1]^{1,1} \subseteq G$ or $[g_0, g_1]^{1,1} \cap$

$\cap G = \emptyset$. *Let us call such a partition a* s p l i t t i n g   p a r t i t i o n *of* $\mathcal{G}$.

(b) $R \subseteq \mathcal{S}he$ *where* R *is the class of graphs defined in Definition* 4.1.

(c) *All finite Shelah graphs are Sierpiński graphs.*

(d) *There is a finite Sierpiński graph which is not a Shelah graph.*

(e) $\mathcal{S}he \not\subseteq S$

We only give hints for the easy proofs. (a) follows from the definition. To see (b) one proves by an easy induction that $R_n \subseteq \mathcal{S}he$. (c) can be seen by induction on the numbers of elements of a Shelah graph and using (a).

$\mathcal{H}_2 = (4, H_2)$, $H_2 = \{\{0,1\}, \{1,2\}, \{2,3\}\}$, a path of length three, is a Sierpiński graph which is not Shelah and hence (d) is true. (e) follows from the fact that $\mathcal{G}_{T_*}$ is a Shelah graph.

We remark that similarly as the class of Sierpiński trees the class $\mathcal{S}he \cap S$ can be caracterized as follows. Assume F is a class of graphs. Let F'' be the class of graphs $\mathcal{G} = (g, G)$ for which there is a disjoint partition $g = \bigcup_{\eta<\omega+1} g_\eta$ such that $|g_\omega| \leq 1$ and a function $f\colon \omega \to 2$ such that $\mathcal{G}(g_n) \in F$ for $n < \omega$,

$$[g_\eta, g_\xi]^{1,1} \subseteq G \qquad \text{for} \quad \eta < \xi < \omega + 1 \quad \text{if} \quad f(\eta) = 0$$

$$[g_\eta, g_\xi]^{1,1} \cap G = \emptyset \quad \text{for} \quad \eta < \xi < \omega + 1 \quad \text{if} \quad f(\eta) = 1.$$

Now $\mathcal{S}he \cap S$ is the class of graphs which can be obtained starting from the empty graph and the one element graph by transfinite iteration of the operation $''$ in less than ω_1 steps. Since we do not actually use this in the paper we save the reader from the cumbersome details.

Now we come to the lemma which expresses the main restrictive effect of Shelah graphs.

LEMMA 5.5. *Assume* $\lambda \geq \aleph_0$ *and let* $\mathcal{G} = (\lambda + 2, G)$ *be the "typical element" of* $P_{\lambda \dotplus 2}$, *i.e., for* $\xi < \eta < \lambda + 2$, $\xi < \lambda$

$$\{\xi, \eta\} \in G \quad \textit{iff} \quad \xi \textit{ is even.}$$

Then $\mathcal{G}$ *is not a* λ*-Shelah graph.*

(Note that we do not specify whether $\{\lambda, \lambda \dotplus 1\} \in G$ or not.)

PROOF. Assume $\mathcal{G} \rightarrowtail G_\lambda$. Then there is a one-to-one sequence $\{x_\alpha\colon \alpha < \lambda \dotplus 2\} \subseteq \mathfrak{L}_\lambda$ such that for all $\alpha < \lambda$, $\alpha < \beta < \lambda \dotplus 2$ $\{x_\alpha, x_\beta\} \in G_\lambda$ iff α is even.

Let $A_\alpha = \{x_\beta\colon \alpha \leq \beta < \lambda + 2\}$, $\bar{A}_\alpha$ the convex closure of A_α in R_λ and $\mathcal{H}_\alpha = \mathcal{G}_\lambda(A_\alpha)$ for $\alpha < \lambda$.

