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ON SET SYSTEMS HAVING PARADOXICAL COVERING PROPERTIES

By

P. ERDŐS (Budapest), member of the Academy, A. HAJNAL (Budapest), corresponding member of the Academy and E. C. MILNER (Calgary)

1. $\aleph_2$-phenomena. Our set theoretic notation will be standard with one exception. Since this paper is largely concerned with powers of ordinals, the symbol ξ^η will always denote ordinal exponentiation for ordinals ξ, η. Thus, in particular, if $\beta \leqq \alpha$, then $\omega_\alpha^{\omega_\beta}$ is an ordinal $<\omega_{\alpha+1}$. When we use cardinal exponentiation we shall either say so or, if there is no danger of confusion, we write $2^{\aleph_\beta}$ or $\aleph_\alpha^{\aleph_\beta}$ (despite the fact that ω_α and $\aleph_\alpha$ otherwise denote the same object). We shall assume the reader is familiar with the special symbols as defined e.g. in [6] to denote ordinary partition relations, polarized partition relations and square bracket relations.

We begin our discussion by recalling a theorem of MILNER and RADO [13] which asserts that, for any cardinal $\varkappa \geqq \omega$,

$$\xi \nrightarrow (\varkappa^n)^1_{n<\omega} \quad \text{if} \quad \xi < \varkappa^+. \tag{1.1}$$

This implies that $\xi(<\varkappa^+)$ is the union of ω "small" sets A_n $(n<\omega)$, where we mean small in the sense that the order type tp $A_n < \varkappa^n$ $(n<\omega)$. Four our present purposes it is usually more convenient to consider another sequence $B=\langle B_n : n<\omega\rangle$ defined by $B_n = A_0 \cup \ldots \cup A_n$ $(n<\omega)$. The sets B_n are still "small", i.e. tp $B_n < \varkappa^n$ $(n<\omega)$, and they have an additional property, which we call the *ω-covering property*, that the union of any ω of these sets is the whole set ξ. For brevity we shall say that a sequence $B=\langle B_n : n<\omega\rangle$ of subsets of ξ is a *paradoxical decomposition* of ξ if it has the two properties (i) tp $B_n < \varkappa^n$ $(n<\omega)$ and (ii) the ω-covering property. The existence of such a paradoxical decomposition (which is only interesting for $\varkappa^\omega \leqq \xi < \varkappa^+$) implies the polarized partition relation

$$\binom{\omega}{\xi} \nrightarrow \binom{1 \quad \omega}{\varkappa^\omega \quad 1}^{1,1} \quad \text{for} \quad \xi < \varkappa^+, \tag{1.2}$$

and also the square bracket relation

$$\xi \nrightarrow [\varkappa^\omega]^1_{\aleph_0, <\aleph_0} \quad \text{for} \quad \xi < \varkappa^+. \tag{1.3}$$

In our paper [7] we investigated the following problem: Let $\eta < \omega_2$ and let $A=\langle A_\alpha : \alpha < \varkappa\rangle$ be a sequence of subsets of η of length $\varkappa=\omega$ or ω_1 such that each set A_α has order type tp $A_\alpha < \sigma$. Under what conditions can we then assert that there is a subsequence $\langle A_{\alpha_\nu} : \nu < \varrho\rangle$ of length ϱ whose union has a "large" complement in η, say tp $(\eta \setminus \bigcup\{A_{\alpha_\nu} : \nu < \varrho\}) \geqq \tau$? This amounts to an investigation of the polarized partition relation

(1.4) $$\binom{\varkappa}{\eta} \to \binom{1 \quad \varrho}{\sigma \quad \tau}^{1,1}$$

for $\eta < \omega_2$ and $\varkappa = \omega$ or ω_1.

In [7] we gave a complete discussion for (1.4) in the case when η is a power of ω, (although even for this case there remain unresolved questions if the "1" in (1.4) is replaced by a larger finite ordinal). Now in combinatorial set theory most theorems like these have higher cardinal analogues which are usually obtained by replacing each cardinal by its successor. However, when writing [7] we realized that an investigation of (1.4) for the "next higher case", i.e. for $\eta < \omega_3$ and $\varkappa = \omega_1$ or ω_2, leads to entirely different results and problems which we refer to as "$\aleph_2$-phenomena". The main reason why we could not simply extend the results of [7] is that one of the principal tools we used there was the Milner—Rado paradoxical decomposition (1.1) or rather its square bracket analogue (1.3),

$$\xi \nrightarrow [\omega_1^{\omega}]^1_{\aleph_0, <\aleph_0} \quad \text{for} \quad \xi < \omega_2.$$

Now the "higher cardinal" analogue of this is

(1.5) $$\xi \nrightarrow [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0} \quad \text{for} \quad \xi < \omega_3,$$

and this is not true (e.g. it is false if we assume $2^{\aleph_1} = \aleph_2$). We summarize here the $\aleph_2$-phenomena as it relates to the relation (1.5). For $\xi < \omega_2^{\omega_2}$ we do get the expected result, i.e.

(1.6) $$\xi \nrightarrow [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0} \quad \text{for} \quad \xi < \omega_2^{\omega_2}.$$

However, we also have the following.

(1.7) (a) *If* $2^{\aleph_1} = \aleph_2$, *then there is some* $\xi < \omega_3$ *such that*

$$\xi \to [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0}.$$

(b) *It is consistent that*

$$2^{\aleph_1} = \aleph_3 \quad \textit{and} \quad \xi \nrightarrow [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0}$$

holds for all $\xi < \omega_3$.

(1.8) *Both the relations*

$$\omega_2^{\omega_2} \nrightarrow [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0} \quad \textit{and} \quad \omega_2^{\omega_2} \to [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0}$$

are true in different models of set theory. (The relation $\nrightarrow$ holds, e.g. in the constructible universe L, and $\to$ holds e.g. if Chang's conjecture is true.)

These "$\aleph_2$-phenomena" enter into almost all the results and problems considered in this paper, and so it is not possible to give an entirely naive presentation. Although we discovered most of these results as early as 1967, the presentation we give here will rely upon more recent work done by others. In particular we will use the methods worked out in the paper by GALVIN and HAJNAL [9], and we shall give references to other results later.

The remainder of this section will be devoted to a detailed description of the $\aleph_2$-phenomena as it relates to the relation

$$P(\gamma)\colon \binom{\omega_1}{\omega_2^\gamma} \to \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1}$$

for $\gamma<\omega_3$. For the sake of clarity this will be done rather slowly and somewhat redundantly.

Clearly $P(\gamma)$ is equivalent to the assertion: *whenever* $A=\langle A_\alpha:\alpha<\omega_1\rangle$ *is a sequence of subsets of* ω_2^γ *such that* $\operatorname{tp} A_\alpha<\omega_2^{\omega_1}\ (\alpha<\omega_1)$, *then* A *does not have the* ω*-covering property, i.e. there is* $D\in[\omega_1]^\omega$ *such that* $\bigcup\{A_\alpha:\alpha\in D\}\neq\omega_2^\gamma$. On the other hand, in order to establish the negation

$$\neg P(\gamma)\colon \binom{\omega_1}{\omega_2^\gamma} \nrightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1},$$

we have to show that there is some sequence $A=\langle A_\alpha:\alpha<\omega_1\rangle$ of subsets of ω_2^γ such that (i) $\operatorname{tp} A_\alpha<\omega_2^{\omega_1}\ (\alpha<\omega_1)$ and (ii) *A has the ω-covering property.* We shall say briefly that the sequence *A establishes* $\neg P(\gamma)$ if (i) and (ii) hold.

Before we state and prove our first relevant result, it is convenient to introduce some special notation. If $\varkappa$ is an infinite cardinal and $0<\gamma<\varkappa^+$ we choose a fixed sequence $S^\gamma=\langle S_\nu^\gamma:\nu<\mu\rangle$ of subsets of $\varkappa^\gamma$ having the following properties:

(1.9) $$\varkappa^\gamma=\bigcup\{S_\nu^\gamma:\nu<\mu\};$$

(1.10) $$S_0^\gamma<S_1^\gamma<\ldots<S_\nu^\gamma<\ldots,$$

where $X<Y$ means that all the elements of the set X precede all the elements of Y in the ordering of $\varkappa^\gamma$;

(1.11) (a) if $\gamma=\delta+1$, then $\mu=\varkappa$ and $\operatorname{tp} S_\nu^\gamma=\varkappa^\delta\ (\nu<\varkappa)$;

(b) if γ is a limit ordinal, then $\mu=\operatorname{cf}(\gamma)$ and $\operatorname{tp} S_\nu^\gamma=\varkappa^{\gamma_\nu}$, where $\langle\gamma_\nu:\nu<\mu\rangle$ is a fixed increasing sequence of ordinals with limit γ.

We call this sequence S^γ *the standard decomposition* of $\varkappa^\gamma$ (although it depends upon the choicce of the γ_ν in (1.11) (b)).

THEOREM 1.1. *$\neg P(\gamma)$ holds for $\gamma<\omega_3$, i.e.*

(1.12) $$\binom{\omega_1}{\omega_2^\gamma} \nrightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1} \quad \text{for} \quad \gamma<\omega_3.$$

REMARK. The following proof can easily be adapted to prove the more general result Theorem 2.1.

PROOF. We prove the result by induction on γ. For $\gamma<\omega_1$ it is obvious that $\neg P(\gamma)$ holds. Now assume that $\omega_1\leq\gamma<\omega_3$ and distinguish the three cases (i) $\gamma=\delta+1$, (ii) $\operatorname{cf}(\gamma)=\omega$ and (iii) $\operatorname{cf}(\gamma)=\omega_1$.

In the first two cases there is no difficulty in carrying out the inductive step. We give the details here, but in later proofs where a similar type of argument is needed we shall omit the trivial details and simply instruct the reader "to take cross sections". The main idea of the proof of this theorem is in establishing the inductive step for case (iii).

Let $S^\gamma=\langle S^\gamma_\nu:\nu<\mu\rangle$ be the standard decomposition for ω_2^γ. By the induction hypothesis for each $\nu<\mu$ there is a sequence $A^\nu=\langle A^\nu_\alpha:\alpha<\omega_1\rangle$ of subsets of S^γ_ν which establishes $\neg P(\gamma_\nu)$, where $\operatorname{tp} S^\gamma_\nu=\omega_2^{\gamma_\nu}$.

Case 1. In this case $\mu=\omega_2$ and the sets S^γ_ν $(\nu<\omega_2)$ are order isomorphic i.e. $\gamma_\nu=\delta$ $(\nu<\omega_2)$. Therefore, for each $\alpha<\omega_1$ we can assume that the sets A^ν_α are also order isomorphic for $\nu<\omega_2$. Now put $A_\alpha=\bigcup\{A^\nu_\alpha:\nu<\omega_2\}$ $(\alpha<\omega_1)$. For each $\alpha<\omega_1$, there is $f(\alpha)<\omega_1$ such that $\operatorname{tp} A^\nu_\alpha<\omega_2^{f(\alpha)}$ $(\nu<\omega_2)$, and therefore $\operatorname{tp} A_\alpha\leqq\omega_2^{f(\alpha)}<\omega_2^{\omega_1}$. Therefore $A=\langle A_\alpha:\alpha<\omega_1\rangle$ establishes $\neg P(\gamma)$ since each A^ν has the ω-covering property for S^γ_ν $(\nu<\omega_2)$.

Case 2. In this case $\mu=\omega$. Again we define $A_\alpha=\bigcup\{A^\nu_\alpha:\nu<\mu\}$. Then $A=\langle A_\alpha:\alpha<\omega_1\rangle$ has the ω-covering property and moreover

$$\operatorname{tp} A_\alpha=\sum\{\operatorname{tp} A^\nu_\alpha:\nu<\omega\}<\omega_2^{\omega_1}\quad(\alpha<\omega_1),$$

since $\operatorname{tp} A^\nu_\alpha<\omega_2^{\omega_1}$ $(\nu<\omega;\ \alpha<\omega_1)$.

Case 3. In this case $\mu=\omega_1$. For each $\nu<\omega_1$, let $B^\nu=\langle B^\nu_n:n<\omega\rangle$ be a paradoxical decomposition of S^γ_ν as described after (1.1). Then B^ν has the ω-covering property (for S^γ_ν) and $\operatorname{tp} B^\nu_n<\omega_2^n$ $(n<\omega)$. Also, for each $\nu<\omega_1$, let Φ_ν denote any one-to-one function from ν into ω. Now put

$$A_\alpha=\bigcup\{A^\nu_\alpha:\nu\leqq\alpha\}\cup\bigcup\{B^\nu_{\Phi_\nu(\alpha)}:\alpha<\nu<\omega_1\}$$

for $\alpha<\omega_1$. We show that $A=\langle A_\alpha:\alpha<\omega_1\rangle$ establishes $\neg P(\gamma)$.

If $\nu\leqq\alpha<\omega_1$, there is some $f(\nu,\alpha)<\omega_1$ such that $\operatorname{tp} A^\nu_\alpha<\omega_2^{f(\nu,\alpha)}$. Also, there is $f(\alpha)<\omega_1$ such that $f(\nu,\alpha)<f(\alpha)$ for all $\nu\leqq\alpha$. Therefore,

$$\operatorname{tp} A_\alpha\leqq\omega_2^{f(\alpha)}\cdot\alpha+\omega_2^\omega\cdot\omega_1<\omega_2^{\omega_1}\quad(\alpha<\omega_1).$$

All that remains is to verify that A has the ω-covering property. Let $D\in[\omega_1]^\omega$. We must show that

$$A(D)=\bigcup\{A_\alpha:\alpha\in D\}=\omega_2^\gamma.$$

For $\nu<\omega_1$, let $D(\nu)=\{\alpha\in D:\alpha<\nu\}$. Then either $D(\nu)$ or $D\setminus D(\nu)$ is infinite. If $D(\nu)$ is infinite then

$$A(D)\supset\bigcup\{B^\nu_{\Phi_\nu(\alpha)}:\alpha\in D(\nu)\}=S^\gamma_\nu,$$

since B^ν has the ω-covering property. Also, if $D\setminus D(\nu)$ is infinite, then

$$A(D)\supset\bigcup\{A^\nu_\alpha:\alpha\in D\setminus D(\nu)\}=S^\gamma_\nu$$

since A^ν also has the ω-covering property. Thus, in either case $A(D)\supset S^\gamma_\nu$ for each $\nu<\omega_1$. It follows that $A(D)=\omega_2^\gamma$.

The inductive step used in the above proof breaks down completely if $\operatorname{cf}(\gamma)=\omega_2$. The trouble is that, unlike case (i), the sequences $A^\nu=\langle A^\nu_\alpha:\alpha<\omega_1\rangle$ $(\nu<\omega_2)$ obtained from the induction assumption are no longer identical copies of each other.

Our next aim is to say something rather more precise about the order types of the sets A_α of a sequence $A=\langle A_\alpha:\alpha<\omega_1\rangle$ which establishes the negative relation $\neg P(\gamma)$. But in order to state our results we must first recall some definitions from [9] concerning the *rank* of an ordinal function (at least in a generality sufficient for our present purposes).

We denote by Stat (ω_1) the set of all the stationary subsets of ω_1. Let $X\in\mathrm{Stat}(\omega_1)$. Then we define a partial order $<_X$ on ${}^{\omega_1}\omega_1$, the set of all functions from ω_1 into ω_1, by the rule

$$f<_X g \Leftrightarrow \{\alpha\in X: f(\alpha)\geqq g(\alpha)\}\notin \mathrm{Stat}(\omega_1).$$

It is easily seen that $<_X$ is well founded, and because of this we can define the rank function, $\|\cdot\|_X$, by

$$\|f\|_X=\sup\{\|g\|_X+1: g<_X f\}.$$

We shall write $\|\cdot\|$ instead of $\|\cdot\|_{\omega_1}$ and $<$ instead of $<_{\omega_1}$. We need the following easy consequences of this definition (see [9], p. 495).

(1.13) $$(\forall\mu<\omega_1)(\|f\|_X\leqq\mu\Leftrightarrow\{\alpha\in X: f(\alpha)\leqq\mu\}\in\mathrm{Stat}(\omega_1)).$$

(1.14) $$\|f\|_X\leqq\omega_1\Leftrightarrow\{\alpha\in X: f(\alpha)\leqq\alpha\}\in\mathrm{Stat}(\omega_1).$$

We need also the following simple fact:

(1.15) If $X\in\mathrm{Stat}(\omega_1)$, and $\{\alpha\in X: g(\alpha)=f(\alpha)+1\}\notin\mathrm{Stat}(\omega_1)$, then $\|g\|_X=\|f\|_X+1$.

PROOF. Let $h<_X g$. Then $h_1<_X h\Rightarrow h_1<_X f$, and so $\|h\|_X\leqq\|f\|_X$. Thus $\|g\|_X\leqq\|f\|_X+1$. But $f<_X g$ and so $\|f\|_X+1\leqq\|g\|_X$.

