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On partitions of N into summands coprime to N

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Introduction

Let $R(n)$ and $R'(n)$ denote the number of partitions of n into summands and distinct summands respectively that are relatively prime to n. The first author proved as a special case of a more general theorem that [1]

$$\log R(n) \sim \pi\sqrt{\tfrac{2}{3}}\phi^{1/2}(n)$$
$$\log R'(n) \sim \pi\sqrt{\tfrac{1}{3}}\phi^{1/2}(n)$$

where $\phi(n)$ denotes Euler's function. The second author proved [2] that the error terms in the above results are $0\{\exp((1+t)(\log 2)(\log n)/\log\log n)\}$ by showing that the asymptotic results of Roth and Szekeres [3] hold for this problem. Unfortunately these asymptotic results are not in terms of the usual elementary or arithmetic functions.

In this note we determine more explicit and more precise asymptotic results than those mentioned above. In general these results are complicated to state (see Theorems 1.1 and 1.2 below) however when $n = q_1 q_2^2$ where $q_1 = 0\{n^{1/6-\epsilon}\}$ we obtain

$$R'(q_1q_2^2) = \frac{1}{2\sqrt{2}}\, n^{-3/4}3^{-1/4}\left(1-\frac{1}{q_1}\right) \times \exp\left\{\frac{\pi}{\sqrt{3}}(q_1-1)^{1/2}q_2 - f_1\left[\exp\left(-\frac{\pi}{2\sqrt{3}}(q_1-1)^{1/2}\right)\right] + f_1\left[\exp\left(-\frac{\pi}{2\sqrt{3}}(q_1-1)^{1/2}\right)\right]\right\}[1+0\{n^{-1/12}\}] \tag{0.1}$$

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where

$$f_1(x)=\sum_{l=1}^{\infty}\frac{(-1)^l}{l}\frac{x^l}{1+x^l}.$$

This case illustrates the nature of our formulae which follow.

Section 1

The following notation is convenient:

Let n be an integer variable.

We let c denote a positive constant, not necessarily always the same constant. ϵ shall denote the same; however relations involving ϵ shall commonly hold for ϵ sufficiently small.

Let p with or without a subscript denote a prime which divides n, and let q denote an arbitrary prime.

Let

$$M=\prod_p p, \qquad r=\sum_p 1$$

that is r is the number of distinct primes dividing n and M is their product.

Let α $(=\alpha(n))$ be the unique positive root of

$$n=\sum_{(l,M)=1}\frac{l}{e^{\alpha l}+1}. \tag{1.1}$$

All following equations and asymptotic relations involving α may only hold for α sufficiently small or equivalently n sufficiently large.

Let A_2 $(=A_2(n))$ be defined by

$$A_2=\sum_{(l,M)=1}\frac{l^2e^{\alpha l}}{e^{\alpha l}+1}$$

In [2] it is shown that

$$\alpha=\pi\left(\frac{\phi(M)}{12M}\right)^{1/2}n^{-1/2}+0\{n^{-1+c/\log\log n}\}$$

$$R'(n)=(2\pi A_2)^{-1/2}\exp\left[\sum_{(l,M)=1}\frac{\alpha l}{e^{\alpha l}+1}+\log(1+e^{\alpha l})\right]\qquad[1+0\{\alpha^{1/2-\epsilon}\}]\quad(1.2$$

First of all we shall require a more accurate estimate for α than the one just quoted. To give this estimate it is convenient to define

$$f(x)=\sum_{j=1}^{\infty}\frac{jx^j}{1+x^j}=\sum b_m x^m, \tag{1.3}$$
$$b_m=\sum_{d/m}(-1)^d d$$

and

$$S_1(n)=\sum_{s=1}^{r}{}'(-1)^s\alpha p_1\cdots p_s f(e^{-\alpha p_1\cdots p_s}) \tag{1.4}$$

where $\sum'$ denotes summation over the range $n^{+1/2}h^{-1}(n)<p_1\cdots p_s<n^{1/2}h(n)$, and $h(n)$ is any monotone function such that $n^{1/12}>h(n)>n^{\epsilon}$ for some constant $\epsilon>0$. Furthermore let

$$T_n=\sum_{s=1}^{r}\sum_{p_1\cdots p_s>n^{1/2}h^{-1}(n)}(-1)^s(p_1\cdots p_s)^{-1} \tag{1.5}$$

Our improved estimate for α is

LEMMA 1.1.