We remind the reader that $Q_\lambda = \{r_\alpha\colon \alpha < \lambda\}$ as in Definition 5.4. If $\gamma = \min\,\{\delta\colon r_\delta \in \bar{A}_\alpha\}$ for some $\alpha < \lambda$ then $A_\alpha = A_\alpha|{<}r_\gamma \cup A_\alpha|{>}r_\gamma$ is a splitting partition of the graph $\mathcal{H}_\alpha$ as defined in Definition 5.4 (a). Now meditation shows that the only splitting partition of $\mathcal{H}_\alpha$ is $A_\alpha = \{x_\alpha\} \cup (A_\alpha \setminus \{x_\alpha\})$. Let $\gamma_\alpha = \min \{\gamma: r_\gamma \in \bar{A}_\alpha\}$ for $\alpha < \lambda$. Then either $A_{\alpha+1} < \{r_{\gamma_\alpha}\}$ or $\{r_{\gamma_\alpha}\} < A_{\alpha+1}$ holds for $\alpha < \lambda$, and as a corollary of this the sequence $\{\gamma_\alpha: \alpha < \lambda\}$ is clearly increasing. On the other hand considering that $x_\lambda, x_{\lambda+1} \in A_\alpha$ for $\alpha < \lambda$ it follows that $\gamma_\alpha \leq \min \{\gamma: r_\gamma \in (x_\lambda, x_{\lambda+1})$ or $r_\gamma \in (x_{\lambda+1}, x_\lambda)\}$, a contradiction.

Definition 5.6. For each $\lambda \geq \omega$, let $\mathcal{H}^\lambda = (h^\lambda, H^\lambda)$ be the following graph.

$h^\lambda = \bigcup_{\alpha \leq \omega+1} h_{\lambda,\alpha}$ where the summands are disjoint,

$|h_{\lambda,n}| = \lambda$ for $n < \omega$; $|h_{\lambda,\omega}| = 1$, and

$[h_{\lambda,n}, h_{\lambda,\alpha}]^{1,1} \subseteq H^\lambda$ for $n \leq \alpha < \omega + 1$ if n is even,

$[h_{\lambda,n}, h_{\lambda,\alpha}]^{1,1} \cap H^\lambda = \emptyset$ for $n \leq \alpha < \omega + 1$ if n is odd.

We have

Corollary 5.6. *Assume $\lambda (\geq \omega)$ is an infinite regular cardinal, and $\mathcal{G} \in \bigcup_{\xi < \lambda^+} P_\xi$ is a λ-Shelah graph. Then $\mathcal{G} \rightarrowtail \mathcal{H}^\lambda$.*

Proof. By Lemma 5.5, the "typical element" of $P_{\lambda+2}$ described in Lemma 5.5 does not embed into $\mathcal{G}$. An easy discussion shows that then $G \rightarrowtail \mathcal{H}^\lambda$.

§ 6. The case $\Delta_0 = \Delta_1 = [\varkappa]$ (continued)

Theorem 6.1. *Assume $\omega \leq \lambda < \varkappa$ are regular cardinalls*

(a) *If $\tau^\lambda < \varkappa$ for all cardinal $\tau < \varkappa$ then*

$$P_{\lambda+1} \subseteq C^*(\varkappa).$$

(b) *If there is a $\varkappa$-Souslin tree then*

$$C^*(\varkappa) \subseteq \bigcup_{\xi < \varkappa} P_\xi.$$

(c) *If there is a $\varkappa$-Souslin tree, $\lambda^+ = 2^\lambda = \varkappa$ and $2^{\underline{\lambda}} = \lambda$ then*

$$C^*(\varkappa) \subseteq \mathrm{Sub}_\alpha(\mathcal{H}^\lambda) \quad \text{where} \quad \aleph_\alpha = \varkappa.$$

Corollary 6.2.

(a) $P_{\omega+1} \subseteq C^*(\aleph_1) = C([\aleph_1], [\aleph_1], \aleph_1)$

(b) *If* $\neg$SH. *then* $C^*(\aleph_1) \subseteq \bigcup_{\xi < \omega_1} P_\xi$

(c) *If* $\neg$S.H. *and* C.H. *holds then*

$$C^*(\aleph_1) \subseteq \mathrm{Sub}\,(\mathcal{H}^\omega)$$

where $\mathcal{H}^\omega$ *is the countable graph given in Definition* 5.6.

PROBLEM 5. Does $\mathcal{H}^\omega \rightarrowtail \mathcal{G}$ hold for all graphs $\mathcal{G}$ establishing $\aleph_1 \nrightarrow (\aleph_1)^2$?