Next we define a special sequence of functions $h_\gamma\in{}^{\omega_1}\omega_1$ for $\gamma<\omega_2$ by transfinite recursion on γ. For each limit ordinal $\gamma<\omega_2$ we fix a strictly increasing sequence $\langle\gamma_\nu:\nu<\mu\rangle$ of length $\mu=\mathrm{cf}(\gamma)$ having limit γ. We agree that this is the same sequence as that associated with the standard decomposition for ω_2^γ appearing in (1.11)(b). Now define h_γ by:

$$h_0\equiv 0;\quad h_{\gamma+1}\equiv h_\gamma+1;$$

$$h_\gamma(\alpha)=\sup\{h_{\gamma_n}(\alpha): n<\omega\}\quad\text{if}\quad \mathrm{cf}(\gamma)=\omega;$$

$$h_\gamma(\alpha)=\sup\{h_{\gamma_\nu}(\alpha):\nu<\alpha\}\quad\text{if}\quad \mathrm{cf}(\gamma)=\omega_1.$$

The function h_γ defined in the case $\mathrm{cf}(\gamma)=\omega_1$ is called the *diagonal supremum* of the h_{γ_ν} $(\nu<\omega_1)$. Note that, if $X\in\mathrm{Stat}(\omega_1)$, if h_γ is the supremum or the diagonal supremum of certain h_{γ_ν}, and if $g<_X h_\gamma$, then $g<_Y h_{\gamma_{\nu_0}}$ for some $\nu_0<\mathrm{cf}(\gamma)$ and $Y\in\mathrm{Stat}(\omega_1)$. This fact ensures that $h_\gamma\restriction X$ is "the γ-th function on X" for any $X\in\mathrm{Stat}(\omega_1)$, i.e.

(1.16) $$\|h_\gamma\|_X=\gamma\quad\text{for}\quad\gamma<\omega_2\quad\text{and}\quad X\in\mathrm{Stat}(\omega_1).$$

As a corollary of this we have, for $\gamma<\omega_2$, $X\in\mathrm{Stat}(\omega_1)$ and $f\in{}^{\omega_1}\omega_1$,

(1.17) $$\|f\|_X\leqq\gamma\Leftrightarrow\{\alpha\in X: f(\alpha)\leqq h_\gamma(\alpha)\}\in\mathrm{Stat}(\omega_1);$$

also,

(1.18) $$h_\gamma<_X h_\delta\quad\text{for}\quad\gamma<\delta<\omega_2.$$

We make one final remark. For a limit ordinal γ $(\omega<\gamma<\omega_2)$ the above sequence $\langle\gamma_\nu:\nu<\mathrm{cf}(\gamma)\rangle$ can be chosen so that $\omega\leqq\gamma_0<\gamma_1<\dots$. This ensures that

(1.19) $$h_\gamma(\alpha)\geqq\omega\quad(0<\alpha<\omega_1;\ \omega\leqq\gamma<\omega_2).$$

We shall also make use of another, stronger partial ordering on ${}^{\omega_1}\omega_1$, $\ll$, defined by

$$f \ll g \Leftrightarrow |\{\alpha < \omega_1 : f(\alpha) \geqq g(\alpha)\}| \leqq \aleph_0,$$

i.e. g eventually exceeds f. Again, it is easily seen that $\ll$ is wellfounded and $f \ll g \Rightarrow f <_X g$ for any $X \in \mathrm{Stat}(\omega_1)$. The functions h_γ $(\gamma < \omega_2)$ defined above are also increasing in this stronger sense, i.e.

$$h_0 \ll h_1 \ll \ldots \ll h_\gamma \ll \ldots . \tag{1.20}$$

We can associate with any sequence $A = \langle A_\alpha : \alpha < \omega_1 \rangle$ of sets of ordinal numbers, an ordinal function f^A defined by

$$f^A(\alpha) = \min\{\delta : \mathrm{tp}\, A_\alpha < \omega_2^\delta\}.$$

Note that, if $A_\alpha \neq \emptyset$, then $f^A(\alpha) = \sigma_\alpha + 1$ for some ordinal σ_α. Also, if A establishes $\neg P(\gamma)$ for some γ, then $f^A \in {}^{\omega_1}\omega_1$. The next theorem shows that, if A establishes $\neg P(\gamma)$ for some large γ, then the associated function f^A is also large in some sense.

THEOREM 1.2. *Let* $\gamma < \omega_2$. *If* $A = \langle A_\alpha : \alpha < \omega_1 \rangle$ *is a sequence of subsets of* ω_2^γ *such that* $\|f^A\| \geqq \gamma$, *then* A *does not have the* ω_1 *covering property.*

PROOF. We prove this by induction on γ. It is trivial for $\gamma = 0$ since, by (1.13), the hypothesis implies that $A_\alpha = \emptyset$ for a stationary set of α's. Now assume $\gamma > 0$.

Case 1. γ is a limit ordinal. We can assume that the sets A_α are non-empty for all but countably many α, and so $f^A(\alpha) = g(\alpha) + 1$ for all but a countable number of α. Therefore, by (1.15), $\|f^A\| = \|g\| + 1$ and hence $\|f^A\| \leqq \gamma' < \gamma$. By the induction hypothesis $A' = \langle A_\alpha \cap \omega_2^{\gamma'} : \alpha < \omega_1 \rangle$ does not have the ω_1-covering property and hence neither does A.

Case 2. $\gamma = \delta + 1$. Let $\langle S_\nu^\gamma : \nu < \omega_2 \rangle$ be the standard decomposition for ω_2^γ. As in Case 1, $f^A(\alpha) = g(\alpha) + 1$ for all but countably many α's and $\|f^A\| = \|g\| + 1 \leqq \delta + 1$, so $\|g\| \leqq \delta$. Now for each α there is $\nu(\alpha) < \omega_2$ such that

$$\mathrm{tp}\,(A_\alpha \cap S_\nu^\gamma) < \omega_2^{g(\alpha)} \quad \text{for} \quad \nu(\alpha) < \nu < \omega_2.$$

There is $\nu_0 < \omega_2$ such that $\nu(\alpha) < \nu_0$ for all $\alpha < \omega_1$. Consider the sequence $A' = \langle A_\alpha \cap S_{\nu_0}^\gamma : \alpha < \omega_1 \rangle$. Clearly[1] $f^{A'} \lesssim g$ and so $\|f^{A'}\| \leqq \|g\| \leqq \delta$. Therefore, by the induction hypothesis A' does not have the ω_1-covering property and hence neither does A.

COROLLARY 1.3.

$$\begin{pmatrix} \omega_1 \\ \omega_2^\gamma \end{pmatrix} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^\gamma & 1 \end{pmatrix}^{1,1}$$

holds for $\gamma < \omega_1$.

PROOF. If $A = \langle A_\alpha : \alpha < \omega_1 \rangle$ is any sequence of subsets of ω_2^γ such that $\mathrm{tp}\, A_\alpha < \omega_2^\gamma$, then $f^A(\alpha) \leqq h_\gamma(\alpha) = \gamma$ $(\alpha < \omega_1)$. Hence $\|f^A\| \leqq \|h_\gamma\| = \gamma$ and so A does not have the ω_1-covering property.

[1] Naturally, $f_1 \lesssim f_2$ means $|\{\alpha : f_1(\alpha) > f_2(\alpha)\}| \leqq \aleph_0$.

COROLLARY 1.4. *If* $\|f\|<\omega_2$ *for all* $f\in{}^{\omega_1}\omega_1$, *then*

$$\binom{\omega_1}{\omega_2^{\omega_2}} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1}.$$

PROOF. Let $A=\langle A_\alpha:\alpha<\omega_1\rangle$ be a sequence of subsets of $\omega_2^{\omega_2}$ such that $\operatorname{tp} A_\alpha<\omega_2^{\omega_1}$. Then $f^A\in{}^{\omega_1}\omega_1$ and so $\|f^A\|<\gamma$ for some $\gamma<\omega_2$. It follows that $\|f^{A'}\|<\gamma$, where $A'=\langle A_\alpha\cap\omega_2^\gamma:\alpha<\omega_1\rangle$. By the theorem A' does not have the ω_1-covering property and so neither does A.

It is easily seen that Theorem 1.2 is best possible for $\gamma<\omega_2$ since there is a system $A^\gamma=\langle A_\alpha^\gamma:\alpha<\omega_1\rangle$ which establishes $\neg P(\gamma)$ and is such that $f^A(\alpha)\leqq h_{\gamma+1}(\alpha)$ $(\alpha<\omega_1)$ and hence by (1.16) and (1.17), $\|f^A\|\leqq\gamma+1$. This result can be proved by exactly the same induction argument that we used to prove Theorem 1.1; we only have to make sure that the A_α^ν chosen in the various places have order types less than $\omega_2^{h_{\gamma_\nu}(\alpha)+1}$ and this ensures that the A_α defined there have order types less than $\omega_2^{h_\nu(\alpha+1)}$. We omit the details since this result is also a Corollary of the following more general result Theorem 1.5.

We make one preliminary remark. We say that a function $g\in{}^{\omega_1}\omega_1$ *establishes* the negative relation $\neg P(\gamma)$,

$$\binom{\omega_1}{\omega_2^\gamma} \not\to \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1},$$

if there is an $A=\langle A_\alpha:\alpha<\omega_1\rangle$ which establishes it and is such that $f^A(\alpha)\leqq g(\alpha)$ for all $\alpha<\omega_1$. Now if g establishes $\neg P(\gamma)$ and $g\leqq h$, then the function h_1 defined by $h_1(\alpha)=\max\{h(\alpha),\omega\}$ also establishes $\neg P(\gamma)$. For suppose $A=\langle A:\alpha<\omega_1\rangle$ establishes $\neg P(\gamma)$ and $f^A(\alpha)\leqq g(\alpha)$ $(\alpha<\omega_1)$. Then there is $\alpha_0<\omega_1$ so that $f^A(\alpha)\leqq h(\alpha)$ $(\alpha_0\leqq\alpha<\omega_1)$. Let $\langle B_n:n<\omega\rangle$ be a paradoxical decomposition of ω_2^γ and consider the system $A'=\langle A'_\alpha:\alpha<\omega_1\rangle$ defined by

$$A'_\alpha=\begin{cases} A_\alpha & \text{for} \quad \alpha_0\leqq\alpha<\omega_1, \\ B_{\psi(\alpha)} & \text{for} \quad \alpha<\alpha_0, \end{cases}$$

where ψ is any one-to-one map from α_0 into ω. Clearly A' establishes $\neg P(\gamma)$ and $f^{A'}(\alpha)\leqq h_1(\alpha)$ $(\alpha<\omega_1)$. Thus h_1 also establishes $\neg P(\gamma)$. It follows from this that, if $g,h\in{}^{\omega_1}\omega_1$, g establishes $\neg P(\gamma)$, $g\ll h$ and $h(\alpha)\geqq\omega$ $(\alpha<\omega_1)$, then h also establishes $\neg P(\gamma)$.

THEOREM 1.5. *Let* $\gamma<\omega_2$ *and suppose that* $\langle f_\sigma:\sigma\leqq\gamma\rangle$ *is a strongly increasing sequence of infinite-valued functions, i.e.* $f_\sigma(\alpha)\geqq\omega$ $(\sigma\leqq\gamma;\ \alpha<\omega_1)$ *and* $f_0\ll f_1\ll\dots$. *Then* $f_\gamma+1$ *establishes* $\neg P(\gamma)$.

PROOF. This is trivial for $\gamma=0$. We now assume that $\gamma>0$ and use induction on γ.

Let $\langle S_\nu^\gamma:\nu<\mu\rangle$ be the standard decomposition for ω_2^γ, where $\operatorname{tp} S_\nu^\gamma=\omega_2^{\gamma_\nu}$ $(\nu<\mu)$. Then $\gamma_\nu<\gamma$ $(\nu<\mu)$ and so by the induction hypothesis there is a system $A^\nu=\langle A_\alpha^\nu:\alpha<\omega_1\rangle$ of subsets of S_ν^γ which has the ω-covering property and is such that

$$\operatorname{tp} A_\alpha^\nu<\omega_2^{f_{\gamma_\nu}(\alpha)+1} \quad (\alpha<\omega_1;\ \nu<\mu).$$

Case 1. $\gamma=\delta+1$. In this case $\mu=\omega_2$ and $\gamma_\nu=\delta$ $(\nu<\omega_2)$. Put $A_\alpha=\bigcup\{A_\alpha^\nu:\nu<\omega_2\}$ $(\alpha<\omega_1)$. Then $A=\langle A_\alpha:\alpha<\omega_1\rangle$ has clearly the ω-covering property. Moreover,

$$\operatorname{tp} A_\alpha \leqq \omega_2^{f_\delta(\alpha)+1} \quad (\alpha<\omega_1).$$

Therefore $f_\delta+2$ establishes $\neg P(\gamma)$ and $f_\delta+2\leqq f_\gamma+1$. Therefore by the remark preceding the theorem $f_\gamma+1$ also establishes $\neg P(\gamma)$.

Case 2. $\operatorname{cf}(\gamma)=\omega$. In this case $\mu=\omega$ and $\gamma_\nu\nearrow\gamma$. Again put $A_\alpha=\bigcup\{A_\alpha^\nu:\nu<\omega\}$ $(\alpha<\omega_1)$. Then $A=\langle A_\alpha:\alpha<\omega_1\rangle$ has the ω-covering property and

$$\operatorname{tp} A_\alpha \leqq \omega_2^{g(\alpha)},$$

where $g(\alpha)=\sup\limits_{\nu<\omega}\left(f_{\gamma_\nu}(\alpha)+1\right)$ $(\alpha<\omega_1)$. Hence $g+1$ establishes $\neg P(\gamma)$ and therefore $f_\gamma+1$ also establishes $\neg P(\gamma)$ since $g\leqq f_\gamma$.

Case 3. $\operatorname{cf}(\gamma)=\omega_1$. In this case $\mu=\omega_1$ and $\gamma_\nu\nearrow\gamma$. Let $B^\nu=\langle B_n^\nu:n<\omega\rangle$ be a paradoxical decomposition for S_ν^γ $(\nu<\omega_1)$, and for each $\nu<\omega_1$ let ψ_ν be a one-to-one map from $\nu+1$ into ω. Put

$$\theta(\alpha)=\min\left(\{\alpha\}\cup\{\varrho<\alpha:\ f_{\gamma_\varrho}\geqq f_\gamma(\alpha)\}\right) \quad (\alpha<\omega_1),$$

and define

$$\tau_\nu=\sup\{\beta<\omega_1:\ \theta(\beta)\leqq\nu\} \quad (\nu<\omega_1).$$

Now define

$$A_\alpha=\bigcup\{A_\nu^\gamma(\alpha):\ \nu<\theta(\alpha)\}\cup\bigcup\{B^\nu_{\psi_{\tau_\nu}(\alpha)}:\ \theta(\alpha)\leqq\nu\leqq\omega_1\}.$$

First we observe that for $\nu<\omega_1$ there are only countably many ordinals $\beta<\omega_1$ which satisfy $\theta(\beta)\leqq\nu$. Otherwise, there would be ordinals β_0 $(\sigma<\omega_1)$ so that $\theta(\beta_\sigma)=\theta\leqq\nu<\beta_0<\beta_1<\ldots<\beta_\sigma<\ldots<\omega_1$. But this implies that $f_{\gamma_\theta}(\beta_\sigma)\geqq f_\gamma(\beta_\sigma)$ for $\sigma<\omega_1$, a contradiction against the hypothesis $f_{\gamma_\theta}\ll f_\gamma$. It follows that there are only countably many $\beta<\omega_1$ for which $\theta(\beta)\leqq\nu$ and so $\tau_\nu<\omega_1$ $(\nu<\omega_1)$. Moreover, if $\alpha<\omega_1$ and $\theta(\alpha)\leqq\nu<\omega_1$, then $\tau_\nu\geqq\alpha$. Thus $\psi_{\tau_\nu}(\alpha)$ is defined and the above definition for A_α is meaningful.

Now we have

$$\operatorname{tp} A_\alpha \leqq \sum\{\omega_2^{f_{\gamma_\nu}(\alpha)+1}:\ \nu<\theta(\alpha)\}+\omega_2^\omega\omega_1\} \quad (\alpha<\omega_1),$$

and since $f_{\gamma_\nu}(\alpha)<f_\gamma(\alpha)$ for $\nu<\theta(\alpha)$, it follows that

$$\operatorname{tp} A_\alpha < \omega_2^{f_\gamma(\alpha)+1} \quad (\alpha<\omega_1).$$

To complete the proof in this case it is enough to verify that $A(D)=\bigcup\{A_\alpha:\alpha\in D\}\supset S_\nu^\gamma$ whenever $\nu<\omega_1$ and $D\in[\omega_1]^\omega$. Let $D_1=\{\alpha\in D:\nu<\theta(\alpha)\}$. If D_1 is infinite, then $S(D)\supset\bigcup\{A_\nu^\gamma(\alpha):\alpha\in D_1\}=S_\nu^\gamma$ since A^ν has the ω-covering property. On the other hand, if D_1 is finite, $S(D)\supset\bigcup\{B^\gamma_{\psi_\nu(\alpha)}:\alpha\in D\setminus D_1\}=S_\nu^\gamma$ since B^ν has the ω-covering property.

Case 4. $\operatorname{cf}\gamma=\omega_2$. In this case $\mu=\omega_2$ and $\gamma_\nu\nearrow\gamma$. Now for each $\nu<\omega_2$ there is $\beta_\nu<\omega_1$ such that

$$f_{\gamma_\nu}(\alpha)<f_\gamma(\alpha) \quad (\beta_\nu\leqq\alpha<\omega_1).$$

Also, there is $\beta<\omega_1$ such that $\beta_\nu=\beta$ for $\aleph_2$ different values of $\nu<\omega_2$. Now put

$S^* = \bigcup\{S_\nu^\gamma : \beta_\nu = \beta\}$, $A_\alpha^* = \bigcup\{A_\alpha^\nu : \beta_\nu = \beta\}$ $(\alpha < \omega_1)$. Then $A^* = \langle A_\alpha^* : \alpha < \omega_1 \rangle$ clearly has the ω-covering property for S^*. Moreover, $\operatorname{tp} S^* = \omega_2^\gamma$ and

$$\operatorname{tp} A_\alpha^* \leqq \sum \{\omega_2^{f_{\gamma_\nu}(\alpha)+1} : \beta_\nu = \beta\} \leqq \omega_2^{g(\alpha)},$$

where $g(\alpha) \equiv f_\gamma(\alpha)$ $(\beta \leqq \alpha < \omega_1)$. Thus $g+1$ establishes $\neg P(\gamma)$ and hence so does $f_\gamma + 1$ since $g \ll f_\gamma$.