$$\alpha=\frac{\pi}{2\sqrt{3}}\left(\frac{\phi(M)}{M}-T_n\right)^{1/2}n^{1/2}+\frac{S_1(n)}{2n}+0\left\{\frac{n^{-1+((1+\epsilon)\log 2/\log\log n)}}{h(n)}\right\}.$$

Proof. We write

$$\sum_{(l,M)=1}\frac{l}{e^{\alpha l}+1}=\sum_{j=1}^{\infty}\frac{j}{e^{\alpha j}+1}-\sum_{(l,M)>1}\frac{l}{e^{\alpha l}+1}.$$

We note that

$$-\sum_{(l,M)>1}\frac{l}{e^{\alpha l}+1}=\sum_{s=1}^{r}(-1)^s\sum_{p_1\cdots p_s}\sum_{j=1}^{\infty}\frac{jp_1p_2\cdots p_s}{e^{\alpha jp_1p_2\cdots p_s}+1}$$

(our notation means the sum over all products of the prime divisors of n considered s at a time).

Now with f as defined in (1.3), we see that

$$\sum_{j=1}^{\infty} \frac{je^{-\alpha ja}}{1+e^{-\alpha ja}} = f(e^{-\alpha a}).$$

It follows from the Euler–Maclaurin sum formula (see [2] for details) that if $a \le n^{1/2}h^{-1}(n)$ where $h(n)$ denotes any monotone function such that $n^{1/12} > h(n) > n^{\epsilon}$ for some constant $\epsilon > 0$, then

$$\sum_{j=1}^{\infty} \frac{ja}{e^{\alpha ja}+1} = \frac{\pi^2}{12a\alpha^2} + 0\{\alpha^{-1}h^{-1}(n)\}$$

and if $a > n^{1/2}h(n)$ then

$$\sum_{j=1}^{\infty} \frac{ja}{e^{\alpha ja}+1} = 0\{e^{-h(n)/2}\}$$

Hence (1.1) becomes (there are fewer than $n^{(1+\epsilon)\log 2/\log\log n}$ terms in (1.1) as shown in [2])

$$n = \frac{\pi}{12\alpha^2}\left[1 + \sum_{s=1}^{r} \sum_{p_1 \cdots p_s < n^{1/2}h^{-1}(n)} (-1)^s (p_1 \cdots p_s)^{-1}\right]$$

$$+ \sum_{s \ge 1} \sum_{n^{1/2}h^{-1}(n) < p_1 \cdots p_s < n^{1/2}h(n)} (-1)^s p_1 \cdots p_s f(e^{-\alpha p_1 \cdots p_s})$$

$$+ 0\left\{\frac{n^{(1+\epsilon)\log 2/\log\log n}}{\alpha h(n)}\right\} + 0\{e^{-h(n)/2} n^{(1+\epsilon)\log 2/\log\log n}\}. \tag{1.6}$$

From (1.2) on the range of summation in $\sum'$

$$\alpha p_1 \cdots p_s = \pi\left(\frac{\phi(M)}{12M}\right)^{1/2} n^{-1/2} p_1 \cdots p_s + 0\{n^{-1/2}h(n)\},$$

thus

$$e^{-\alpha p_1 \cdots p_s} = \exp\left[-\pi\left(\frac{\phi(M)}{12M}\right)^{1/2} n^{-1/2} p_1 \cdots p_s\right][1 + 0\{n^{-1/2}h(n)\}].$$

Now on the range we are considering $ch^{-1}(n)(\log\log n)^{-1/2} < \alpha p_1 \cdots p_s < ch(n)$.

From the mean value theorem

$$f(e^{-\alpha p_1 \cdots p_s}) = f\left(\exp\left[-\pi\left(\frac{\phi(M)}{12M}\right)^{1/2} n^{-1/2} p_1 \cdots p_s\right]\right)$$
$$+0\left\{f'\left(\exp\left[-\frac{\log\log n}{h(n)}\right]\right)n^{-1/2}h(n)\right.$$
$$\left.\times \exp\left[-\pi\left(\frac{\phi M}{12M}\right)^{1/2} n^{-1/2} p_1 \cdots p_s\right]\right\}$$

As $x \to 0$ however it is well-known that $f(e^{-x}) \sim cx^{-2}$, $f'(e^{-x}) \sim cx^{-3}$; hence

$$f(e^{-\alpha p_1 \cdots p_s}) = f\left(\exp\left[-\pi\left(\frac{\phi M}{12M}\right)^{1/2} n^{-1/2} p_1 \cdots p_s\right]\right)$$
$$+0\{n^{-1/2}h^3(n)\}\,(=0\{n^{-1/4}\}).$$