Note that $\mathcal{H}^\omega \in S \cap \mathcal{S}he \cap P_{\omega^2\dot{+}1}$.

PROOF of Theorem 6.1. First we prove the "negative" parts. If $(T, \prec)$ is a $\varkappa$-Souslin tree, then both $\mathcal{G}_T$ and $\neg\,\mathcal{G}_T$ establish $\varkappa \nrightarrow (\varkappa)^2_2$. If $\mathcal{G} \rightarrowtail \mathcal{G}_T$ and $\mathcal{G} \rightarrowtail \neg\,\mathcal{G}_T$ then $\neg\,\mathcal{G} \rightarrowtail \mathcal{G}_T$, hence $\mathcal{G} \in P$. Thus $|\mathcal{G}| < \varkappa$ implies $\mathcal{G} \in \bigcup_{\xi<\varkappa} P_\xi$ and this proves part (b).

To see (c) assume $\mathcal{G} \in C^*(\varkappa)$. Then, by Shelah's theorem 5.3 $\mathcal{G} \rightarrowtail \mathcal{G}_\lambda$, hence $\mathcal{G}$ is a λ-Shelah graph. Then, by Corollary 5.6., $\mathcal{G} \rightarrowtail \mathcal{H}^\lambda$.

The positive part (a) of Theorem 6.1 is a reformulation of an old result of us, the proof can be carried out with the methods of [6] and will be given in details in the book [5]. For the convenience of the reader we outline the proof. In fact the following stronger statement is true.

PROPOSITION 6.3. *Assume* $f\colon [\varkappa]^2 \to 2$ *is a 2-partition of* $\varkappa$ *with 2 colors, such that there is no homogeneous subset stationary in* $\varkappa$. *Assume further that,* $\lambda, \varkappa$ *satisfy the assumptions of* 6.1. (a). *Then all elements of* $P_{\lambda\dot{+}1}$ *embed into the graph corresponding to* f.

First we can define the canonical partition tree $\prec$ on $\varkappa$ associated with f satisfying the following conditions:

(1) $\xi \prec \eta \Rightarrow \xi < \eta$.

(2) If $\xi \prec \eta \prec \zeta$ then $f(\{\xi, \eta\}) = f(\{\xi, \zeta\})$.

(3) If $\xi < \eta$, $\dot{\xi} = \dot{\eta} \cap \xi$ and $\forall \zeta < \dot{\xi}(f(\{\zeta, \xi\}) = f(\{\zeta, \eta\}))$ then $\xi \prec \eta$. Note that $\dot{\xi} = \{\zeta \in \varkappa : \zeta \prec \xi\}$. Put $L(\xi, i) = \{\eta \in \dot{\xi} : f(\{\eta, \xi\}) = i\}$ for $i < 2$. It is easy to see that (2) and (3) imply

(4) $L(\xi, i) = L(\eta, i) \wedge \xi < \eta \Rightarrow \xi \prec \eta$ for $i < 2$.

By Lemma 5.1 it is clearly sufficient to see that there is a $\xi < \varkappa$ with $cf(\xi) = \lambda$ such that $L(\xi, i)$ is a cofinal subset of ξ for $i > 2$.

Now start with the remark that $\{\xi < \varkappa : cf(\xi) = \lambda\} = A$ is a stationary subset of $\varkappa$. If the claim is not true, then there is a stationary $B(\subseteq A)$ and an $i(< 2)$ such that $\sup L(\xi, i) < \xi$ for all $\xi(\in B)$. Then there is a stationary subset $C(\subseteq B)$ and a $\varrho(< \varkappa)$ such that $L(\xi, i) \subseteq \varrho$ for $\xi \in C$. Now using the cardinal assumption $|\varrho|^\lambda < \varkappa$ we would get a stationary subset $D(\subseteq C)$ such that $L(\xi, i) = L(\eta, i)$ for $\xi, \eta \in D$. However this would imply by (4) that D is a chain in $\prec$. Then (2) would imply that there is a stationary $E(\subseteq D)$ which is homogeneous for $1 - i$, a contradiction.