COROLLARY 1.6. *For* $\gamma < \omega_2$ *the function* $h_\gamma + 1$ *establishes* $\neg P(\gamma)$.

PROOF. For $\gamma < \omega_1$ this is obvious since $h_\gamma \equiv \gamma$. For $\gamma \geqq \omega_1$ the result follows from the theorem and the observation that $h_\omega \ll h_{\omega+1} \ll \ldots \ll h_\gamma$ is a strongly increasing sequence of length γ and the values $h_\gamma(\alpha)$ are all infinite for $\alpha < \omega_1$ and $\omega \leqq \nu$ by (1.19).

COROLLARY 1.7. *If there is a function* $h \in {}^{\omega_1}\omega_1$ *so that* $h_\nu \ll h$ *for all* $\nu < \omega_2$, *then*

$$\binom{\omega_1}{\omega_2^{\omega_1}} \nrightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1}.$$

PROOF. This follows from the theorem and the fact that

$$h_0 \ll h_1 \ll \ldots \ll h_\gamma \ll \ldots \ll h \quad (\gamma < \omega_2).$$

The results of this section concerning the $\aleph_2$-phenomena for the relation

$$P(\gamma) : \binom{\omega_1}{\omega_2^\gamma} \rightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1}$$

are summarized in the following theorem.

THEOREM 1.8. (a) $\neg P(\gamma)$ *holds for* $\gamma < \omega_2$.

(b) *If* $\gamma < \omega_3$ *and there is a strongly increasing sequence* $\langle f_\sigma : \sigma \leqq \gamma \rangle$ *of length* $\gamma + 1$ *in* ${}^{\omega_1}\omega_1$ *(i.e.* $f_0 \ll f_1 \ll \ldots \ll f_\gamma$*), then* $\neg P(\gamma)$.

(c) *If* $2^{\aleph_1} = \aleph_2$, *then there is some* $\gamma < \omega_3$ *such that* $P(\gamma)$ *is true.*

(d) *It is consistent that* $2^{\aleph_1} > \aleph_2$ *and* $\neg P(\gamma)$ *for all* $\gamma < \omega_3$.

(e) $P(\omega_2)$ *fails in* L *and* $P(\omega_2)$ *holds if Chang's conjecture is true.*

PROOF. (a) and (b) are respectively Theorems 1.1 and 1.5. The first part of (e) follows from Corollary 1.7 since, by a theorem of BAUMGARTNER [1] in L there is an $h \in {}^{\omega_1}\omega_1$ such that $h_\nu \ll h$ for all $\nu < \omega_2$. The second part of (e) follows from Corollary 1.4 and a result of BAUMGARTNER [2] and BENDA [4] which tells us that Chang's conjecture implies $\|f\| < \omega_2$ for all $f \in {}^{\omega_1}\omega_1$. (d) follows from Theorem 1.5 and a theorem of LAVER [11] (and BAUMGARTNER [2]) which says that it is consistent with ZFC and $2^{\aleph_1} > \aleph_2$ that whenever $F \subset {}^{\omega_1}\omega_1$, $|F| = \aleph_2$ then there is some $g \in {}^{\omega_1}\omega_1$ which eventually majorizes every $f \in F$, i.e. $f \ll g$ for all $f \in F$.

To see (c) let us remark that by Lemma 3 of [9] $\|\underline{\omega_1}\| < (2^{\aleph_1})^+$, where $\underline{\omega_1}$ is the constant function ω_1. Hence by the hypothesis of (c), $\|f\| \leqq \|\underline{\omega_1}\| = \xi < \omega_3$ for all $f \in {}^{\omega_1}\omega_1$ and then by Theorem 1.2, $P(\xi)$ is true.

We do not know if the converse of Theorem 1.5 is true, i.e. *if* $\gamma < \omega_3$ *and* $\neg P(\gamma)$ *holds, does it follow that there is a strongly increasing sequence of functions* $f_\sigma \in {}^{\omega_1}\omega_1$ $(\sigma \leqq \gamma)$ *of length* $\gamma + 1$? However, as the next theorem shows, we can prove that under the stated hypothesis there is a *weakly* increasing sequence of length $\gamma + 1$.

THEOREM 1.9. *If* $\gamma<\omega_3$ *and* $\neg P(\gamma)$, *then there are functions* $f_\sigma\in{}^{\omega_1}\omega_1$ $(\sigma\leqq\gamma)$ *such that* $f_0<f_1<\ldots<f_\gamma$.

This is an immediate consequence of Theorem 1.2 and the following theorem on the rank function which has an independent interest.

THEOREM 1.10. *Let* $f\in{}^{\omega_1}\omega_1$, $\|f\|=\gamma<\omega_3$. *Then there are functions* $f_\sigma\in{}^{\omega_1}\omega_1$ $(\sigma\leqq\gamma)$ *such that* $f_0<f_1<\ldots<f_\gamma$, *and moreover,*

$$f_\gamma(\alpha)=\omega^{\omega^{f(\alpha)}}f(\alpha)\quad(\alpha<\omega_1). \tag{1.21}$$

PROOF. For any function $g\in{}^{\omega_1}\omega_1$, let $\hat g$ denote the function ω^{ω^g}. The result is true for $\gamma\leqq\omega_2$ by (1.17) and (1.18) since $h_\nu<f<\hat f\cdot f$ for $\nu<\gamma$. We now prove the theorem by transfinite induction on γ. Assume $\omega_2<\gamma<\omega_3$ and that the result holds for all smaller ordinals. We distinguish the two cases (1) $\gamma=\delta+1$, (2) γ is a limit ordinal.

Before giving the induction details we make a remark about the choice of f_γ in (1.21). We use two elementary facts about ordinal exponentiation

(a) $\xi<\eta\Rightarrow\omega^\xi<\omega^\eta$,

(b) $\varrho<\omega^{\omega^\xi}\Rightarrow\varrho^2<\omega^{\omega^\xi}$.

Property (b) actually characterizes ordinals of the form ω^{ω^ξ}, and it is precisely this which allows our induction proof to work. To see (b), suppose $\varrho<\omega^{\omega^\xi}$. If $\xi=0$, then $\varrho<\omega$ and $\varrho^2<\omega$. If $\xi>0$, then ω^ξ is a limit ordinal and so $\varrho<\omega^\sigma$ for some $\sigma<\omega^\xi$. Then $\varrho^2\leqq\omega^{\sigma\cdot 2}<\omega^{\omega^\xi}$ by (a).

Case 1. $\gamma=\delta+1$. There is $f'<f$ such that $\|f'\|=\delta$. Now the result follows immediately from the induction hypothesis since $\hat{f'}\cdot f'<\hat f\cdot f$.

Case 2. γ a limit ordinal. Let $\mathrm{cf}(\gamma)=\mu$. By assumption $\mu=\omega$, ω_1 or ω_2. Let $\langle\gamma_\nu:\nu<\mu\rangle$ be the fixed increasing sequence of ordinals with limit γ mentioned in (1.11) (b), and let $f_\nu\in{}^{\omega_1}\omega_1$ $(\nu<\mu)$ be functions such that $f_\nu<f$ and $\|f_\nu\|=\gamma$ $(\nu<\mu)$. By the induction hypothesis, there are functions $f_\sigma^\nu\in{}^{\omega_1}\omega_1$ $(\sigma<\gamma_\nu)$ such that

$$f_0^\nu<f_1^\nu<\ldots<f_\sigma^\nu<\ldots<\hat f_\nu\cdot f_\nu\quad(\sigma<\gamma_\nu).$$

Let $N=\{(\sigma,\nu):\sigma<\gamma_\nu\wedge\nu<\mu\}$. Then the order type of N under the usual anti-lexicographic ordering, $<_0$, is $\mathrm{tp}\,N(<_0)=\gamma$. Thus it is sufficient to define functions $f_{(\sigma,\nu)}$ for $(\sigma,\nu)\in N$ so that $f_{(\sigma,\nu)}<\hat f\cdot f$ and

$$(\sigma,\nu)<_0(\sigma',\nu')\Rightarrow f_{(\sigma,\nu)}<f_{(\sigma',\nu')}. \tag{1.22}$$

For any ordinals ξ,η, there is an order preserving map $\varphi_{\xi,\eta}$ from $\xi\times\eta$ (ordered by $<_0$) onto the ordinal $\xi\cdot\eta$.

We now define $f_{(\sigma,\nu)}\in{}^{\omega_1}\omega_1$ for $(\sigma,\nu)\in N$ by

$$f_{(\sigma,\nu)}(\alpha)=\begin{cases}\varphi_{\hat f(\alpha),f(\alpha)}(f_\sigma^\nu(\alpha),h_\nu(\alpha)) & \text{if}\quad (f_\sigma^\nu(\alpha),h_\nu(\alpha))\in\hat f(\alpha)\times f(\alpha),\\ 0, & \text{otherwise.}\end{cases}$$

Note that

$$f_{(\sigma,\nu)}(\alpha)=\varphi_{\hat f(\alpha),f(\alpha)}(f_\sigma^\nu(\alpha),h_\nu(\alpha))$$

holds for all but a non-stationary set of α's, since $f_\sigma^\nu<\hat f_\nu\cdot f_\nu<\hat f_\nu\cdot f_\nu<\hat f$ and $h_\nu<f$.

Thus $f_{(\sigma,\nu)} < f \cdot f$ for $(\sigma, \nu) \in N$. Also, if $(\sigma, \nu), (\sigma', \nu') \in N$ and $(\sigma, \nu) <_0 (\sigma', \nu')$, then either (i) $\nu < \nu'$ or (ii) $\nu = \nu'$ and $\sigma < \sigma'$, and hence

$$(f_\sigma^\nu(\alpha), h_\nu(\alpha)) <_0 (f_{\sigma'}^{\nu'}(\alpha), h_{\nu'}(\alpha))$$

holds for all but a non-stationary set of α's. Thus $f_{(\sigma,\nu)} < f_{(\sigma',\nu')}$.

We remark that, analogously to the rank functions $\|\cdot\|$ defined before (1.13), we could also define a rank function $|||\cdot|||$ corresponding to the partial well-ordering $\ll$. This rank function has not, however, been so thoroughly investigated as the ordinary rank function $\|\cdot\|$. The reason for this is that the ideal of the non-stationary sets is normal and this accounts for many of the pleasant properties of $\|\cdot\|$. But the ideal of the countable sets is not normal, and we cannot expect $|||\cdot|||$ to behave so well. In particular the functions h_ν do not have the nice properties (1.17), (1.18) for this new rank.

It is clear that $|||f||| \leq \|f\|$ since $g \ll h \Rightarrow g < h$, we and remark that we could improve Theorem 1.2 slightly by replacing $\|f\|$ by $|||f|||$. However, this would not help to solve the problem mentioned before Theorem 1.9 since it is not known (in ZFC) whether $|||f||| = \omega_2$ and $f \in {}^{\omega_1}\omega_1$ implies the existence of a strongly increasing sequence of functions $f_\sigma \in {}^{\omega_1}\omega_1$ $(\sigma \leq \omega_2)$ of length ω_2.

2. Some extensions of the results of the previous section. General lemmas. In § 3 and § 4 we are going to give discussions of the relations

$$\binom{\omega_1}{\omega_2^\delta} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^\sigma & \omega_2^\tau \end{pmatrix}^{1,1} \tag{2.1}$$

and

$$\binom{\omega_1}{\omega_2^\delta} \to \begin{pmatrix} 1 & \omega_0 \\ \omega_2^\sigma & \omega_2^\tau \end{pmatrix}^{1,1} \tag{2.2}$$

for $\delta, \tau < \omega_3$ and $\sigma < \omega_1$. The restriction to the case $\sigma < \omega_1$ is not entirely necessary, but an analysis for the case $\sigma \geq \omega_1$ will inevitably be complicated by the same kind of $\aleph_2$-phenomena that we encountered in § 1 in connection with the case $\sigma = \omega_1$, $\tau = 0$. In fact most of the results in § 1 find natural extensions to higher order types and we begin this section with a brief indication of these.

The following is an easy extension of Theorem 1.1.

THEOREM 2.1. *If* $\sigma < \omega_3$, $\mathrm{cf}(\sigma) = \omega_1$ *and* $\gamma < \omega_2$ *then*

$$\binom{\omega_1}{\omega_2^{\sigma+\gamma}} \nrightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^\sigma & 1 \end{pmatrix}^{1,1}.$$

PROOF. We prove this only for the case $\gamma = 0$. The general result follows by induction on γ just as in the proof of Theorem 1.1. Let $\langle S_\nu : \nu < \omega_1 \rangle$ be the standard decomposition of ω_2^σ as described in § 1. For $\nu < \omega_1$, let $\langle B_n : n < \omega \rangle$ be a paradoxical decomposition of S_ν and let φ_ν be a one-to-one map from ν into ω. Now consider the system $\langle A_\alpha : \alpha < \omega_1 \rangle$ of subsets of ω_2^σ, where

$$A_\alpha = \bigcup \{S_\nu : \nu \leq \alpha\} \cup \bigcup \{B_{\varphi_\nu(\alpha)}^\nu : \alpha < \nu < \omega_1\}.$$

This system clearly establishes Theorem 2.1. for $\gamma = 0$.

There is a sharpening of this last theorem which is analogous to Theorem 1.5.

THEOREM 2.2. *Let* $\sigma<\omega_2$, $\operatorname{cf}(\sigma)=\omega_1$, *and let* $\langle\sigma_\nu:\nu<\omega_1\rangle$ *be a closed, cofinal, strictly increasing sequence in* σ. *Let* $\gamma<\omega_3$ *and suppose that* $\langle f^\nu_\gamma:\nu\leqq\gamma\rangle$ *is a strongly increasing sequence of functions in* ${}^{\omega_1}\omega_1$ *of length* $\gamma+1$. *Then there is a system* $A=\langle A_\alpha:\alpha<\omega_1\rangle$ *of* ω_1 *subsets of* $\omega_2^{\sigma+\gamma}$ *having the* ω*-covering property and such that*

$$\operatorname{tp} A_\alpha<\omega_2^{\sigma_\alpha+f^\gamma_\gamma(\alpha)+1}\quad(\alpha<\omega_1).$$

PROOF. The case $\gamma=0$ follows from the last proof, for, by the definition of A_α above, we have

$$\operatorname{tp} A_\alpha\leqq\omega_2^{\sigma_\alpha}+\omega_2^{\omega}\omega_1<\omega_2^{\sigma_\alpha+1}\leqq\omega_2^{\sigma_\alpha+f^0_0(\alpha)+1}.$$

Now an induction argument similar to that used to prove Theorem 1.5 works here as well.

Theorem 1.2 has a similar generalization.

THEOREM 2.3. *Let* $\omega\leqq\tau<\omega_3$, $\gamma<\omega_3$, $f\in{}^{\omega_1}\omega_1$. *If* $\|f\|\leqq\gamma$ *and* $A=\langle A_\alpha:\alpha<\omega_1\rangle$ *is a sequence of subsets of* $\omega_2^{\tau+\gamma}$ *such that*

$$\operatorname{tp} A_\alpha<\omega_2^{\omega+f(\alpha)}\quad(\alpha<\omega_1),$$

then there is $D\in[\omega_1]^{\omega_1}$ *such that*

$$\operatorname{tp}(\omega_2^{\tau+\gamma}\setminus\cup\{A_\alpha:\alpha\in D\})\geqq\omega_2^{\tau}. \tag{2.3}$$

PROOF. For $\gamma=0$ this is obvious since $f(\alpha)=0$ for all but a non-stationary set of α's and hence there is an uncountable set $D\subset\omega_1$ such that $\operatorname{tp} A_\alpha<\omega_2^n$ $(\alpha\in D)$, for some fixed $n<\omega$. This implies (2.3). The general result follows by an induction argument similar to that used to prove Theorem 1.2.

These results enable us to state generalizations of Theorem 1.8. Thus, from Theorem 2.2 it follows that

$$\begin{pmatrix}\omega_1\\ \omega_2^{\sigma+\omega_2}\end{pmatrix}\nrightarrow\begin{pmatrix}1 & \omega\\ \omega_2^{\sigma} & 1\end{pmatrix}^{1,1} \tag{2.4}$$

holds in L if $\sigma<\omega_3$ and $\operatorname{cf}\sigma=\omega_1$. Whereas, from Theorem 2.3 we see that Chang's conjecture implies that

$$\begin{pmatrix}\omega_1\\ \omega_2^{\tau+\omega_2}\end{pmatrix}\rightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega_1} & \omega_2^{\tau}\end{pmatrix}^{1,1} \tag{2.5}$$

holds for all $\tau<\omega_3$. It is interesting to note that (2.5) and $2^{\aleph_0}=\aleph_1$ implies a strong negation of (2.4) with $\sigma=\omega_1$, namely

$$\begin{pmatrix}\omega_1\\ \omega_2^{\omega_2}\end{pmatrix}\rightarrow\begin{pmatrix}1 & \omega\\ \omega_2^{\omega_1} & \omega_2^{\omega_2}\end{pmatrix}^{1,1}. \tag{2.6}$$

For suppose that (2.6) is false. Then there is a sequence $A=\langle A_\alpha:\alpha<\omega_1\rangle$ of subsets of $\omega_2^{\omega_2}$ such that $\operatorname{tp} A_\alpha<\omega_2^{\omega_1}$ $(\alpha<\omega_1)$ and such that

$$\operatorname{tp}(\omega_2^{\omega_2}\setminus\cup\{A_\alpha:\alpha\in D\})<\omega_2^{\omega_2}$$

holds for all $D\in[\omega_1]^\omega$. In view of the hypothesis $2^{\aleph_0}=\aleph_1$, it follows from this that there is some $\tau<\omega_2$ such that

$$\operatorname{tp}(\omega_2^{\omega_2}\setminus\bigcup\{A_\alpha\colon \alpha\in D\})<\omega_2^\tau$$

also holds for all $D\in[\omega_1]^\omega$. But this contradicts (2.5). Hence (2.6) is consistent (with Chang's conjecture and say G. C. H.). However, we do know that the stronger relation

$$\begin{pmatrix}\omega_1\\ \omega_2^{\omega_2}\end{pmatrix}\to\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega_1} & \omega_2^{\omega_2}\end{pmatrix}^{1,1}$$

is false assuming $2^{\aleph_1}=\aleph_2$ (see Theorem 3.1 (a)).