Hence

$$S_1(n) = \frac{\pi}{\alpha n^{1/2}}\left(\frac{\phi M}{12M}\right)^{1/2} \sum_{s=1} (-1)^s p_1 \cdots p_s f\left(\exp\left[-\pi\left(\frac{\phi M}{12M}\right)^{1/2} n^{-1/2} p_1 \cdots p_s\right]\right)$$
$$+0\left\{\frac{h^3(n)}{\alpha n}\right\}\,(=0\{n^{1/4+\epsilon}\}).$$

Then (1.6) becomes

$$n = \frac{\pi^2}{12\alpha^2}\left(\frac{\phi(M)}{12M} - T_n\right) + \frac{S_1(n)}{\alpha} + 0\left\{\frac{n^{(1+\epsilon)\log 2/\log\log n}}{\alpha h(n)}\right\} \tag{1.7}$$

If one substitutes

$$\alpha = \frac{\pi}{2\sqrt{3}}\left(\frac{\phi(M)}{M} - T_n\right)^{-1/2} n^{-1/2}(1+f(n))$$

into (1.7) one obtains

$$f(n) = \frac{S_1(n)}{n^{1/2}} \frac{\sqrt{3}}{\pi}\left(\frac{\phi(M)}{M} - T_n\right)^{-1/2} + 0\{n^{-3/2+\epsilon}\}.$$

Hence we have Lemma 1.1.

In a very similar way we may estimate

$$\sum_{(l,M)=1} \log(1+e^{\alpha l}) = \sum_{j=1}^{\infty} \frac{j}{e^{\alpha j}+1} - \sum_{s=1}^{r} (-1)^s \sum_{p_1 \cdots p_s} \sum_{j=1}^{\infty} \log(1+e^{\alpha j p_1 \cdots p_s})$$

We let

$$\sum_{j=1}^{\infty} \log(1+e^{-\alpha j p_1 \cdots p_s}) = f_1(e^{-\alpha p_1 \cdots p_s})$$

where

$$f_1(x) = \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \frac{x^l}{1+x^l} = \sum_{l=1}^{\infty} \left(\sum_{m/l} \frac{(-1)^{m+1}}{m} \right) x^l \tag{1.8}$$

that is $xf_1'(x) = f(x)$ where $f(x)$ is defined by (1.3). Again it follows from the Euler–Maclaurin formula (see [2] for details) that if $a < n^{1/2}h^{-1}(n)$

$$\sum_{j=1}^{\infty} \log(1+e^{-\alpha j a}) = \frac{\pi^2}{12a\alpha} - \frac{\log 2}{2} + 0\{h^{-1}(n)\},$$

and if $a > n^{1/2}h(n)$

$$\sum_{j=1}^{\infty} \log(1+e^{-\alpha j a}) = 0\{e^{-h(n)/2}\}.$$

We obtain as before:

LEMMA 1.2.

$$\begin{aligned}\sum_{(l,M)=1} \log(1+e^{-\alpha l}) &= \frac{\pi^2}{12\alpha}\left(\frac{\phi(M)}{M} - T_n\right) \\ &+ \frac{\log 2}{2}\left(1 + \sum_{s=1}^{r} (-1)^s \sum_{p_1 \cdots p_s < nh^{-1}(n)} 1\right) \\ &+ \sum_{s=1}^{r}{}' (-1)^s f_1(e^{-\alpha p_1 \cdots p_s}) + 0\left\{\frac{n^{(1+\epsilon)\log 2/\log\log n}}{h(n)}\right\}.\end{aligned}$$

It has been shown in [2] that the following result holds.

LEMMA 1.3.

$$A_2 = \frac{\phi(M)}{6M\alpha^3}[1+0(\alpha)] = \frac{\pi 3^{-3/4}}{4}\left(\frac{\phi(M)}{12M}\right)^{3/4} n^{-3/4}[1+0\{n^{-1/2+\epsilon}\}].$$

Hence if we let $\sum'$ mean the same as in (1.4) and define

$$U(n) = 1 + \sum_{s=1}^{r} (-1)^s \sum_{p_1 \cdots p_s < n^{1/2}h^{-1}(n)} 1$$

$$S(n) = \sum^{(r)} (-1)^s f_1\left(\exp\left[-\pi\left(\frac{\phi(M)}{12M}\right)^{1/2} n^{-1/2} p_1 \cdots p_s\right]\right) \tag{1.9}$$