We think it is time to mention some other problems for embedding uncountable graphs. Assume $2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_2$. Shelah's result mentioned in the introduction implies only that

$C([\aleph_1, \aleph_2], [\aleph_1, \aleph_2], \aleph_2)$ is not the class of all graphs of cardinality $\aleph_1$, but this does not exclude all embedding theorems. E. g. the result just proved says that

$$P_{\omega_1 \dot{+} 1} \subseteq C([\aleph_2], [\aleph_2], \aleph_2).$$

PROBLEM 6.

(a) Can one prove in ZFC $+ 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$ that $C([\aleph_1, \aleph_2], [\aleph_1, \aleph_2], \aleph_2)$ does not contain all graphs of cardinality $\aleph_1$?

(b) Assume $\mathcal{G}$ is a graph establishing $\omega_2 \nrightarrow (\omega_1 \dot{+} \omega)^2_2$. Do all graphs of cardinality $\aleph_1$ embed into $\mathcal{G}$?

In [8] Shelah conjectures that there is a positive answer to (a). The existence of graphs establishing $\omega_2 \nrightarrow (\omega_1 \dot{+} \omega)^2_2$ is known to be consistent with ZFC $+ (2^{\aleph_0} = \aleph_1) \wedge (2^{\aleph_1} = \aleph_2)$ (see e.g. [4]) and Shelah's result does not exclude a positive answer to (b).

§ 7. A discussion of the results for the classes $C_0(\Delta_0, \Delta_1, \aleph_1)$; the case $\Delta_0 = [\aleph_0, \aleph_1]$, $\Delta_1 = [\aleph_1]$; a consequence of Martin's axiom

Let us first recall our results concerning finite graphs. Assuming $\neg$ SH we had a complete characterization in each of the cases considered up to now.

$C_0([\aleph_1/\aleph_1], [\aleph_1, \aleph_1], \aleph_1) =$ the class of all finite graphs. By Theorem 1.1, $C_0([\aleph_1, \aleph_1], [\aleph_1, \aleph_1], \aleph_1) =$ the class of finite Sierpiński graphs, by Theorem 2.1.

Assuming $\neg$ SH $C_0([\aleph_1/\aleph_1], [\aleph_1], \aleph_1) = C_0([\aleph_1, \aleph_1], [\aleph_1], \aleph_1) =$ the class of all finite p-trees.

This follows from Theorem 4.1 since, e.g. by Theorem 3.3, all finite p-trees are Sierpiński graphs.

Assuming $\neg$ SH

$$C_0([\aleph_1], [\aleph_1], \aleph_1) =$$

$=$ the class of all finite graphs $\mathcal{G}$ where both $\mathcal{G}$ and $\neg\,\mathcal{G}$ are p-trees $\in \bigcup_{n<\omega} P_n$. This follows from Corollary 6.2, and Lemma 5.1 describes these graphs completely.

We turn back to problems left open in case we do not assume $\neg$ SH later in this chapter.

First we want to point out that the only case not convered by these results is $\Delta_0 = [\aleph_0, \aleph_1]$, $\Delta_1 = [\aleph_1]$.

In this case we have very little information, since we do not know any graph G establishing $\aleph_1 \nrightarrow ([\aleph_0, \aleph_1], [\aleph_1])^2$ (in any consistent extension of ZFC) which is not universal for countable graph.

However we do know that $C_0([\aleph_0, \aleph_1], [\aleph_1], \aleph_1)$ is "larger" than $C_0([\aleph_1/\aleph_1], [\aleph_1], \aleph_1)$ since we will prove in ZFC that it contains the finite elements of R, hence it contains $\mathcal{H}_0$ which is not a p-tree. This will be the consequence of the following

THEOREM 7.1. $R \subseteq C([\aleph_0, \aleph_1], [\aleph_1], \aleph_1)$.