We need the following corollary of Theorem 2.3.

COROLLARY 2.4. *If* $\omega\leqq\tau<\omega_3$ *and* $\gamma<\omega_1$ *then*

$$\begin{pmatrix}\omega_1\\ \omega_2^{\tau+\gamma}\end{pmatrix}\to\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+\gamma} & \omega_2^{\tau}\end{pmatrix}^{1,1}.$$

We now describe a general method to obtain polarized partition relations. First we introduce a new partition relation.

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\to\left(\begin{matrix}\xi_0\\ \eta_0\end{matrix},\begin{bmatrix}\xi_1\\ \eta_1\end{bmatrix}_{\varkappa,<\lambda}\right)^{1,1}. \tag{2.7}$$

By definition, (2.7) means that the following statement is true: *If f is a partial function from $\xi\times\eta$ into $\varkappa$ then* EITHER *there is $X_0\times Y_0\subset\xi\times\eta$ such that* $\operatorname{tp} X_0=\xi_0$, $\operatorname{tp} Y_0=\eta_0$ and *$X_0\times Y_0$ is disjoint from $D(f)$, the domain of f*, OR *there is $X_1\times Y_1\subset D(f)$ such that* $\operatorname{tp} X_1=\xi_1$, $\operatorname{tp} Y_1=\eta_1$ *and* $|f''(X_1\times\{v\})|<\lambda$ *for all* $v\in Y_1$.

A more general symbol than (2.7) can be defined but we do not bother to do this since (2.7) is sufficiently general for most of our present purposes.

We now give two lemmas establishing connections between polarized partition relations and the new relation (2.7).

LEMMA 2.5. *Suppose $\varkappa$ is an infinite cardinal and*

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\nrightarrow\left(\begin{matrix}1\\ \eta_0\end{matrix}\ \begin{bmatrix}\xi_1\\ \eta_1\end{bmatrix}_{\omega,<\omega}\right)^{1,1}. \tag{2.8}$$

Let Ξ_ν $(\nu<\eta)$ be ordinals such that $\Xi_\nu<\varkappa^+$ and let $\Xi=\sum\{\Xi_\nu\colon \nu<\eta\}$. Let Θ denote the set of all ordinals of the form $\varphi'=\sum\{\varphi_\nu\colon \nu<\eta\}$, where $\varphi_\nu\leqq\Xi_\nu$ and

a) $\varphi_\nu\neq\Xi_\nu\Rightarrow\varphi_\nu<\varkappa^\omega$ $(\nu<\eta)$,

b) $\operatorname{tp}\{\nu<\eta\colon \varphi_\nu=\Xi_\nu\}<\eta_0$.

Let $\Phi=\sup\{\varphi'+1\colon \varphi'\in\Theta\}$ *and* $\Psi=\sup\{\sum\{\Xi_\nu\colon \nu\in Y\}+1\colon Y\subset\eta\wedge\operatorname{tp} Y<\eta_1\}$. *Then*

$$\begin{pmatrix}\xi\\ \Xi\end{pmatrix}\nrightarrow\begin{pmatrix}1 & \xi_1\\ \Phi & \Psi\end{pmatrix}^{1,1}. \tag{2.9}$$

PROOF. By the hypothesis (2.8) there is a partial function f from $\xi\times\eta$ into ω such that

(i) $(\{\alpha\}\times Y_0)\cap D(f)\neq\emptyset$ whenever $\alpha\in\xi$ and $Y_0\subset\eta$, $\operatorname{tp} Y_0=\eta_0$, and

(ii) whenever $X_1\times Y_1\subset\xi\times\eta$, $\operatorname{tp} X_1=\xi_1$, $\operatorname{tp} Y_1=\eta_1$, then there is some $v\in Y$ such that either $X_1\times\{v\}\not\subset D(f)$ or $|f''(X_1\times\{v\})|=\omega$.

Let $\langle S_v: v<\eta\rangle$ be a decomposition of Ξ such that $\operatorname{tp} S_v=\Xi_v$ $(v<\eta)$ and $S_0<S_1<\dots$. Let $B^v=\langle B_n^v: n<\omega\rangle$ be a paradoxical decomposition of S_v such that $\operatorname{tp} B_n^v<\varkappa^n$ for $n<\omega$. Now consider the subsets $A_\alpha\subset\Xi$ $(\alpha<\xi)$ given by

$$A_\alpha=\bigcup\{S_v:\ (\alpha,v)\notin D(f)\}\cup\bigcup\{B^v_{f(\alpha,v)}:\ (\alpha,v)\in D(f)\}.$$

By (i) it follows that $\operatorname{tp} A_\alpha\in\Theta$ $(\alpha<\xi)$ and hence $\operatorname{tp} A_\alpha<\Phi$ $(\alpha<\xi)$. Now let $X_1\subset\xi$, $B\subset\Xi$ with $\operatorname{tp} X_1=\xi_1$ and $\operatorname{tp} B=\Psi$. Put $Y_1=\{v<\eta: B\cap S_v\neq\emptyset\}$. Then $\operatorname{tp} Y_1\geq\eta_1$, by the definition of Ψ. Hence, by (ii), there is $v\in Y_1$ such that either $X_1\times\{v\}\not\subset D(f)$ or $|f''(X_1\times\{v\})|=\omega$. Since B^v has the ω-covering property for S_v, this implies that in either case $S_v\subset\bigcup\{A_\alpha:\alpha\in X_1\}$ and hence $B\cap\bigcup\{A_\alpha:\alpha\in X_1\}\neq\emptyset$.

LEMMA 2.6. *Suppose that $\varkappa$ is an infinite cardinal and*

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\rightarrow\begin{pmatrix}1\\ \eta_0\end{pmatrix},\begin{bmatrix}\xi_1\\ \eta_1\end{bmatrix}_{\omega,<\omega}\Bigg)^{1,1} \tag{2.10}$$

holds. Let Ξ_v $(v<\eta)$ be ordinals such that $\omega\leq\Xi_v<\varkappa^+$ $(v<\eta)$ and let $\Xi=\sum\{\varkappa^{\Xi_v}: v<\eta\}$. Let $\Psi=\min\{\sum\{\varkappa^{\Xi_v}: v\in Y\}: Y\subset\eta,\ \operatorname{tp} Y=\eta_1\}$. Then

$$\begin{pmatrix}\xi\\ \Xi\end{pmatrix}\rightarrow\begin{pmatrix}1 & \xi_1\\ \varkappa^{\omega}\cdot\eta_0 & \Psi\end{pmatrix}^{1,1} \tag{2.11}$$

holds.

PROOF. Let $\langle A_\alpha:\alpha<\xi\rangle$ be a system of subsets of Ξ such that $\operatorname{tp} A_\alpha<\varkappa^\omega\eta_0$ $(\alpha<\xi)$. Let $\langle S_v: v<\eta\rangle$ be a decomposition of Ξ with $\operatorname{tp} S_v=\varkappa^{\Xi_v}$ $(v<\eta)$ and $S_0<S_1<\dots$. Define a partial function f from $\xi\times\eta$ into ω having domain

$$D(f)=\{(\alpha,v):\operatorname{tp}(A_\alpha\cap S_v)<\varkappa^\omega\}$$

and such that

$$f(\alpha,v)=\min\{n:\ \operatorname{tp}(A_\alpha\cap S_v)<\varkappa^n\}$$

for $(\alpha,v)\in D(f)$.

By (2.10), and by the fact that $\operatorname{tp} A_\alpha<\varkappa^\omega\eta_0$ $(\alpha<\xi)$ it follows that there are $X_1\subset\xi$ and $Y_1\subset\eta$ such that $\operatorname{tp} X_1=\xi_1$, $\operatorname{tp} Y_1=\eta_1$, and $X_1\times Y_1\subset D(f)$ and $|f''(X_1\times\{v\})|<\omega$ for all $v\in Y_1$.

For each $v\in Y_1$, the set $\bigcup\{A_\alpha\cap S_v:\alpha\in X_1\}$ has order type less than $\varkappa^\omega$ and hence $S_v\setminus\bigcup\{A_\alpha\cap S_v:\alpha\in X_1\}$ has order type $\varkappa^{\Xi_v}$. Put $B=\bigcup\{S_v: v\in Y\}\setminus\bigcup\{A_\alpha:\alpha\in X_1\}$. Then $\operatorname{tp} B\geq\Psi$ and $B\cap A_\alpha=0$ for $\alpha\in X_1$. This establishes (2.11).

3. Discussion of the relation (2.1). The aim of this chapter is to give a discussion of the relation

$$\begin{pmatrix}\omega_1\\ \omega_2^\varrho\end{pmatrix}\rightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^\sigma & \omega_2^\tau\end{pmatrix}^{1,1} \tag{3.1}$$

for $\varrho,\tau<\omega_3$ and $\sigma<\omega_1$. We are going to give a complete discussion under the assumptions $2^{\aleph_0}=\aleph_1$, $2^{\aleph_1}=\aleph_2$.

Our main positive result is

THEOREM 3.1.

a) $\binom{\omega_1}{\omega_2^\tau} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^n & \omega_2^\tau \end{pmatrix}^{1,1}$ *for* $n < \omega$ *and* $n \leqq \tau \leqq \omega_3$

b) $\binom{\omega_1}{\omega_2^{\tau+\gamma}} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+\gamma} & \omega_2^\tau \end{pmatrix}^{1,1}$ *for* $\tau < \omega_3$, $\tau+\gamma \geqq \omega+\gamma$ *and* $\gamma < \omega_1$

c) $\binom{\omega_1}{\omega_2^{\tau+\gamma}} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+1+\gamma} & \omega_2^\tau \end{pmatrix}^{1,1}$ *for* $\omega+1 \leqq \tau < \omega_3$, $\mathrm{cf}(\omega_2^\tau) = \omega_2$ *and* $\gamma < \omega$.

PROOF. a) is a trivial consequence of the fact that the union of ω_1 sets of type $< \omega_2^n$ has type $< \omega_2^n$.

b) is a restatement of Corollary 2.4 if $\omega \leqq \tau$. If $\tau < \omega$ then $\tau+\gamma \geqq \omega+\gamma$ implies $\gamma = \omega+\gamma$ and hence the statement is true by the special case $\binom{1}{\omega_2^{\omega+\gamma}} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+\gamma} & \omega_2^\omega \end{pmatrix}$ of Corollary 2.4.

We prove c) by induction on γ. In case $\gamma=0$ let $\langle S_\nu : \nu < \omega_2 \rangle$ be the standard decomposition of ω_2^τ and assume that $A_\alpha \subset \omega_2^\tau$, $\mathrm{tp}\, A_\alpha < \omega_2^{\omega+1}$ for $\alpha < \omega_1$.

Now for each $\alpha < \omega_1$ there are $n(\alpha) < \omega$ and $\nu(\alpha) < \omega_2$ such that

$$\mathrm{tp}\,(A_\alpha \cap S_\nu) < \omega_2^{n(\alpha)} \quad \text{for} \quad \nu(\alpha) < \nu < \omega_2.$$

There is a $D \in [\omega_1]^{\omega_1}$ such that $n(\alpha) = n$ for $\alpha \in D$, where n is some fixed integer, and then it is easily seen that

$$\mathrm{tp}\,(\omega_2^\tau \setminus \cup \{A_\alpha : \alpha \in D\}) = \omega_2^\tau.$$

We now prove c) by induction on $\gamma < \omega$. Assume the statement is true for γ. Let $\langle S_\nu : \nu < \omega_2 \rangle$ be a standard decomposition of $\omega_2^{\tau+\gamma+1}$; and assume $A_\alpha \subset \omega_2^{\tau+\gamma+1}$, $\mathrm{tp}\, A_\alpha < \omega_2^{\omega+1+\gamma+1}$ $(\alpha < \omega_1)$. Then for each $\alpha < \omega_1$ there is $\nu(\alpha)$ such that $\mathrm{tp}\,(A_\alpha \cap S_\nu) < \omega_2^{\omega+1+\gamma}$ for $\nu(\alpha) < \nu < \omega_2$. Choose ν so that $\mathrm{tp}\,(A_\alpha \cap S_\nu) < \omega_2^{\omega+1+\gamma}$ for all $\alpha < \omega_1$. By the induction hypothesis there is a $D \in [\omega_1]^{\omega_1}$ such that

$$\mathrm{tp}\,(S_\nu \setminus \{A_\alpha : \alpha \in D\}) \geqq \omega_2^\tau.$$

Now our aim is to show that the rather simple positive results of the last theorem are best possible.

THEOREM 3.2. *Assume* $\tau < \omega_3$.

(3.2) *If* $2^{\aleph_0} = \aleph_1$ *and* $\mathrm{cf}(\tau) = \omega$ *then*

$$\binom{\omega_1}{\omega_2^\tau} \nrightarrow \begin{pmatrix} 1 & \omega_1 \\ \omega_2^\omega+1 & \omega_2^\tau \end{pmatrix}^{1,1}.$$

(3.3) *If* $\mathrm{cf}(\tau) = \omega_1$ *then*

$$\binom{\omega_1}{\omega_2^\tau} \nrightarrow \begin{pmatrix} 1 & \omega_1 \\ \omega_2^\omega \omega_1 & \omega_2^\tau \end{pmatrix}^{1,1}.$$

(3.4) *If* $2^{\aleph_1} = \aleph_2$ *and* $\mathrm{cf}(\omega_2^\tau) = \omega_2$ *then*

$$\binom{\omega_2}{\omega_2^\tau} \nrightarrow \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+1}+1 & \omega_2^\tau \end{pmatrix}^{1,1}.$$

PROOF OF (3.2). Let $\langle S_n : n < \omega \rangle$ be the standard decomposition of ω_2^ξ and let tp $S_n = \omega_2^{\xi_n}$ $(n < \omega)$. We apply Lemma 2.5 with $\xi = \omega_1$, $\eta = \omega$, $\eta_0 = 1$, $\xi_1 = \omega_1$, $\eta_1 = \omega$. With this choice of the parameters the Φ of the lemma is $\omega_2^\omega + 1$, while $\Psi = \omega_2^\xi$. Hence, by the lemma, we only have to prove that $2^{\aleph_0} = \aleph_1$ implies

$$(3.5) \qquad \binom{\omega_1}{\omega} \nrightarrow \begin{pmatrix} 1 & \begin{bmatrix} \omega_1 \\ \omega \end{bmatrix}_{\omega, < \omega} \\ 1 & \end{pmatrix}^{1,1}.$$

We prove this relation under the weaker hypothesis: *There is a sequence of functions* $f_\alpha \in {}^\omega\omega$ $(\alpha < \omega_1)$ *having the property that for any* $g \in {}^\omega\omega$, *there is* $\beta < \omega_1$ *such that* $\{n < \omega : g(n) \geqq f_\alpha(n)\}$ *is finite for all* $\alpha > \beta$ (i.e. there is an ω_1-scale).

Assuming the f_α $(\alpha < \omega_1)$ satisfy the above, we now define a function $f : \omega_1 \times \times \omega \to \omega$ by

$$(3.6) \qquad f(\alpha, \nu) = f_\alpha(\nu).$$

Let $D \in [\omega_1]^{\omega_1}$, $N \in [\omega]^\omega$ and suppose that $f''(D \times \{n\})$ is finite for all $n \in N$. Then there are integers $g(n)$ $(n < \omega)$ such that $f_\alpha(n) < g(n)$ for all $\alpha \in D$ and $n \in N$, a contradiction. Thus $\{\nu < \omega : |f''(D \times \{\nu\})| < \omega\}$ must be finite.

PROOF OF (3.3). The idea of the proof is very similar to that used in the proof of Lemma 2.5, and suggests how that lemma can be generalized. We did not bother to state the generalization since this is the only instance where the stronger statement would be needed.

First we prove:

(3.7) *There is a function* $f : \omega_1 \times \omega_1 \to \omega$ *such that for all* $A, B \in [\omega_1]^{\omega_1}$ *there is* $\nu \in B$ *with* $|f''((A \setminus \nu) \times \{\nu\})| = \omega$.

Incidentally, we remark that (3.7) implies the relation

$$\omega_1 \nrightarrow [\omega_1]^2_{\omega, < \omega}$$

which does not seem to have been noted previously.

To see (3.7) choose a sequence of functions $f_\alpha : \alpha \to \omega$ for $\alpha < \omega_1$ in such a way that, for all $\beta < \alpha < \omega_1$, $f_\beta(\nu) \neq f_\alpha(\nu)$ for all but finitely many ν. Now put

$$f(\alpha, \nu) = \begin{cases} f_\alpha(\nu) & \text{for} \quad \nu < \alpha < \omega_1, \\ 0 & \text{otherwise.} \end{cases}$$

Let now $A \in [\omega_1]^{\omega_1}$ and put $T = \{\nu < \omega_1 : |\{f(\alpha, \nu) : \alpha \in A \setminus \nu + 1\}| < \omega\}$. We want to verify that $B \setminus T \neq \emptyset$ for $B \in [\omega_1]^{\omega_1}$ and this follows if we show that $|T| \leqq \omega$. Assume $|T| = \omega_1$. Then there are $T' \in [T]^{\omega_1}$ and $n < \omega$ such that

$$|\{f_\alpha(\nu) : \alpha \in A \setminus \nu + 1\}| < n \quad \text{for} \quad \nu \in T'.$$

Let $\nu_0 < \ldots < \nu_k < \ldots < \alpha_0 < \ldots < \alpha_n$ be ordinals such that $\nu_k \in T'$ for $k < \omega$ and $\alpha_i \in A$ for $i \leqq n$. Then there are $i \neq j \leqq n$ such that $f_{\alpha_i}(\nu_k) = f_{\alpha_j}(\nu_k)$ for infinitely many k, a contradiction.