($f_1(x)$ is defined by (1.8)), we obtain from Lemmas 1.1, 1.2, 1.3:

THEOREM 2.1.

$$R'(n) = \frac{1}{2\sqrt{2}} 3^{1/4}\left(\frac{\phi(M)}{12M}\right)^{1/4} n^{-3/4}$$
$$\times \exp\left\{\left(\frac{\phi(M)}{M} - T_n\right)^{1/2} n^{1/2}\pi/\sqrt{3} + S(n) + \frac{\log 2}{2}U(n)\right\}$$
$$\times\left[1 + 0\left\{\frac{n^{c/\log\log n}}{h(n)}\right\}\right]$$

where $S(n)$ and $U(n)$ are defined in (1.9) and $h(n)$ is any monotone function $n^{1/12} > h(n) > n^{\epsilon}$. Also T_n is defined by (1.5).

If $n = q^2$ then

$$\frac{\phi(M)}{M} = \frac{\phi(n)}{n} = 1 - \frac{1}{q}$$

We may choose $h(n) = n^{1/12}$, then

$$S(n) = -f_1(e^{(\pi/2\sqrt{3})}), \qquad U(n) = 1;$$

hence

COROLLARY 1.1.

$$R'(q^2) = \frac{1}{2\sqrt{2}} n^{-3/4} 3^{-1/4} \exp\left\{\frac{\pi}{\sqrt{3}}\phi^{1/2}(n) + \frac{\log 2}{2} - f_1(e^{-(\pi/2\sqrt{3})})\right\}[1 + 0(q^{-1/6})].$$

If $n = q_1 q_2^2$ where $q_1 = 0\{n^{1/6-\epsilon}\}$ we can again take $h(n) = n^{1/12}$ and

$$\frac{\phi(M)}{M} = \left(1-\frac{1}{q_1}\right)\left(1-\frac{1}{q_2}\right), \qquad T_n = -\frac{1}{q_2} + \frac{1}{q_1 q_2}$$

$$S(n) = -f_1\left(\exp\left[-\frac{\pi}{2\sqrt{3}}\frac{(q_1-1)^{1/2}}{q_1}\right]\right) + f_1\left(\exp\left[-\frac{\pi}{2\sqrt{3}}(q_1-1)^{1/2}\right]\right),$$

$$U_n = -1+1=$$

and we obtain equation (0.1). Similarly we could consider integers of the form $p_1^{\alpha_1} \cdots p_r^{\alpha_r} p_{r+1}^2$ where $p_1 \cdots p_r = 0\{n^{1/6-\epsilon}\}$ and obtain an asymptotic formula involving sums of 2^r terms. It may well be that this is all one can hope for.

Finally let $R(n)$ denote the number of ways of writing n as the sum of integers coprime to n. Then by very similar methods one obtains

THEOREM 2.2.

$$R(n) = \frac{\pi}{4 \cdot 6^{3/4}}\left(\frac{\phi(M)}{M}\right)^{1/2} n^{-3/4} \exp\left\{\pi\left(\frac{2n}{3}\right)^{1/2}\left(\frac{\phi(M)}{M} - T_n\right)^{1/2} + W(n) + Y(n)\right\}\left\{1 + 0\left\{\frac{n^{-(c/\log\log n)}}{h(n)}\right\}\right\}$$

where

$$W(n) = \frac{1}{2}\sum_{s=1}^{r} \sum_{p_1 \cdots p_s \leq n^{1/2}h^{-1}(n)} \log(p_1 \cdots p_s)$$

$$Y(n) = \sum{}' (-1)^s (p_1 \cdots p_s) g\left(\exp\left[-\frac{\pi}{\sqrt{6}}\left(\frac{\phi(M)}{M} - T_n\right)^{1/2}\right] n^{-1/2} p_1 \cdots p_s\right)$$

and

$$g(x) = \sum_{j=1}^{\infty} j^{-1} \frac{x^j}{1-x^j} = \sum_{m=1}^{\infty} c_m x^m,$$

$$c_m = \sum_{d/m} d^{-1}.$$

We would like to close with a couple of problems. Let us say an integer n is maximal if $R'(n) > R'(m)$ for all $m < n$. It seems likely that the primes are the only maximal numbers however this would imply that between an integer n and $n + n^{1/2+\epsilon}$ there is a prime. Moreover we say that an integer m is minimal if

$R'(m) > R'(n)$ for all $m > n$. To characterize the minimal integers also seems difficult.

This problem is connected with the highly composite and highly abundant numbers [4].

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