PROOF. Assume $\mathcal{G} = (\omega_1, G)$ establishes $\aleph_1 \nrightarrow ([\aleph_0, \aleph_1], \aleph_1)^2$. Theorem 4.5 tells us that either $R \subseteq \mathrm{Sub}\,(\mathcal{G})$ or there is a Souslin tree. But in fact the proof of Theorem 4.5 tells us that either $R \subseteq \mathrm{Sub}\,(\mathcal{G})$ or there is a $T(\in [\omega_1]^{\aleph_1})$ such that $\mathcal{G}(T)$ is the projection of a Souslin tree. But this is impossible since the projection of a Souslin tree contains $[\aleph_0, \aleph_1]$.

The following are the simplest unsolved cases

PROBLEM 7. Let $\mathcal{H}_1$ be the pentagon without diagonals, i.e., $h_1 = 5$, $H_1 = \{\{0, 1\}, \ldots, \{1,4\}, \{0, 4\}\}$. Is $\mathcal{H}_1 \in C_0([\aleph_0, \aleph_1], [\aleph_1], \aleph_1)$? Or is

$$\mathcal{H}_2 \in C_0([\aleph_0, \aleph_1], [\aleph_1], \aleph_1)?$$

$\mathcal{H}_1$ is the simplest graph which is not Sierpiński and $\mathcal{H}_2$ defined on p. 221 is a Sierpiński graph which is not an element of R.

As we have already mentioned when stating Problem 4 it might happen that $C([\aleph_1], [\aleph_1], \aleph_1) = S$ in some consistent estension of ZFC. However we know that SH is not strong enough to imply this since as a corollary of Shelah's result 5.3 we know that

$$2^{\aleph_0} = \aleph_1 \Rightarrow C([\aleph_1], [\aleph_1], \aleph_1) \subseteq S \cap \mathcal{She}.$$

It would be nice to prove, as a generalization of Theorem 4.5 that

$$\neg\,\mathrm{SH} \Rightarrow C([\aleph_1], [\aleph_1], \aleph_1) = S \cap \mathcal{She}$$

and

$$\mathrm{MA} + 2^{\aleph_0} > \aleph_1 \Rightarrow C([\aleph_1], [\aleph_1], \aleph_1) = S$$

but we have no hope to do so.

For MA = Martin's axiom see [9]. We will prove one more theorem working in this direction which is by no means best possible.

THEOREM 7.2. *Assume* $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$. *Then* $\mathcal{H} \in C([\aleph_1], [\aleph_1], \aleph_1)$ *for all finite Shelah graphs* $\mathcal{H}$.

PROOF. We prove this by induction on $|h|$. Assume $\mathcal{H} = (h, H)$ is a finite Shelah graph, and we know the statement for all Shelah graphs $\mathcal{H}'$ with $|h'| < n$.

By 5.4 (a) we know that there is a splitting partition $h = h_0 \cup h_1$ of $\mathcal{H}$, and by symmetry we may assume that $[h_0, h_1]^{1,1} \subseteq H$. Assume $\mathcal{G} = (\omega_1, G)$ establishes $\aleph_1 \nrightarrow (\aleph_1)^2_2$.

Let us call a finite function $f: \omega_1 \to \omega$ a good coloring of G if

$$\forall \alpha, \beta \in D(f) \left(\{\alpha, \beta\} \in G \Rightarrow f(\alpha) \neq f(\beta)\right).$$

We know that the chromatic number of $\mathcal{G}$ is $\aleph_1$, otherwise $[\aleph_1] \subseteq \neg \mathcal{G}$. Then, by Lemma 5.2 of [2] and by MA $+ 2^{\aleph_0} > \aleph_1$ there is a sequence $f_\alpha: \alpha < \omega_1$ of good colorings of $\mathcal{G}$ such that $f_\alpha \cup f_\beta$ is not a good coloring of $\mathcal{G}$ for $\alpha \neq \beta < \omega_1$.

From this we get as in Lemma 5.3 of [2] that there exists a sequence $D_\alpha \subseteq \omega_1 (\alpha < \omega_1)$ of pairwise disjoint subsets ω_1 such that $[D_\alpha, D_\beta]^{1,1}$ contains an edge of G for all $\alpha < \beta < \omega_1$ and $|D_\alpha| = n$ for some $n < \omega$ and for all $\alpha < \omega_1$.