To finish the proof of (3.3) let $\langle S_\nu : \nu < \omega_1 \rangle$ be a standard decomposition of ω_2^ξ. For $\nu < \omega_1$ let $\langle B_n^\nu : n < \omega \rangle$ be a paradoxical decomposition of S_ν for $\nu < \omega_1$. For $\alpha < \omega_1$ let $A_\alpha = \bigcup \{B_{f(\alpha,\nu)}^\nu : \nu < \alpha\}$, where f satisfies (3.7).

It follows, just as in the proof of Lemma 2.5, that $\langle A_\alpha:\alpha<\omega_1\rangle$ establishes the negative relation (3.3).

Proof of (3.4). Let $\langle S_\nu:\nu<\omega_2\rangle$ be the standard decomposition of ω_2^ξ, where $\operatorname{tp} S_\nu=\omega_2^{\xi_\nu}$ for $\nu<\omega_2$. We apply Lemma 2.5 with $\xi=\omega_2$, $\eta=\omega_2$, $\eta_0=1$, $\xi_1=\omega_1$, $\eta_1=\omega_2$. Then Φ and Ψ of the lemma are respectively $\omega_2^{\omega+1}+1$ and ω_2^ξ.

Hence by the lemma it is sufficient to prove that $2^{\aleph_1}=\aleph_2$ implies

$$\binom{\omega_2}{\omega_2}\not\to\begin{pmatrix}1 & \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}_{\omega,<\omega_1}\\ 1 & \end{pmatrix}^{1,1}. \tag{3.8}$$

This is a trivial corollary of the fact ([6], Theorem 17A) that $2^{\aleph_1}=\aleph_2$ implies

$$\binom{\omega_2}{\omega_2}\not\to\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}_{\omega}^{1,1}. \tag{3.9}$$

We need the following extension of Theorem 3.2.

Theorem 3.3. *Assume* $\tau<\omega_3$, $\gamma<\omega_1$.

(a) *If* $2^{\aleph_0}=\aleph_1$ *and* $\operatorname{cf}(\tau)=\omega$ *then*

$$\binom{\omega_1}{\omega_2^{\xi+\gamma}}\not\to\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+\gamma}+1 & \omega_2^\xi\end{pmatrix}^{1,1}. \tag{3.10}$$

(b) *If* $\operatorname{cf}(\tau)=\omega_1$ *and* $\gamma>0$ *then*

$$\binom{\omega_1}{\omega_2^{\xi+\gamma}}\not\to\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1}+1 & \omega_2^\xi\end{pmatrix}^{1,1}. \tag{3.11}$$

(c) *If* $2^{\aleph_1}=\aleph_2$ *and* $\operatorname{cf}(\omega_2^\xi)=\omega_2$ *then*

$$\binom{\omega_2}{\omega_2^{\xi+\gamma}}\not\to\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1+\gamma}+1 & \omega_2^\xi\end{pmatrix}^{1,1}. \tag{3.12}$$

Proof. We prove all these statements by induction on $\gamma<\omega_1$. Since there are notable differences it will be convenient to give the proofs separately.

Proof of (3.10). For $\gamma=0$ this is (3.2). Assume $\gamma>0$. In the case $\gamma=\delta+1$ we can take identical cross sections. Now assume $\operatorname{cf}(\gamma)=\omega$. Let $\langle S_n:n<\omega\rangle$ be the standard decomposition of $\omega_2^{\xi+\gamma}$, $\operatorname{tp} S_n=\omega_2^{\xi+\gamma_n}$. Let $A^n=\langle A_\alpha^n:\alpha<\omega_1\rangle$ establish

$$\binom{\omega_1}{\omega_2^{\xi+\gamma_n}}\not\to\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+\gamma_n}+1 & \omega_2^\xi\end{pmatrix}^{1,1} \quad\text{in } S_n \quad\text{for } n<\omega.$$

Let $\langle B_k^n:k<\omega\rangle$ be a paradoxical decomposition of S_n for $n<\omega$. Let $f:\omega_1\times\omega\to\omega$ satisfy (3.6). For $\alpha<\omega_1$ put $A_\alpha=\bigcup\{A_\alpha^n:n<\omega\}\cup\{B_{f(\alpha,n)}^n:n<\omega\}$. Clearly $\operatorname{tp}(A_\alpha\cap(\bigcup\{S_i:i\le n\}))<\omega_2^{\omega+\gamma}$ for $n<\omega$, and hence $\operatorname{tp} A_\alpha\le\omega_2^{\omega+\gamma}$ $(\alpha<\omega_1)$. Assume $D\in[\omega_1]^{\omega_1}$. Then, by the choice of the A_α^n,

$$\operatorname{tp}(S_n\setminus\bigcup\{A_\alpha:\ \alpha\in D\})<\omega_2^\xi$$

for all $n<\omega$. Also, by the choice of the B_k^n and f,

$$\bigcup\{B_{f(\alpha,n)}^n:\ \alpha\in D,\ n<\omega\}$$

covers an end section of $\omega_2^{\tau+\gamma}$. It follows that

$$\operatorname{tp}\left(\omega_2^{\tau+\gamma} \setminus \bigcup\{A_\alpha : \alpha \in D\}\right) < \omega_2^{\tau}.$$

Proof of (3.11). Let $\langle S_\nu : \nu < \omega_1 \rangle$ be a standard decomposition of $\omega_2^{\tau+\gamma}$. In case $\gamma = 1$ take identical cross sections of systems establishing (3.3). In case $\gamma = \delta + 1 > 1$ take identical cross sections of the inductive systems. In case $\operatorname{cf}(\gamma) = \omega$ take cross sections. It is easy to check that this system satisfies all the requirements. (In case $\operatorname{cf}(\gamma) = \omega$ we need the fact that $\operatorname{cf}(\tau) = \omega_1 > \omega$.)

Proof of (3.12). For $\gamma = 0$ this is 3.4. For $\operatorname{cf}(\gamma) = \omega$ take cross sections. Now suppose that $\gamma = \delta + 1$ and let $\langle S_\nu : \nu < \omega_2 \rangle$ be a standard decomposition of $\omega_2^{\tau+\gamma}$. Let $A^\nu = \langle A_\alpha^\nu : \alpha < \omega_2 \rangle$ be identical copies esatblishing $\begin{pmatrix} \omega_2 \\ \omega_2^{\tau+\delta} \end{pmatrix} \to \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega+1+\delta}+1 & \omega_2^{\tau} \end{pmatrix}^{1,1}$ on S_ν for $\nu < \omega_2$. Let $\langle B_n^\nu : n < \omega \rangle$ be a paradoxical decomposition S^ν for $\nu < \omega_2$. Let $f : \omega_2 \times \omega_2 \to \omega$ establish (3.8). For $\alpha < \omega_2$ let

$$A_\alpha = \bigcup\{A_\alpha^\nu : \nu < \omega_2\} \cup \bigcup\{B_{f(\alpha,\nu)}^\nu : \nu < \omega_2\}.$$

For each $\nu < \omega_2$, $A_\alpha \cap \bigcup\{S_\mu : \mu < \nu\} \leqq \omega_2^{\omega+1+\delta} \cdot (\nu+1)$, and hence $\operatorname{tp} A_\alpha \leqq \omega_2^{\omega+1+\delta+1}$ for $\alpha < \omega_2$. Now, by the choice of B_n^ν and f the union of every ω_1 A_α covers an endsection of $\omega_2^{\tau+\delta}$. By the choice of the A_α^ν the union of every ω_1 A_α omits a set of type less than ω_2^{τ} from each S_ν $(\nu < \omega_2)$. Since $\operatorname{cf}(\omega_2^{\tau}) = \omega_2$ it follows that

$$\operatorname{tp}\left(\omega_2^{\tau+\gamma} \setminus \bigcup\{A_\alpha : \alpha \in D\}\right) < \omega_2^{\tau}$$

for all $D \in [\omega_1]^{\omega_1}$.

We claim that, assuming $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, Theorems 3.1—3.3 provide a complete discussion of (3.1). We may of course assume that $\sigma, \tau \leqq \varrho$. In case $\sigma \leqq \omega$ (3.1) is true by Theorem 3.1 a) and b). So we only have to investigate the relation

$$\begin{pmatrix} \omega_1 \\ \omega_2^{\varrho} \end{pmatrix} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+1+\gamma} & \omega_2^{\tau} \end{pmatrix}^{1,1} \tag{3.13}$$

for $\tau \leqq \varrho < \omega_3$, $\gamma < \omega_1$ and $\omega + 1 + \gamma \leqq \varrho$. If $\tau \leqq \omega$ the statement is true by Theorem 3.1 b). Hence we may assume $\tau > \omega$. ϱ can uniquely be written as $\varrho = \tau + \varrho'$. If $\operatorname{cf}(\tau) = \omega$ or $\operatorname{cf}(\tau) = \omega_1$, then (3.13) holds iff $1 + \gamma \leqq \varrho'$. If $\operatorname{cf}(\omega_2^{\tau}) = \omega_2$, then (3.13) holds iff $\gamma \leqq \varrho'$. These follow from Theorems 3.1 and 3.3 and the elementary fact that $\varrho' < \gamma \Leftrightarrow \delta + \varrho' < \delta + \gamma$ for any δ. To conclude this chapter we give some results about possible improvements of our theorems.

An easy iteration gives the following improvement of Theorem 3.1 c):

If $\omega + 1 \leqq \tau < \omega_3$, $\operatorname{cf}(\omega_2^{\tau}) = \omega_2$, $0 < \gamma < \omega_1$, *and* $n < \omega$, *then*

$$\begin{pmatrix} \omega_1 \\ \omega_2^{\tau+\gamma} \end{pmatrix} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+1+\gamma} & \omega_2^{\tau} n \end{pmatrix}^{1,1}.$$

In particular,

$$\begin{pmatrix} \omega_1 \\ \omega_2^{\omega+2} \end{pmatrix} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+2} & \omega_2^{\omega+1} n \end{pmatrix}^{1,1} \quad \text{for} \quad n < \omega.$$

We omit the proof of this, but it is intriguing to note that this cannot be improved by replacing n by ω.

THEOREM 3.4.

$$\begin{pmatrix}\omega_1\\ \omega_2^{\omega+2}\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1}+1 & \omega_2^{\omega+1}\omega\end{pmatrix}^{1,1}$$

is consistent with G.C.H. (e.g. holds in L).

PROOF. We apply Lemma 2.5 with $\varkappa=\omega_2$, $\xi=\omega_1$, $\eta=\omega_2$, $\eta_0=1$, $\xi_1=\omega_1$, $\eta_1=\omega$, $\Xi=\omega_2^{\omega+2}$, $\Xi_\nu=\omega_2^{\omega+1}$ for $\nu<\omega_2$. It is easy to check that $\Phi=\omega_2^{\omega+1}+1$, $\Psi=\omega_2^{\omega+1}\omega$. Hence we only have to establish the consistency of

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \begin{bmatrix}\omega_1\\ \omega\end{bmatrix}_{\omega,<\omega}\\ 1 & \end{pmatrix}^{1,1}. \tag{3.15}$$

This follows from the following statement:

(3.16) *There is a function $f:\omega_1\times\omega_2\to\omega$ such that for all $A\in[\omega_1]^{\omega_1}$, $B\in[\omega_2]^{\omega}$ there is a $\nu\in B$ so that $f''(A\times\{\nu\})=\omega$.*

This has been proved to be consistent with G.C.H. by PRIKRY [14]. Later JENSEN [10] showed using morasses that Prikry's result (3.16) holds in L.

Finally we are going to prove that (3.3) of Theorem 3.2 is best possible.

THEOREM 3.5. *Assume* $\tau<\omega_3$, $\mathrm{cf}(\tau)=\omega_1$, $\xi<\omega_1$. *Then*

$$\begin{pmatrix}\omega_1\\ \omega_2^{\xi}\end{pmatrix} \to \begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega}\xi & \omega_2^{\xi}\end{pmatrix}^{1,1}.$$

In order to prove this we need a lemma on set mappings which is similar to a theorem of ERDŐS and SPECKER [15].

LEMMA 3.6. *Let $\xi<\omega_1$ and let $f:\omega_1\to P(\omega_1)$ be a set mapping on ω_1 such that $\mathrm{tp}\, f(x)<\xi$ for all $x\in\omega_1$. Then there are $X, Y\in[\omega_1]^{\omega_1}$ such that $f(X)\cap\overline{Y}=\emptyset$, where $\overline{Y}$ denotes the closure of Y in ω_1.*

PROOF. We will assume that the lemma is false and obtain a contradiction.

For $U, V\subset\omega_1$, let $S(U, V)=\{x\in U: f(x)\cap\overline{V}=\emptyset\}$. Suppose $A, B\in[\omega_1]^{\omega_1}$ and that

$$|S(A, \overline{Y})| = \aleph_1 \quad \textit{whenever} \quad Y\subset B \quad \textit{and} \quad 0<|Y|\leqq\aleph_0. \tag{3.17}$$

Choose $x_0\in A$, $y_0\in B$ so that $f(x_0)<\{y_0\}$. Now let $0<\nu<\omega_1$ and suppose that we have already chosen $x_\mu\in A$, $y_\mu\in B$ for $\mu<\nu$ so that $f(X_\nu)\cap\overline{Y}_\nu=\emptyset$, where $X_\nu=\{x_\mu:\mu<\nu\}$ and $Y_\nu=\{y_\mu:\mu<\nu\}$. By (3.17) there is $x_\nu\in S(A, \overline{Y}_\nu)\setminus X_\nu$. Choose $y_\nu\in B$ so that $Y_\nu\cup f(X_\nu\cup\{x_\nu\})<\{y_\nu\}$. Then, contrary to our assumption, the lemma is satisfied with $X=\{x_\nu:\nu<\omega_1\}$ and $Y=\{y_\nu:\nu<\omega_1\}$. It follows that, whenever $A, B\in[\omega_1]^{\omega_1}$, then there are $x(A, B)\in A$ and $Y(A, B)\subset B$ such that $0<|Y(A, B)|\leqq\aleph_0$ and

$$f(x)\cap\overline{Y(A, B)}\neq\emptyset \quad \textit{for all} \quad x\in A \quad \textit{such that} \quad x\geqq x(A).$$

We now define ordinals $\alpha_\nu<\omega_1$ for $\nu<\omega_1$ by transfinite induction as follows. Let $\nu<\omega_1$ and suppose that we have already defined α_μ for $\mu<\nu$. Let $\beta_\nu=\sup\{\alpha_\mu:\mu<\nu\}$, $B_\nu=\{\varrho:\beta_\nu\leqq\varrho<\omega_1\}$. Let $Y_\nu=Y(B_\nu, B_\nu)$, $x_\nu=x(B_\nu, B_\nu)$ and choose $\alpha_\nu < \omega_1$ so that $\bar{Y}_\nu \cup \{x_\nu\} \cup f(x_\nu) < \{\alpha_\nu\}$. This defines Y_ν and α_ν for all $\nu < \omega_1$ so that $\bar{Y}_0 < \bar{Y}_1 < \dots$ and

$$f(x) \cap \bar{Y}_\mu \neq \emptyset \quad for \quad \mu \leqq \nu \quad and \quad \alpha_\nu \leqq x < \omega_1.$$

It follows from this that $\operatorname{tp} f(\alpha_\xi) \geqq \xi$, and this is the desired contradiction.

Proof of Theorem 3.5. Let $\langle S_\nu : \nu < \omega_1 \rangle$ be the standard decomposition of ω_2^ξ, and let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a system of subsets of ω_2^ξ such that $\operatorname{tp} A_\alpha < \omega_2^\omega \xi$ $(\alpha < \omega_1)$.

For $\alpha < \omega_1$, let $f(\alpha) = \{\nu < \omega_1 : \operatorname{tp}(A_\alpha \cap S_\nu) \geqq \omega_2^\omega\}$. Then $\operatorname{tp} f(\alpha) < \xi$ $(\alpha < \omega_1)$. Therefore, by Lemma 3.6 we can assume that

$$\operatorname{tp}(A_\alpha \cap S_\nu) < \omega_2^\omega \quad \text{for} \quad \alpha, \nu < \omega_1.$$

There is no loss of generality if we assume that $A_\alpha \neq \emptyset$ $(\alpha < \omega_1)$ and then for $\alpha, \nu < \omega_1$ there is an integer $n(\alpha, \nu)$ such that

$$\omega_2^{n(\alpha,\nu)} \leqq \operatorname{tp}(A_\alpha \cap S_\nu) < \omega_2^{n(\alpha,\nu)+1}. \tag{3.18}$$

Now for $\alpha < \omega_1$ there are $\sigma(\alpha) < \xi$ and $\tau(\alpha) < \omega_2^\omega$ such that $\operatorname{tp} A_\alpha = \omega_2^\omega \sigma(\alpha) + \tau(\alpha)$. Hence, there are ordinals μ_σ^α $(\sigma \leqq \sigma(\alpha))$ such that $\mu_0^\alpha < \mu_1^\alpha < \dots, g(\alpha) = \{\mu_\sigma^\alpha : \sigma \leqq \sigma(\alpha)\}$ is closed in ω_1 and

$$\operatorname{tp}(A_\alpha \cap \bigcup \{S_\nu : \mu_\sigma^\alpha \leqq \nu < \mu_{\sigma+1}^\alpha\}) = \omega_2^\omega \quad \text{for} \quad \sigma < \sigma(\alpha),$$

$$\operatorname{tp}(A_\alpha \cap \bigcup \{S_\nu : \mu_{\sigma(\alpha)}^\alpha \leqq \nu < \omega_1\}) = \tau(\alpha).$$

Since $\operatorname{tp} g(\alpha) \leqq \xi$, it follows from Lemma 3.6 that there are $D, E \in [\omega_1]^{\omega_1}$ such that $g(D) \cap \bar{E} = \emptyset$.

