(v) *Suppose that $f(z)$ is transcendental,* (3) *is satisfied and $p>k$. Then*

(a) *$f(z)$ has at most a finite number of poles if and only if $h(z)$ has at most a finite number of poles and if M, N denote the poles of $f(z)$, $h(z)$ respectively then $(p-k)M \leq N \leq (p+k)M$.*

(b) *$\rho_h(\infty) = p\rho_f(\infty)$, where $\rho_f(\infty) = \overline{\lim}_{r\to\infty} (\log n(r, f))/(\log r)$, where $n(r, f)$ is the counting function of poles used in Nevanlinna theory with similar meaning given to $\rho_h(\infty)$. It is deduced that ∞ is a Borel exceptional value of $f(z)$ if and only if it is a Borel exceptional value of $h(z)$.*

(c) *If, for any $\sigma > 0$*

$$\overline{\lim_{r\to\infty}} \frac{T(\sigma r, f)}{T(r, f)} = \sigma^{\rho}, \quad (\rho = \rho_f),$$

then

$$\frac{(p+k)\,\delta(\infty, f) - 2k}{p-k} \leq \delta(\infty, h) \leq \frac{(p-k)\,\delta(\infty, f) + 2k}{p+k}$$

where $\delta(\infty, \;)$ denotes the Nevanlinna deficiency of the value ∞. (In particular $\delta(\infty, f) = 1$ if and only if $\delta(\infty, h) = 1$.)

(vi) *Suppose that $f(z)$ has infinitely many poles and* (3) *is satisfied then*

$$\overline{\lim_{r\to\infty}} \frac{n(r, h)}{n(r, f)} = \infty$$

unless $n(r, f) = O((\log r)^K)$ for some constant K $(K>1)$.

Results (i)–(vi) remain valid, except the Remark in (iii), if (1) and (3) are replaced by (2) and (4) respectively.

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On a geometric property of Lemniscates

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In the Euclidean space R^3, we define the product

$$p_n(w, w_k) = \prod_{k=1}^{n} |w - w_k|,$$

where $w=(w_1, w_2, w_3)$, $w_k=(w_{k1}, w_{k2}, w_{k3})$, and $|w-w_k|$ is the distance between w and w_k. Let $C(n)$ be the class of all such products with the same degree n. For any product p, we call $E(p)=\{w: p(w)\leq 1\}$ the lemniscate of p. With the help of those definitions, we prove the following

THEOREM. *Let* $p_n(w, w_k)$ *and* $p_n^*(w, w_k^*)$ *be two products in* $C(n)$ *such that* $E(p_n)\subseteq E(p_n^*)$. *If all zeros* w_k *of* p_n *lie on the same plane, then we have* $p_n(w, w_k)\equiv p_n^*(w, w_k^*)$.

The condition that all zeros w_k of p_n lie on the same plane is necessary. Without it, the theorem is no longer true.

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