Let $D = \bigcup_{\alpha < \omega_1} D_\alpha$, and let U be a uniform ultrafilter on ω_1. For each $x \in D$, $x \in D_\alpha$ let

$$r(x) = \{\beta < \omega_1 : \alpha < \beta \wedge \mathcal{G}(x, D) \cap D_\beta \neq \emptyset\}.$$

Let $A = \{x \in D: r(x) \in U\}$. By the assumptions A meets each D_α, hence $|A| = \aleph_1$.

Using that $|D_\alpha| = n$ for $\alpha < \omega_1$ it now follows that there is a $B (\in [A]^{\aleph_1})$ such that

$$|\bigcap \{\mathcal{G}(x, D) : x \in F\}| = \aleph_1$$

for all finite subsets F of B.

Since $\mathcal{G}(B)$ establishes $\aleph_1 \nrightarrow (\aleph_1)^2_2$, by the induction hypothesis, there is an $F_0 (\in [B]^{<\aleph_0})$ such that $\mathcal{H}(h_0)$ is isomorphic to $\mathcal{G}(F_0)$. Let now $C = \bigcap \{\mathcal{G}(x, D): x \in F_0\}$. Then $|C| = \aleph_1$ and $\mathcal{G}(C)$ establishes $\aleph_1 \nrightarrow (\aleph_1)^2_2$, hence again by the induction hypothesis there is an $F_1 \in [C]^{<\aleph_0}$ such that $\mathcal{H}(h_1)$ is isomorphic to $\mathcal{G}(F_1)$.

Then $\mathcal{H}$ is isomorphic to $\mathcal{G}(F_0 \cup F_1)$, hence $\mathcal{H} \rightarrowtail \mathcal{G}$.

§ 8. On weak embeddings

DEFINITION 8.1. Where $\mathcal{F}$ is a class of graphs, let $\mathcal{C}o(F) = \{\neg \mathcal{H} : \mathcal{H} \in \mathcal{F}\}$

We have already seen that as a corollary of Theorems 1.1 and 2.1

$$Cw([\aleph_1/\aleph_1], [\aleph_1, \aleph_1], \aleph_1) = \text{the class of countable graphs},$$

$$Cw([\aleph_1, \aleph_1], [\aleph_1, \aleph_1], \aleph_1) = S.$$

Let us see first that the cases $\Delta_0 = [\aleph_1/\aleph_1]$ or $[\aleph_1, \aleph_1]$, $\Delta_1 = \aleph_1$ do not lead to genuine new problems.

By 3.2.

$$Cw([\aleph_1/\aleph], [\aleph_1], \aleph_1) \subseteq \mathcal{T} \cup \mathcal{C}o(\mathcal{T})$$

$$Cw([\aleph_1, \aleph_1], [\aleph_1], \aleph_1) \subseteq (\mathcal{T} \cup \mathcal{C}o(\mathcal{T})) \cap \mathcal{S}$$

provided $\neg$ S.H. holds.

On the other hand, by Theorem 4.1

$$\mathcal{T}_\omega \cup \mathcal{C}o(\mathcal{T}_\omega) \subseteq Cw([\aleph_1/\aleph_1], [\aleph_1], \aleph_1) \quad \text{and}$$

$$(\mathcal{T}_\omega \cap S) \cup \mathcal{C}o(\mathcal{T}_\omega \cap S) \subseteq Cw\ ([\aleph_1, \aleph_1], [\aleph_1], \aleph_1).$$

Note that $\mathcal{C}o(\mathcal{T}_\omega \cap S) = \mathcal{C}o(\mathcal{T}_\omega) \cap S$, and we are back to Problem 2. In fact we do not even know if

$$\mathcal{T}_{\omega+1} \cap S \subseteq Cw([\aleph, \aleph_1], [\aleph_1], \aleph_1)$$

or

$$\mathcal{T}_{\omega+1} \subseteq Cw([\aleph_1/\aleph_1], [\aleph_1], \aleph_1)$$

holds.

However we can prove a theorem in case $\Delta_0 = \Delta_1 = [\aleph_1]$.

THEOREM 8.1.