Suppose that for some $\alpha \in D$ the set $\{n(\alpha, \nu) : \nu \in E\}$ is unbounded. Then there are $\nu_i \in E$ such that $\nu_0 < \nu_1 < \dots$ and $n(\alpha, \nu_0) < n(\alpha, \nu_1) < \dots$. Therefore, by (3.18),

$$\operatorname{tp}(A_\alpha \cap \bigcup \{S_{\nu_i} : i < \omega\}) = \omega_2^\omega$$

and hence $\nu = \sup \{\nu_i : i < \omega\} \in g(\alpha)$. This contradicts the fact that $g(\alpha) \cap \bar{E} = \emptyset$. It follows that, for each $\alpha \in D$ there is $n(\alpha) < \omega$ such that $n(\alpha, \nu) < n(\alpha)$ for all $\nu \in E$. There is $D_1 \in [D]^{\omega_1}$ such that $n(\alpha) = n$ for all $\alpha \in D_1$ and hence $\operatorname{tp}(S_\nu \cap \bigcup \bigcup \{A_\alpha : \alpha \in D_1\}) < \omega_2^n$ $(\nu \in E)$. It follows that $\operatorname{tp}(\omega_2^\xi \setminus \bigcup \{A_\alpha : \alpha \in D_1\}) = \omega_2^\xi$.

4. A discussion of (2.2). In this section we discuss what happens when the term ω_1 on the right-hand side of (3.1) is replaced by ω, i.e. we are going to investigate relations of the form

$$\binom{\omega_1}{\omega_2^\varrho} \to \binom{1 \quad \omega}{\omega_2^\tau \quad \omega_2^\xi}^{1,1} \tag{4.1}$$

for $\varrho, \tau < \omega_3$ and $\sigma < \omega_1$. We have already indicated in § 2 the need to restrict our attention to the case $\sigma < \omega_1$. For this case we shall give a complete analysis of (4.1) under the assumption $2^{\aleph_0} = \aleph_1$. The main part of this section and the next is devoted to proving the positive relations stated in Theorem 4.1. The negative relations given in Theorem 4.6 are much simpler to prove.

THEOREM 4.1. *Assume* $2^{\aleph_0}=\aleph_1$. *Let* $\sigma<\omega_1$, $\varrho=\omega_1\xi+\gamma$, *where* $\xi<\omega_2$ *and* $\gamma<\omega_1$. *Then*

$$\binom{\omega_1}{\omega_2^\varrho}\rightarrow\binom{\omega}{\omega_2^\varrho}_k^{1,1} \quad \textit{if} \quad k<\omega \quad \textit{and} \quad \xi=0, \tag{4.2}$$

$$\binom{\omega_1}{\omega_2^\varrho}\rightarrow\begin{pmatrix}1 & \omega\\ \omega_2^\sigma & \omega_2^\varrho\end{pmatrix}^{1,1} \quad \textit{if} \quad \mathrm{cf}(\xi)=\omega \quad \textit{or} \quad \omega_2, \tag{4.3}$$

$$\binom{\omega_1}{\omega_2^\varrho}\rightarrow\begin{pmatrix}1 & \omega\\ \omega_2^{\omega+\gamma} & \omega_2^\varrho\end{pmatrix}^{1,1} \quad \textit{if} \quad \xi>0. \tag{4.4}$$

REMARKS. 1. We do not know if the 1 on the right-hand sides of (4.3) and (4.4) can be replaced by ω.

2. The relations (4.3), (4.4) show that the situation here is significantly different from the case of (2.1) (at least under the assumptions $2^{\aleph_0}=\aleph_1$ and $2^{\aleph_1}=\aleph_2$). By Theorem 3.1 the relation

$$\binom{\omega_1}{\omega_2^\gamma}\rightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^\gamma & \omega_2^\gamma\end{pmatrix}$$

holds for $\gamma\leqq\omega+1$, but it is false for $\gamma=\omega+2$. In fact, by Theorem 3.3, we even have the stronger negative relation that (assuming $2^{\aleph_1}=\aleph_2$)

$$\binom{\omega_2}{\omega_2^{\delta+1}}\not\rightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1}+1 & \omega_2^{\delta+1}\end{pmatrix}$$

holds for all $\delta<\omega_2$.

Our proofs of (4.2)—(4.4) are quite complicated, and we need several lemmas. The main idea is most clearly seen in the proof of (4.2). The *reduction lemma* (Lemma 4.3) enables us to faithfully represent a system of $\aleph_1$ "small" subsets of ω_2^γ $(<\omega_2^{\omega_1})$ by a system of "small" subsets of a countable set of type ω^γ. Using this we easily reduce (4.2) to a known theorem for countable ordinals due to BAUMGARTNER and HAJNAL [3] that

$$\binom{\omega_1}{\omega^\gamma}\rightarrow\binom{\omega}{\omega^\gamma}_k^{1,1} \quad (k<\omega;\ \gamma<\omega_1). \tag{4.5}$$

The same ideas are needed to prove (4.3) and (4.4) but we need generalizations of (4.5) (Theorem 4.2) and the reduction lemma (Lemma 4.3).

We first introduce some special notation. Let $1\leqq n<\omega$ and let $\boldsymbol{\xi}=(\xi_0,\dots,\xi_{n-1})$ be a sequence of indecomposable ordinals of length n. Put $\Pi(\boldsymbol{\xi})=\xi_0\times\dots\times\xi_{n-1}$, the cartesian product. If $n>1$, we write $\hat{\boldsymbol{\xi}}$ to denote the sequence $(\xi_0,\dots,\xi_{n-2})$ obtained by deleting the last term. Also, for $X\subset\Pi(\boldsymbol{\xi})$ we denote by $\hat{X}$ the projection of X into $\Pi(\hat{\boldsymbol{\xi}})$. For $u\in\xi_{n-1}$ and $X\subset\Pi(\boldsymbol{\xi})$, define $X^u=X\cap(\xi_0\times\dots\times\xi_{n-2}\times\{u\})$ and $\hat{X}^{(u)}=\widehat{X^u}$.

We now generalize the concept of a *full-sized* subset of an ordinal, i.e. we shall define a relation $X\in F(\boldsymbol{\xi})$ for subsets of $\Pi(\boldsymbol{\xi})$ by induction on the length of $\boldsymbol{\xi}=(\xi_0,\dots,\xi_{n-1})$. For $n=1$, $X\in F(\boldsymbol{\xi})$ if and only if X is a subset of ξ_0 such that $\mathrm{tp}\,X=\xi_0$. Now assume that $n>1$ and that $F(\boldsymbol{\eta})$ has been defined for sequences of indecomposable ordinals of length $n-1$. Then $X\in F(\boldsymbol{\xi})$ if and only if $X\subset\Pi(\boldsymbol{\xi})$ and $\mathrm{tp}\,\{u\in\xi_{n-1}:\hat{X}^{(u)}\in F(\hat{\boldsymbol{\xi}})\}=\xi_{n-1}$.

We now define the polarized partition relation

$$\binom{\omega_1}{\xi} \to \binom{\omega}{\xi}_k^{1,1} \tag{4.6}$$

for a sequence $\xi=(\xi_0, \dots, \xi_{n-1})$ of indecomposable ordinals as follows. The relation (4.6) means that *whenever* $f:\omega_1\times\xi\to k$ *then there are* $D\in[\omega_1]^\omega$ *and a full-sized set* $X\in F(\xi)$ *such that* $D\times X$ *is homogeneous for* f. In the case $n=1$ this definition agrees with the normal polarized partition relation.

In order to prove (4.3) and (4.4) we need (4.7), a generalization of (4.5), and (4.8), a consequence of (4.5).

THEOREM 4.2. (a) *Let* $1\leqq n$, $k<\omega$ *and let* $\xi=(\xi_0, \dots, \xi_{n-1})$ *be a sequence of indecomposable denumerable ordinals. Then*

$$\binom{\omega_1}{\xi} \to \binom{\omega}{\xi}_k^{1,1}. \tag{4.7}$$

(b) *If* $\gamma<\omega_1$, *then*

$$\binom{\omega_1}{\omega^\gamma} \to \left(\begin{matrix}\omega \\ \omega^\gamma\end{matrix}, \begin{bmatrix}\omega \\ \omega^\gamma\end{bmatrix}_{\omega,<\omega^\gamma}\right)^{1,1}. \tag{4.8}$$

We postpone the proof of Theorem 4.2 until the next section and proceed with a statement and proof of the representation lemma (Lemma 4.3) and its generalization (Lemma 4.4).

LEMMA 4.3. (Reduction lemma.) *Let* $\gamma<\omega_1$ *and let* $\langle A_\alpha:\alpha<\omega_1\rangle$ *be a sequence of subsets of* ω_2^γ *such that* $\operatorname{tp} A_\alpha<\omega_2^\gamma$. *Then there is a countable set* $X\subset\omega_2^\gamma$ *such that* $\operatorname{tp} X=\omega^\gamma$ *and* $\operatorname{tp}(A\cap X)<\omega^\gamma$ *for all* $\alpha<\omega_1$.

PROOF. We will prove a slightly stronger statement. Let $\varrho_\alpha=\min\{\varrho:\operatorname{tp} A_\alpha<\omega_2^\varrho\}$ $(\alpha<\omega_1)$. Then there is $X\subset\omega_2^\gamma$ such that $\operatorname{tp} X=\omega^\gamma$ and

$$\operatorname{tp}(A_\alpha\cap X)<\omega^{\varrho_\alpha} \quad (\alpha<\omega_1). \tag{3.4}$$

The proof is by induction on γ. For $\gamma=0$ the statement is trivial. Now let $0<\gamma<\omega_1$ and assume the result is true for smaller ordinals. We can assume that $A_\alpha\neq\emptyset$ $(\alpha<\omega_1)$ and hence that $\varrho_\alpha=\varrho_\alpha'+1$. We distinguish the two cases (1) $\gamma=\delta+1$ and (2) $\operatorname{cf}(\gamma)=\omega$.

Case 1. Let $\langle S_\nu:\nu<\omega_2\rangle$ be the standard decomposition of ω_2^γ. Then $\operatorname{tp} S_\nu=\omega_2^\delta$ $(\nu<\omega_2)$. Now for each $\alpha<\omega_1$ there is $\nu_\alpha<\omega_2$ such that $\operatorname{tp}(A_\alpha\cap S_\nu)<\omega_2^{\varrho_\alpha'}$ for $\nu_\alpha<\nu<\omega_2$. Choose a set $D\subset\omega_2$ such that $\operatorname{tp} D=\omega$ and such that $\nu_\alpha<\nu$ for all $\alpha<\omega_1$ and $\nu\in D$. By the induction hypothesis, for each $\nu\in D$ there is a set $X_\nu\subset S_\nu$ such that $\operatorname{tp} X_\nu=\omega^\delta$ and $\operatorname{tp}(A_\alpha\cap X_\nu)<\omega^{\varrho_\alpha'}$ $(\alpha<\omega_1;\ \nu\in D)$. Put $X=\bigcup\{X_\nu:\nu\in D\}$. Then $\operatorname{tp} X=\omega^\gamma$ and (3.4) holds.

Case 2. Let $\langle S_n:n<\omega\rangle$ be the standard decomposition of ω_2^γ. We can assume that $\operatorname{tp} S_n=\omega_2^{\gamma_n+1}$ $(n<\omega)$, where $\langle\gamma_n:n<\omega\rangle$ is an increasing sequence of ordinals with limit γ. Let $\langle S_{n\nu}:\nu<\omega_2\rangle$ be the standard decomposition of S_n. Then $\operatorname{tp} S_{n\nu}=\omega_2^{\gamma_n}$ $(n<\omega;\ \nu<\omega_2)$. For each $\alpha<\omega_1$ there is $\nu_\alpha<\omega_2$ such that $\operatorname{tp}(A_\alpha\cap S_{n\nu})<\omega_2^{\varrho_\alpha'}$ for all $n<\omega$ and $\nu_\alpha<\nu<\omega_2$. Choose $\nu^*<\omega_2$ so that $\nu_\alpha<\nu^*$ for all $\alpha<\omega_1$. By the induction hypothesis there are sets $X_n\subset S_{n\nu^*}$ $(n<\omega)$ such that $\operatorname{tp} X_n=\omega^{\gamma_n}$

$(n<\omega)$ and $\operatorname{tp}(A_\alpha \cap X_n) < \omega^{\varrho'_\alpha}$ $(\alpha<\omega_1;\ n<\omega)$. Put $X=\bigcup\{X_n : n<\omega\}$. Then $\operatorname{tp} X = \omega^\gamma$ and $\operatorname{tp}(A_\alpha \cap X) \leqq \omega^{\varrho'_\alpha} < \omega^{\varrho_\alpha}$ $(\alpha<\omega_1)$.

We now prove the generalized reduction lemma as follows.

LEMMA 4.4. *Let* $1 \leqq n < \omega$, $\boldsymbol{\xi}=(\xi_0, \dots, \xi_{n-1})$, *where* $\xi_i = \varkappa_i^{\gamma_i}$, $\gamma_i<\omega_1$ *and* $\varkappa_i \in \{\omega, \omega_2\}$ *for* $i<n$. *Suppose* $\langle A_\alpha : \alpha<\omega_1\rangle$ *is a sequence of subsets of* $\Pi(\boldsymbol{\xi})$ *such that no* A_α *is full-sized in* $\Pi(\boldsymbol{\xi})$. *Then there is a sequence* $\mathbf{X}=(X_0, \dots, X_{n-1})$ *such that* $X_i \subset \varkappa_i^{\gamma_i}$, $\operatorname{tp} X_i = \omega^{\gamma_i}$ $(i<n)$ *and no set* $A_\alpha \cap \Pi(\mathbf{X})$ *is full-sized in* $\Pi(\mathbf{X})$.

PROOF. For $n=1$ the statement is either trivial (if $\varkappa_0=\omega$) or follows from Lemma 4.3 (if $\varkappa_0=\omega_2$). We now assume that $n>1$ and use induction.

For each $\alpha<\omega_1$, put $B_\alpha=\{u \in \xi_{n-1} : \hat{A}_\alpha^{(u)} \in F(\hat{\boldsymbol{\xi}})\}$. Then by the assumption that $A_\alpha \notin F(\boldsymbol{\xi})$, it follows that $\operatorname{tp} B_\alpha < \xi_{n-1}$. Therefore, by Lemma 4.3, there is $X_{n-1} \subset \xi_{n-1}$ such that $\operatorname{tp} X_{n-1} = \omega^{\gamma_{n-1}}$ and $\operatorname{tp}(B_\alpha \cap X_{n-1}) < \omega^{\gamma_{n-1}}$ for all $\alpha<\omega_1$.

Now consider the system of sets $\langle \hat{A}_\alpha^{(u)} : \alpha<\omega_1,\ u \in D_\alpha\rangle$, where $D_\alpha = \{u \in X_{n-1} : \hat{A}_\alpha^{(u)} \notin F(\hat{\boldsymbol{\xi}})\}$. By the induction hypothesis it follows that there are sets $X_i \subset \xi_i$ for $i<n-1$ such that $\operatorname{tp} X_i = \omega^{\gamma_i}$ $(i<n-1)$ and such that

$$\hat{A}_\alpha^{(u)} \cap \Pi(\hat{\mathbf{X}}) \notin F(\hat{\mathbf{X}}) \quad (\alpha<\omega_1;\ u \in D_\alpha),$$

where $\mathbf{X}=(X_0, \dots, X_{n-1})$ and $\hat{\mathbf{X}}=(X_0, \dots, X_{n-2})$.

To complete the proof we have to show that for each $\alpha<\omega_1$ the set $A_\alpha \cap \Pi(\mathbf{X})$ is not full-sized, i.e. we have to verify that the set

$$C_\alpha = \{u \in X_{n-1} : \hat{A}_\alpha^{(u)} \cap \Pi(\hat{\mathbf{X}}) \in F(\hat{\mathbf{X}})\}$$

has order type less than $\omega^{\gamma_{n-1}}$. Now, if $u \in C_\alpha$, then $u \in X_{n-1} \setminus D_\alpha$ and so $\hat{A}_\alpha^{(u)} \in F(\hat{\boldsymbol{\xi}})$. Thus $C_\alpha \subset B_\alpha \cap X_{n-1}$ and so has type less than $\omega^{\gamma_{n-1}}$.

We now use the reduction lemmas to obtain "higher" analogues of (4.7) and (4.8). The special case $n=1$, $\varkappa_0=\omega_2$ of Theorem 4.5 (a) gives (4.2).

THEOREM 4.5. *Assume* $2^{\aleph_0}=\aleph_1$.

(a) *If* $1 \leqq n$, $k<\omega$, $\boldsymbol{\xi}=(\xi_0, \dots, \xi_{n-1})$, $\xi_i=\varkappa_i^{\gamma_i}$, $\gamma_i<\omega_1$, $\varkappa_i \in \{\omega, \omega_2\}$ *for* $i<n$, *then*

$$\binom{\omega_1}{\boldsymbol{\xi}} \to \binom{\omega}{\boldsymbol{\xi}}_k^{1,1}. \tag{4.9}$$

(b) *If* $\gamma<\omega_1$, *then*

$$\binom{\omega_1}{\omega_2^\gamma} \to \left(\begin{matrix}\omega \\ \omega_2^\gamma\end{matrix},\ \begin{bmatrix}\omega \\ \omega_2^\gamma\end{bmatrix}_{\omega, <\omega_1}\right)^{1,1}. \tag{4.10}$$

PROOF. (a) Assume this is false. Then there is a function $f : \omega_1 \times \Pi(\boldsymbol{\xi}) \to k$ which disproves (4.9).

For each $D \in [\omega_1]^\omega$ and $j<k$, put

$$A(D, j) = \{x \in \Pi(\boldsymbol{\xi}) : f''(D \times \{x\}) = \{j\}\}$$

Since f disproves (4.9) it follows that the sets $A(D, j)$ are not full-sized in $\Pi(\boldsymbol{\xi})$. By the hypothesis $2^{\aleph_0}=\aleph_1$ there are at most $\aleph_1$ sets $A(D, j)$ and hence by Lemma 4.4 there is a sequence $\mathbf{X}=(X_0, \dots, X_{n-1})$ such that $X_i \subset \varkappa_i^{\gamma_i}$, $\operatorname{tp} X_i = \omega^{\gamma_i}$ $(i<n)$

and $A(D, j)\cap\Pi(\mathbf{X})$ is not full-sized in $\Pi(\mathbf{X})$ for $D\in[\omega_1]^\omega$ and $j<k$. It follows that $f\upharpoonright\omega_1\times\Pi(\mathbf{X})$ disproves

$$\binom{\omega_1}{\boldsymbol{\eta}}\to\binom{\omega}{\boldsymbol{\eta}}_k^{1,1},$$

where $\boldsymbol{\eta}=(\omega^{\gamma_0},\dots,\omega^{\gamma_{n-1}})$, and this contradicts Theorem 4.2(a).

(b) The proof is essentially the same as the proof of (a). Assume that there is a partial function f from $\omega_1\times\omega_2^\gamma$ into ω which disproves (4.10). For each $D\in[\omega_1]^\omega$ let $A(D)=\{\xi<\omega_2^\gamma:(D\times\{\xi\})\cap D(f)=\emptyset\}$ and $B(D)=\{\xi<\omega_2^\gamma: D\times\{\xi\}\subset D(f)\wedge\wedge|f''(D\times\{\xi\})|<\omega\}$. By the assumption on f, $A(D)$ and $B(D)$ have order type $<\omega_2^\gamma$ for $D\in[\omega_1]^\omega$. Therefore, by Lemma 4.3, there is a set $X\subset\omega_2^\gamma$ such that $\operatorname{tp} X=\omega^\gamma$ and $A(D)\cap X$ and $B(D)\cap X$ both have type $<\omega^\gamma$ for each $D\in[\omega_1]^\omega$. Thus $f\upharpoonright\omega_1\times X$ disproves

$$\binom{\omega_1}{\omega^\gamma}\to\left(\begin{matrix}\omega\\ \omega^\gamma\end{matrix},\begin{bmatrix}\omega\\ \omega^\gamma\end{bmatrix}_{\omega,<\omega}\right)^{1,1},$$

contrary to Theorem 4.2(b).

We now prove the main result.

Proof of Theorem 4.1. As we already remarked, (4.2) is a special case of Theorem 4.5(a).

Also, (4.4) follows immediately from (4.10) and Lemma 2.6 (apply the lemma with $\varkappa=\omega_2$, $\xi=\omega_1$, $\xi_1=\omega$, $\eta=\eta_0=\eta_1=\omega_2^\gamma$ and $\Xi_\nu=\varrho-\gamma$ $(\nu<\omega_2^\gamma)$).

It remains to prove (4.3). Let $\varkappa=\operatorname{cf}(\xi)$. Then $\varkappa=\omega$ or ω_2. Let $\langle\tau_\nu:\nu<\varkappa\rangle$ be a strictly increasing sequence of ordinals with limit $\omega_1\xi$. Then $\tau_\nu+\sigma\nearrow\omega_1\xi$ $(\nu<\varkappa)$ also. Thus we may write $\omega_2^\varrho=\omega_2^{\omega_1\xi+\gamma}=\left(\sum\{\omega_2^{\tau_\nu+\sigma}:\nu<\varkappa\}\right)\cdot\omega_2^\gamma$. Put $\xi_0=\omega_2^\sigma$, $\xi_1=\varkappa$, $\xi_2=\omega_2^\gamma$, $\boldsymbol{\xi}=(\xi_0,\xi_1,\xi_2)$. Let $<_0$ denote the antilexicographic ordering of $\Pi(\boldsymbol{\xi})$. Now there are pairwise disjoint sets $S_{(\mu,\nu,\theta)}$ $((\mu,\nu,\theta)\in\Pi(\boldsymbol{\xi}))$ such that

$$\text{(4.11)}\qquad \begin{cases}\omega_2^\varrho=\bigcup\{S_{(\mu,\nu,\theta)}:(\mu,\nu,\theta)\in\Pi(\boldsymbol{\xi})\},\ \operatorname{tp} S_{(\mu,\nu,\theta)}=\omega^{\tau_\nu},\\ S_{(\mu,\nu,\theta)}<S_{(\mu',\nu',\theta')}\quad\text{for}\quad(\mu,\nu,\theta)<_0(\mu',\nu',\theta').\end{cases}$$

Suppose $\langle A_\alpha:\alpha<\omega_1\rangle$ is a sequence of subsets of ω_2^ϱ such that $\operatorname{tp} A_\alpha<\omega_2^\varrho$ $(\alpha<\omega_1)$. In order to prove (4.3) we have to show that there are $D\in[\omega_1]^\omega$ and $C\subset\omega_2^\varrho$ such that $\operatorname{tp} C=\omega_2^\varrho$ and $A_\alpha\cap C=\emptyset$ for all $\alpha\in D$.

For $\alpha<\omega_1$, let $B_\alpha=\{(\mu,\nu,\theta)\in\Pi(\boldsymbol{\xi}):A_\alpha\cap S_{(\mu,\nu,\theta)}\neq\emptyset\}$. Clearly, for fixed $\nu<\varkappa$ and $\theta<\omega_2^\gamma$,

$$\operatorname{tp}\{\mu<\omega_2^\sigma:(\mu,\nu,\theta)\in B_\alpha\}\leqq\operatorname{tp} A_\alpha<\omega_2^\varrho\quad(\alpha<\omega_1),$$

and so the sets B_α $(\alpha<\omega_1)$ are not full-sized in $\Pi(\boldsymbol{\xi})$. By Theorem 4.5(a) (for $k=2$) it follows that there are $D\in[\omega_1]^\omega$ and $B\subset\Pi(\boldsymbol{\xi})$ such that B is full-sized and $B_\alpha\cap B=\emptyset$ for all $\alpha\in D$. Put $C=\bigcup\{S_{(\mu,\nu,\theta)}:(\mu,\nu,\theta)\in B\}$. Then, by (4.11) and the fact that B is full-sized in $\Pi(\boldsymbol{\xi})$, we see that $\operatorname{tp} C=\omega_2^\varrho$. Moreover, by the definition of B_α, we have $A_\alpha\cap C=\emptyset$ for all $\alpha\in D$.

We conclude this chapter by showing that the positive results of Theorem 4.1 are best possible.

Theorem 4.6. *Let* $\tau<\omega_3$, $\operatorname{cf}(\tau)=\omega_1$, $0<\gamma<\omega_1$. *Then*

$$\text{(4.12)}\qquad \binom{\omega_1}{\omega_2^\tau}\nrightarrow\begin{pmatrix}1 & \omega\\ \omega_2^\omega\omega_1+1 & \omega_2^\tau\end{pmatrix}^{1,1},$$

and

$$(4.13)\qquad \begin{pmatrix}\omega_1\\ \omega_2^{\tau+\gamma}\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \omega\\ \omega_2^{\omega+\gamma}+1 & \omega_2^{\tau}\end{pmatrix}^{1,1}.$$

Proof. It is easy to see that (4.12) follows from Lemma 2.5 and the relation

$$(4.14)\qquad \begin{pmatrix}\omega_1\\ \omega_1\end{pmatrix} \nrightarrow \left(\begin{matrix}1\\ 1\end{matrix}, \begin{bmatrix}\omega\\ \omega_1\end{bmatrix}_{\omega, <\omega}\right)^{1,1}.$$

(Apply Lemma 2.5 with $\xi=\omega_1$, $\xi_1=\omega$, $\eta=\eta_1=\omega_1$, $\eta_0=1$, $\Xi_\nu=\omega_2^{\tau_\nu}$ $(\nu<\omega_1)$, where $\tau_\nu \nearrow \tau$. An easy computation shows that $\Xi=\Psi=\omega_2^\tau$ and $\Phi=\omega_2^\omega\omega_1+1$). Now in order to prove (4.14) consider the function $f:\omega_1\times\omega_1\to\omega$ given by $f(\alpha,\nu)=$ $=f_\alpha(\nu)$ where the f_α are pairwise almost disjoint functions in ${}^{\omega_1}\omega$. Now for $A\in[\omega_1]^\omega$ and $B\in[\omega_1]^{\omega_1}$, there is $\nu_0\in B$ such that $f_\alpha(\nu_0)\neq f_{\alpha'}(\nu_0)$ whenever α, α' are distinct elements of A. Thus $f''(A\times\{\nu_0\})$ is infinite. This proves (4.14) and hence (4.12).

To prove (4.13) use induction on γ. If $\gamma=\delta+1$ take identical cross sections of the inductive systems establishing the negative relation for $\omega_2^{\tau+\delta}$. If $\mathrm{cf}(\gamma)=\omega$ take cross sections.

We remark that if we assume the transversal hypothesis, $TH(\omega_1)$: *there are ω_2 almost disjoint functions in ${}^{\omega_1}\omega$* (which is known to be true in L and false if Chang's conjecture holds), then the argument used above yields the following stronger statements:

$$(4.15)\qquad \begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix} \nrightarrow \left(\begin{matrix}1\\ 1\end{matrix}, \begin{bmatrix}\omega\\ \omega_1\end{bmatrix}_{\omega, <\omega}\right)^{1,1},$$

$$(4.16)\qquad \begin{pmatrix}\omega_2\\ \omega_2^{\tau}\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \omega\\ \omega_2^{\omega}\omega_1+1 & \omega_2^{\tau}\end{pmatrix}^{1,1} \quad \text{if} \quad \mathrm{cf}(\tau)=\omega_1,$$

$$(4.17)\qquad \begin{pmatrix}\omega_2\\ \omega_2^{\tau+\gamma}\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \omega\\ \omega_2^{\omega+\gamma}+1 & \omega_2^{\tau}\end{pmatrix}^{1,1} \quad \text{if} \quad \mathrm{cf}(\tau)=\omega_1,\ 0<\gamma<\omega_1.$$

Finally, we note that in the case $\tau<\omega_3$, $\mathrm{cf}(\tau)=\omega_1$ there is a gap between the positive result (assuming $2^{\aleph_0}=\aleph_1$)

$$\begin{pmatrix}\omega_1\\ \omega_2^{\tau}\end{pmatrix} \to \begin{pmatrix}1 & \omega\\ \omega_2^{\omega} & \omega_2^{\tau}\end{pmatrix}^{1,1}$$

given by (4.4) and the negative result given by (4.12). This gap is easy to fill as the following theorem shows. Although this does not follow from Lemma 2.6, the proof is similar.

Theorem 4.7. *If $\tau<\omega_3$, $\mathrm{cf}(\tau)=\omega_1$, and $\delta<\omega_1$, then*

$$\begin{pmatrix}\omega_1\\ \omega_2^{\tau}\end{pmatrix} \to \begin{pmatrix}1 & \delta\\ \omega_2^{\omega}\omega_1 & \omega_2^{\tau}\end{pmatrix}^{1,1}.$$

Proof. Let $\langle S_\nu:\nu<\omega_1\rangle$ be the standard decomposition of ω_2^τ and let $\langle A_\alpha:\alpha<\omega_1\rangle$ be a system of subsets of ω_2^τ such that $\mathrm{tp}\, A_\alpha<\omega_2^\omega\omega_1$. For each $\alpha<\omega_1$ there are $\nu(\alpha)<\omega_1$ and $n(\alpha)<\omega$ such that $\mathrm{tp}\,(A_\alpha\cap S_\nu)<\omega_2^{n(\alpha)}$ for $\nu(\alpha)<\nu<\omega_1$. Now there is $D\subset\omega_1$ such that $\mathrm{tp}\, D=\delta$ and $n(\alpha)=n$ for all $\alpha\in D$. Clearly $\mathrm{tp}\,\{\omega_2^\tau\setminus\bigcup\{A_\alpha:\alpha\in D\}\}=\omega_2^\tau$.

5. Proof of Theorem 4.2. As we have already mentioned, both statements of Theorem 4.2 are generalizations of (4.4) which is a result of BAUMGARTNER and HAJNAL [3], and both can be proved with the methods used there. Since [3] appeared F. GALVIN [8] developed an elementary method to obtain the results of [3] and after we obtained Theorem 4.2 he kindly informed us that his method can be used to prove this theorem as well. However, we decided to give our original proof since it can be explained in less space.

As in [3] both statements will be proved first under the assumption that $MA_{\aleph_1}$ holds (for Martin's axiom see e.g. [12]). Then we exhibit partial orders which are well-founded iff the corresponding statements are false. This will show that if the statements are true in the standard model of Solovay and Tennenbaum yielding the consistency of $MA_{\aleph_1}$, then they are true in the ground model i.e. they are true in ZFC.

First we describe the partial orders.

a) Let $\xi=(\omega^{\gamma_i}:i<n)$, $0<\gamma_i<\omega_1$ $(i<n)$ and let $f:\omega_1\times\Pi(\xi)\to k$ be given. Let φ be a one-to-one mapping of ω onto $\Pi(\xi)$. For $m<\omega$ and $i<n$ we write $\varphi(m,i)$ to denote the i-th coordinate of $\varphi(m)\in\Pi(\xi)$. Let P be the set of all pairs of sequences $((\alpha_j: j<l), (\mathbf{u}_j: j<l))$, where $l<\omega$, the α_j are different ordinals $<\omega_1$ for $j<l$, $\mathbf{u}_j=(u_{j,0},\dots,u_{j,n-1})\in\Pi(\xi)$ for $j<l$,

$$u_{j,i}<u_{j',i} \text{ iff } \varphi(j,i)<\varphi(j',i) \text{ for } j<j'<l,$$

and such that f is homogeneous on $\{\alpha_j:j<l\}\times\{\mathbf{u}_j:j<l\}$. The partial order is defined on P by the rule that $((\alpha'_j:j<l'), (\mathbf{u}'_j:j<l'))$ is an extension of $((\alpha_j:j<l), (\mathbf{u}_j:j<l))$ iff $(\alpha'_j:j<l')$ is an extension of $(\alpha_j:j<l)$ and $(\mathbf{u}'_j:j<l')$ is an extension of $(\mathbf{u}_j:j<l)$.

b) Let $\gamma<\omega_1$ and let f be a partial function from $\omega_1\times\omega^\gamma$ into ω. Let φ be a one-to-one mapping from ω onto ω^γ. Let P consist of all pairs $((\alpha_j:j<n), (u_j:j<l))$, where $l<\omega$, the α_j are different ordinals $<\omega_1$ for $j<l$, $u_j<\omega^\gamma$, $u_j<u_{j'}$ iff $\varphi(j)<\varphi(j')$ for $j<j'<l$, and such that *either* $(\{\alpha_j:j<l\}\times\{u_j:j<l\})\cap D(f)=0$ holds *or* $\{\alpha_j:j<l\}\times\{u_j:j<l\}\subset D(f)$ and $f(\alpha_{j'},u_j)=f(a_{j''},u_j)$ for $j<j', j''<l$. The extension is defined as in case a).

We leave the reader to check that these partial orders have the desired properties and proceed to derive the statements (4.7) and (4.8) from $MA_{\aleph_1}$.

PROOF OF (4.7) FROM $MA_{\aleph_1}$. Let $\xi=(\xi_i:i<n)$ be given, where $\xi_i=\omega^{\gamma_i}$, $\gamma_i<\omega_1$ for $i<n<\omega$.

For $X\subset\Pi(\xi)$ we denote by $\tilde{X}$ the projection of X into ξ_{n-1}. A subset $Y\subset X$ will be called a *section* of X (in $\Pi(\xi)$) if $X=\bigcup\{X^u:u\in S\}$ and S is a section of $\tilde{X}$.

Instead of (4.7) we are going to prove the following stronger statement.

(5.1) *Assume $f:\omega_1\times\Pi(\xi)\to k$, where $k<\omega$. Then there are a full-sized subset $X\subset\Pi(\xi)$, increasing sections $X_0\subset X_1\subset\dots$ of X* (in $\Pi(\xi)$), *and functions $m:\omega_1\to\omega$, $j:\omega_1\to k$ such that* (i) $X=\bigcup\{X_i:i<\omega\}$ *and* (ii) $f''(\{\alpha\}\times(X\setminus X_{m(\alpha)}))=\{j(\alpha)\}$ (i.e. $\{\alpha\}\times(X\setminus X_{m(\alpha)})$ is homogeneous for f in the colour $j(\alpha)$).

We prove (5.1) by induction on n, the length of ξ. For $n=1$ this is just Lemma 3 of [3]. Now assume $n>1$.