(a) $Cw([\aleph_1], [\aleph_1], \aleph_1) \subseteq \mathcal{T} \cup \mathcal{C}o\,(\mathcal{T})$ *if* $\neg$ SH *holds,*

(b) $Cw([\aleph_1,] [\aleph_1], \aleph_1) \subseteq \mathcal{S}he$ *if* CH *holds,*

(c) $Cw([\aleph_1], [\aleph_1], \aleph_1) \subseteq S$,

(d) $\mathcal{T}_\omega \cap S \subseteq Cw([\aleph_1], [\aleph_1], \aleph_1)$.

COROLLARY 8.2. *Assume* $\neg$ SH. *Then*

$$Cw_0([\aleph_1], [\aleph_1], \aleph_1) = Cw_0([\aleph_1/\aleph_1], [\aleph_1], \aleph_1) = \bigcup_{n<\omega} \mathcal{T}_n \vee \mathcal{C}o(\bigcup_{n<\omega} \mathcal{T}_n).$$

i.e., this class consists of finite p-trees and their complements.

In [4] 4. a p. 286 we made the mistake of saying that this class consists of the finite p-trees which is obviously impossible since it is closed under complementation.

Corollary 8.2. follows from Theorem 8.1 and the remarks made about

$$Cw([\aleph_1/\aleph_1], [\aleph_1], \aleph_1).$$

PROOF of Theorem 8.1. (a), (b), (c) are restatements of earlier results. For the proof of (d) assume that $\mathcal{G} = (\omega_1, G)$ establishes $\aleph_1 \nrightarrow (\aleph_1)^2_2$. If there is a $C(\in [\omega_1]^{\aleph_1})$ such that $G(C)$ establishes $\aleph_1 \nrightarrow (\aleph_1, [\aleph_1, \aleph_1])^2$, then by Theorem 4.1(a) either $S \subseteq \mathrm{Sub}(\mathcal{G})$ or $\mathcal{G}_{T_*} \rightarrowtail \neg\mathcal{G}$.

Hence we may assume that

$$\forall C \in [\omega_1]^{\aleph_1} \, \exists A, B \in [C]^{\aleph_1} \; [A, B]^{1,1} \cap \mathcal{G} = \emptyset.$$

Then, by Lemma 4.3, $\mathcal{G}_{T_*} \rightarrowtail \mathcal{G}$. This proves the stronger statement that for all graphs $\mathcal{G}$ establishing $\aleph_1 \nrightarrow (\aleph_1)^2_2$ either $S \subseteq \mathrm{Sub}(\mathcal{G}) \cap \mathrm{Sub}\,(\neg\mathcal{G})$ or $\mathcal{G}_{T_*} \overset{w}{\rightarrowtail} G$.

Note that (a) (b) of Theorem 8.2 strongly restrict the class $Cw([\aleph_1], [\aleph_1], \aleph_1)$.

Definition 8.2. Let $\hat{\mathfrak{F}}_{\omega\dot{+}1}$ = the class of projections of those trees $(T, \prec)$ which do not contain a chain $\mathcal{C}$, such that $\mathrm{tp}\,\mathcal{C}\,(\prec) = \omega \dot{+} 2$ so that the n-th element of the chain has incomparable successors for $n < \omega$.

Corollary 8.3. *Assume* $\neg$SH *and* CH *Then*

$$Cw([\aleph_1], [\aleph_1], \aleph_1) \subseteq \hat{\mathfrak{F}}_{\omega\dot{+}1} \cup \mathcal{C}o(\hat{T}_{\omega\dot{+}1}).$$

Proof. Assume

$$\mathcal{H} = (h, H) \in Cw([\aleph_1], [\aleph_1], \aleph_1).$$

Then by Theorem 8.1 (a) either $\mathcal{H}$ or $\neg\mathcal{H}$ is the projection of a tree $(h, \prec)$. If $(h, \prec) \in \hat{\mathfrak{F}}_{\omega\dot{+}1}$ then $(h, \prec)$ contains a subtree whose projection is isomorphic to the "typical element" of $P_{\omega\dot{+}1}$ as described in Lemma 5.5. But then, by this lemma, $\mathcal{H}$ is not a Shelah graph, and this contradicts to Theorem 8.1 (c).

At this point we do not try to describe the class of graphs for which the problem remains open. We only remark that $\mathcal{H}^\omega$ as defined in 5.6 is an element of $\hat{\mathfrak{F}}_{\omega\dot{+}1}$. It is clear that if $\mathcal{H}^\omega \overset{w}{\rightarrowtail} \mathcal{G}$ then $\mathcal{H}^\omega \rightarrowtail \mathcal{G}$ for all $\mathcal{G}$. Hence the problem if $\mathcal{H}^\omega \in Cw([\aleph_1], [\aleph_1], \aleph_1)$ is equivalent to the problem if $\mathcal{H}^\omega \in C([\aleph_1], [\aleph_1], \aleph_1)$ (see Problem 6).

*

A list of the graphs and classes of graphs used in the paper

$[\varkappa]$	the complete $\varkappa$-graph	Def. 0.1 (e) p. 206
$[\varkappa, \lambda]$	the complete $(\varkappa, \lambda)$-bipartite graph	Def. 0.1 (e) p. 206
$[\varkappa/\varkappa]$	the half $(\varkappa, \varkappa)$-bipartite graph	Def. 0.1 (g) p. 206
$C_\alpha(\Delta_0, \Delta_1, \varkappa)$, $C(\Delta_0, \Delta_1, \varkappa)$		Def. 0.5 (a) p. 207
$Cw_\alpha(\Delta_0, \Delta_1, \varkappa)$, $C\omega(\Delta_0, \Delta_1, \varkappa)$		Def. 0.5 (b) p. 208
$\mathrm{Sub}_\alpha(G)$, $\mathrm{Sub}\,(G)$		Def. 0.5 (c) p. 208
Sierpiński graphs		Def. 0.6 (a) p. 208

S	the class of countable Sierpiński graphs	Def. 2.1	p. 211
$\mathcal{G}_T$	the projection of the tree $(T, \prec)$	Def. 3.2	p. 213
p-tree		Def. 3.2	p. 213
$\mathfrak{T}$	the class of countable p-trees	Def. 3.3	p. 213
$\mathfrak{T}_\alpha$	the class of p-tree which are projections of trees of length $\leq \alpha$	Def. 3.3	p. 213
Sierpiński trees	$S \cap \mathfrak{T}$	Def. 3.3	p. 213
Z_ϱ	a class of trees	Def. 3.4 (b)	p. 214
$\mathfrak{S}_\varrho$	the projections of the elements of Z_ϱ	Def. 3.4 (d)	p. 214
T_0	the complete binary ω-tree	Def. 3.4 (e)	p. 214
$\mathfrak{T}_s$	the class of countable trees which embed into a Sierpiński graph		p. 215
$\mathcal{H}_0$	the quadrilateral without diagonals	Def. 4.1	p. 218
R	a class of countable graphs	Def. 4.1	p. 218
P	the class of p-trees whose complement is a p-tree as well	Def. 5.1	p. 219
P_ξ	a subclass of P	Def. 5.2	p. 219
$C^*(\varkappa)$	$C_\alpha([\varkappa], [\varkappa], \varkappa)$ for $\aleph_\alpha = \varkappa$	Def. 5.1	p. 219
$\mathcal{G}_\lambda$	a λ-Shelah graph	Def. 5.4	p. 220
		Def. 5.5	p. 220
$\mathcal{She}_\lambda$, $\mathcal{She}$	the class of λ-Shelah graphs and Shelah graphs, respectively	Def. 55	p. 220
$\mathcal{H}_2$	a 4-point Sierpiński graph	5.4 (d)	p. 221
$\mathcal{H}^\lambda$	a special λ-Shelah graph	Def. 5.6	p. 222
$\mathcal{H}_1$	the pentagon without diagonals	Problem 7.	p. 225
$\mathcal{Co}(\mathfrak{F})$	$\{\neg \mathcal{G} : \mathcal{G} \in \mathfrak{F}\}$	Def. 8.1	p. 226
$\hat{\mathfrak{S}}_{\omega+1}$	a class of trees	Def. 8.2	p. 228

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