Since f may be considered as a map from $(\omega_1\times\xi_{n-1})\times\Pi(\hat{\xi})$ into k, and since $\omega_1\times\xi_{n-1}$ has cardinality ω_1, it follows from the induction hypothesis that there are sets Y, Y_i $(i<\omega)$ and functions $m_1:\omega_1\times\xi_{n-1}\to\omega$ and $j_1:\omega_1\times\xi_{n-1}\to k$ such that Y is a full-sized subset of $\Pi(\hat{\xi})$, the Y_i are sections of Y (in $\Pi(\hat{\xi})$), $Y_0\subset Y_1\subset\ldots$, $Y=\bigcup\{Y_i:i<\omega\}$ and

(5.2) $$f''(\{\alpha\}\times(Y\setminus Y_{m_1(\alpha,u)})\times\{u\})=\{j_1(\alpha,u)\}$$

holds for all $\alpha<\omega_1$ and $u\in\xi_{n-1}$.

Applying the induction hypothesis once more for the function $j_1:\omega_1\times\xi_{n-1}\to k$, it follows also that there are sets Z, Z_i $(i<\omega)$ and functions $m_2:\omega_1\to\omega$ and $j:\omega_1\to k$ such that Z is a full-sized subset of ξ_{n-1} (i.e. has type ξ_{n-1}), the Z_i are sections of Z, $Z_0\subset Z_1\subset\ldots$, $Z=\bigcup\{Z_i:i<\omega\}$ and

(5.3) $j_1(\alpha,u)=j(\alpha)$ for $\alpha<\omega_1$ and $u\in Z\setminus Z_{m_2(\alpha)}$.

Put $\varphi_\alpha(u)=m_1(\alpha,u)$ for $\alpha<\omega_1$ and $u\in Z$. By a lemma of K. KUNEN (see e.g. [3]), $MA_{\aleph_1}$ implies that there is a function $\varphi:Z\to\omega$ such that

(5.4) $\varphi_\alpha(u)<\varphi(u)$ *for all but finitely many* $u\in Z$

holds for each $\alpha<\omega_1$.

Let X be the full-sized subset of $\Pi(\xi)$ such that $\bar{X}=Z$ and $\hat{X}^{(u)}=Y\setminus Y_{\varphi(u)}$ $(u\in Z)$. Let X_i be the section of X determined by Z_i $(i<\omega)$. By (5.4) there is $m:\omega_1\to\omega$ such that $m(\alpha)\geq m_2(\alpha)$ $(\alpha<\omega_1)$ and such that $\varphi_\alpha(u)<\varphi(u)$ for all $u\in Z\setminus Z_{m(\alpha)}$ $(\alpha<\omega_1)$. Now for $\alpha<\omega_1$ and $u\in Z\setminus Z_{m(\alpha)}$, we have $Y\setminus Y_{\varphi_\alpha(u)}\supset\supset Y\setminus Y_{\varphi(u)}=\hat{X}^{(u)}$ and hence, by (5.2) and (5.3),

$$f''(\{\alpha\}\times\hat{X}^{(u)}\times\{u\})=\{j_1(\alpha,u)\}=\{j(\alpha)\},$$

i.e.

$$f''(\{\alpha\}\times(X\setminus X_{m(\alpha)}))=\{j(\alpha)\}.$$

PROOF OF (4.8) FROM $MA_{\aleph_1}$. Let now f be a partial function from $\omega_1\times\omega^\gamma$ into ω. By $MA_{\aleph_1}$ we know that

$$\binom{\omega_1}{\omega^\gamma}\to\binom{\omega_1}{\omega^\gamma}^{1,1}_k.$$

This is a theorem of [3] and in fact is a corollary of (4.7) just proved. Therefore, we may assume that $D(f)=\omega_1\times\omega^\gamma$. We can also assume that $\gamma>0$. Let $\{x_n:n<\omega\}=\omega^\gamma$ be a one-to-one enumeration of ω^γ. We define a sequence $\{y_n:n<\omega\}$ of elements of ω_1 and a sequence $\{Y_n:n<\omega\}$ of subsets of ω_1 by induction on $n<\omega$ as follows. $Y_0=\omega_1$. Assume that $n<\omega$ and that $Y_n\in[\omega_1]^{\omega_1}$ and y_j $(j<n)$ have already been defined. Choose a subset $Z_n\in[Y_n]^{\omega_1}$ such that $Z_n\times\{x_n\}$ is homogeneous for f. Let y_n be an arbitrary element of Z_n and $Y_{n+1}=Z_n\setminus\{y_n\}$. Let $A_1=\{y_n:n<\omega\}$ and $B_1=\omega^\gamma$. If $x_n\in B_1$, then $f''(A_1\times\{x_n\})=f''(\{y_i:i\leq n\}\times\{x_n\})$ is clearly finite. This proves (4.8).

Note that in both (4.7) and (4.8) the ω on the right-hand sides can be replaced by any $\alpha<\omega_1$, but we do not need this.

6. The case of $\aleph_2$ sets. The first aim of this chapter is to prove an analogue of Theorem 1.1 for the case of $\aleph_2$ sets. This will show that the critical number for which the "$\aleph_2$-phenomena" appears in this case is ω_1+2 instead of ω_1.

THEOREM 6.1. *If* $\gamma \leqq \omega_2$, *then*

$$\binom{\omega_2}{\omega_2^\gamma} \nrightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1+2} & 1 \end{pmatrix}^{1,1}. \tag{6.1}$$

PROOF. We use induction on γ. The statement is clearly true for $\gamma \leqq \omega_1+1$. We now assume $\omega_1+1 < \gamma \leqq \omega_2$ and that the statement is true for all $\delta < \gamma$. Let $\varkappa = \mathrm{cf}(\omega_2^\gamma)$ and let $\langle S_\nu : \nu < \varkappa \rangle$ be a standard decomposition of ω_2^γ, $\mathrm{tp}\, S_\nu = \omega_2^{\gamma_\nu}$, $(\nu < \varkappa)$. By the induction hypothesis we can choose a system of subsets $\langle A_\alpha^\nu : \alpha < \omega_2 \rangle$ of S_ν $(\nu < \varkappa)$ such that $\mathrm{tp}\, A_\alpha^\nu < \omega_2^{\omega_1+2}$ and $S_\nu \subset \bigcup \{A_\alpha^\nu : \alpha \in D\}$ for $D \in [\omega_2]^\omega$. We now distinguish the three cases 1 $\varkappa = \omega_2$, (2) $\varkappa = \omega$, (3) $\varkappa = \omega_1$.

Case 1. Since $\gamma_\nu < \omega_2$ for $\nu < \omega_2$, by Theorem 1.1, we can find sets $B_\beta^\nu \subset S^\nu$ $(\beta < \omega_1)$ establishing

$$\binom{\omega_1}{\omega_2^{\gamma_\nu}} \nrightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1}.$$

We choose a one-to-one mapping φ_ν of ν into ω_1 for $\nu < \omega_2$. Now for $\alpha < \omega_2$, let $A_\alpha = \bigcup \{A_\alpha^\nu : \nu < \alpha\} \cup \bigcup \{B_{\varphi_\nu(\alpha)}^\nu : \alpha < \nu\}$. Clearly, $\mathrm{tp}\, A_\alpha < \omega_2^{\omega_1+2}$ for $\alpha < \omega_2$. Let $D \in [\omega_2]^\omega$, $\nu < \omega_2$. Put $D_0 = \{\alpha \in D : \nu < \alpha\}$, $D_1 = \{\alpha \in D : \alpha < \nu\}$. Since either D_0 or D_1 is infinite, either $\bigcup \{\alpha \in D_0 : A_\alpha^\nu\}$ or $\bigcup \{\alpha \in D_1 : B_{\varphi_\nu(\alpha)}^\nu\}$ covers S_ν. Thus $\bigcup \{A_\alpha : \alpha \in D\}$ covers S_ν in any case. It follows that $\omega_2^\gamma \subset \bigcup \{A_\alpha : \alpha \in D\}$. Hence the system $\langle A_\alpha : \alpha < \omega_2 \rangle$ establishes (6.1).

For cases 2, 3 take cross sections.

The partition relation just established shows the same "$\aleph_2$-phenomena" as the relation $\binom{\omega_1}{\omega_2^\gamma} \to \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1}$. We do not go into details, but we will show the following equivalence:

$$\binom{\omega_2}{\omega_2^{\omega_1+1}} \to \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1+2} & 1 \end{pmatrix}^{1,1} \Leftrightarrow \binom{\omega_1}{\omega_2^{\omega_1}} \to \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1}. \tag{6.2}$$

The implication from the left is implicitly contained in the proof of Theorem 6.1, and the reverse implication is an easy corollary of the following lemma. We leave it to the reader to derive (6.2) from the lemma which we prove in detail since this will be needed for other purposes.

LEMMA 6.2. *Assume* $\delta < \omega_3$ *and that* $\mathrm{cf}(\omega_2^\delta)$ *is either* ω *or* ω_2. *Let* $\langle A_\alpha : \alpha < \omega_2 \rangle$ *be a sequence of subsets of* ω_2^δ *such that* $\mathrm{tp}\, A_\alpha < \omega_2^{\omega_1+1}$ *for* $\alpha < \omega_2$. *Then there exist* $X \subset \omega_2^\delta$, $\gamma < \omega_1$ *and* $D \in [\omega_2]^{\omega_1}$ *such that* $\mathrm{tp}\, X = \omega_2^\delta$ *and* $\mathrm{tp}\, A_\alpha \cap X < \omega_2^\gamma$ *for all* $\alpha \in D$.

PROOF. Let $\varkappa = \mathrm{cf}(\omega_2^\delta)$ and let $\langle S_\nu : \nu < \varkappa \rangle$ be a standard decomposition of ω_2^δ. We distinguish the two cases (1) $\varkappa = \omega$, (2) $\varkappa = \omega_2$.

Case 1. We may assume that $\mathrm{tp}\, S_n = \omega_2^{\delta_n+1}$ for $n < \omega = \varkappa$. Let $\langle S_{n,\varrho} : \varrho < \omega_2 \rangle$ be a standard decomposition of S_n for $n < \omega$. Let $\alpha < \omega_2$. Using the fact that $\mathrm{tp}\, A_\alpha < \omega_2^{\omega_1+1}$, for each $n < \omega$ there are $\beta(\alpha, n) < \omega_1$ and $\varrho(\alpha, n) < \omega_2$ such that $\mathrm{tp}\,(A_\alpha \cap S_{n,\varrho}) < \omega^{\beta(\alpha,n)}$ for $\varrho(\alpha, n) < \varrho < \omega_2$. Put $\beta(\alpha) = \sup\{\beta(\alpha, n) : n < \omega\}$ and $\varrho(\alpha) = \sup\{\varrho(\alpha, n) : n < \omega\}$. Then $\beta(\alpha) < \omega_1$ and $\varrho(\alpha) < \omega_2$ for $\alpha < \omega_2$. Clearly there are $D \in [\omega_2]^{\omega_1}$, $\gamma < \omega_1$ and $\sigma < \omega_2$ such that

$$\beta(\alpha)+1 = \gamma \quad \text{and} \quad \varrho(\alpha) < \sigma \quad \text{for} \quad \alpha \in D.$$

Let $X = \bigcup \{S_{n,\sigma} : n < \omega\}$. Then X, γ and D satisfy the requirements of the lemma.

Case 2. For each $\alpha<\omega_2$ there are $\beta(\alpha)<\omega_1$ and $\varrho(\alpha)<\omega_2$ such that $\operatorname{tp}(S_\nu\cap A_\alpha)<\omega_2^{\beta(\alpha)}$ for $\varrho(\alpha)<\nu<\omega_2=\varkappa$. There are $D\in[\omega_2]^{\omega_1}$, $\gamma<\omega_1$ and $\sigma<\omega_2$ such that $\beta(\alpha)+1=\gamma$ and $\varrho(\alpha)<\sigma$ for $\alpha\in D$. Put $X=\bigcup\{S_\nu:\sigma\leqq\nu<\omega_2\}$. Then X,γ and D satisfy the requirements.

We now try to investigate the analogue of (3.1) for the case of $\aleph_2$-sets, i.e. we consider the relation

$$\binom{\omega_2}{\omega_2^\varrho}\to\binom{1\quad\omega_1}{\omega_2^\gamma\quad\omega_2^\tau}^{1,1} \tag{6.3}$$

for $\varrho,\tau<\omega_3$ and $\gamma<\omega_1+2$.

First we give an extension of the positive result Theorem 3.1.

THEOREM 6.2. *Assume* $\tau<\omega_3$, $\tau+\gamma\geqq\omega+\gamma$, *and* $\gamma<\omega_1+2$. *Then*

$$\binom{\omega_2}{\omega_2^{\tau+\gamma}}\to\binom{1\quad\omega_1}{\omega_2^{\omega+\gamma}\quad\omega_2^\tau}^{1,1}.$$

Note that for $\gamma<\omega_1$ Theorem 3.1(b) yields a stronger result. For $\gamma=\omega_1$ or ω_1+1 this can be considered as a generalization of c) of Theorem 3.1 as well, since, $1+\gamma=\gamma$ holds in this case. In case $\gamma=\omega_1$ Theorem 6.2 is trivial since among $\aleph_2$ sets of type $<\omega_2^{\omega_1}$ there are $\aleph_2$ with type $<\omega_2^\delta$ for some $\delta<\omega_1$. Assume now that $A_\alpha\subset\omega_2^{\tau+\omega_1+1}$ and $\operatorname{tp}A_\alpha<\omega_2^{\omega_1+1}$ for $\alpha<\omega_2$. Then, by Lemma 6.2, there are $X\subset\subset\omega_2^{\tau+\omega_1+1}$, $D\in[\omega_2]^{\omega_1}$ and $\gamma<\omega_1$ such that

$$\operatorname{tp}X=\omega_2^{\tau+\omega_1+1},\quad\text{and}\quad\operatorname{tp}A_\alpha\cap X<\omega_2^\gamma$$

for $\alpha\in D$. Hence our claim follows from Theorem 3.1(b).

One could conjecture that, at least assuming $2^{\aleph_0}=\aleph_1$ and $2^{\aleph_1}=\aleph_2$, Theorem 6.2 gives the best possible positive result. However, we were not able to prove this. We now discuss the problem by considering separately the different possible values of $\operatorname{cf}(\tau)$. In case $\operatorname{cf}(\tau)=\omega_1$ we already noted ((4.16) and (4.17)) that if $TH(\omega_1)$ holds, then

$$\binom{\omega_2}{\omega_2^\tau}\not\to\binom{1\quad\omega}{\omega_2^\omega\omega_1+1\quad\omega_2^\tau}\quad for\quad\tau<\omega_3,\quad\operatorname{cf}(\tau)=\omega_1 \tag{6.4}$$

and

$$\binom{\omega_2}{\omega_2^{\tau+\gamma}}\not\to\binom{1\quad\omega}{\omega_2^{\omega+\gamma}+1\quad\omega_2^\tau}\quad for\quad\tau<\omega_3,\quad\operatorname{cf}(\tau)=\omega_1,\quad 0<\gamma<\omega_1. \tag{6.5}$$

We are going to prove the following extension of this.

THEOREM 6.4. *Assume* $TH(\omega_1)$. *If* $\tau<\omega_3$, $\operatorname{cf}(\tau)=\omega_1$ *and* $0<\gamma<\omega_1+2$, *then*

$$\binom{\omega_2}{\omega_2^{\tau+\gamma}}\not\to\binom{1\quad\omega}{\omega_2^{\omega+\gamma}+1\quad\omega_2^\tau}^{1,1}.$$

PROOF. In view of (6.5) we only have to prove this for $\gamma=\omega_1$ and $\gamma=\omega_1+1$. Suppose $\gamma=\omega_1$. Let $\langle S_\nu:\nu<\omega_1\rangle$ be a standard decomposition of $\omega_2^{\tau+\omega_1}$ such that $\operatorname{tp}S_\nu=\omega_2^{\omega_1+\nu}$ for $\nu<\omega_1$. For each $\nu<\omega_1$ choose sets $A_\alpha^\nu\subset S_\nu$ $(\alpha<\omega_2)$ establishing

$$\binom{\omega_2}{\omega_2^{\tau+\nu}}\not\to\binom{1\quad\omega}{\omega_2^{\omega+\nu}+1\quad\omega_2^\tau}^{1,1}.$$

For each $\nu<\omega_1$ choose a paradoxical decomposition $\langle B_n^\nu : n<\omega\rangle$ of S_ν. By $TH(\omega_1)$ there are ω_2 almost disjoint functions $f_\alpha:\omega_1\to\omega$ $(\alpha<\omega_2)$. Now for $\alpha<\omega_2$ put

$$A'_\alpha=\bigcup\{A_\alpha^\nu:\ \nu<\omega_1\},\quad A''_\alpha=\bigcup\{B^\nu_{f_\alpha(\nu)}:\ \nu<\omega_1\},$$

$A_\alpha=A'_\alpha\cup A''_\alpha$. Clearly $\operatorname{tp} A_\alpha<\omega_2^{\omega_1}+1$. The union of countably many A'_α omits a set of type $<\omega_2^\xi$ from each S_ν $(\nu<\omega_1)$. The union of countably many A''_α covers an end section of $\omega_2^{\xi+\omega_1}$. It follows that

$$\operatorname{tp}(\omega_2^{\xi+\omega_1}\setminus\bigcup\{A_\alpha:\alpha\in D\})<\omega_2^\xi$$

for all $D\in[\omega_2]^\omega$.

For the case $\gamma=\omega_1+1$, take identical cross sections of the systems obtained for $\gamma=\omega_1$.

We now make some remarks about these results.

First of all, it is not worth continuing the induction of the last theorem beyond $\gamma=\omega_1+1$. For, by combining the methods used to prove Theorems 6.1 and 6.2, we can easily prove:

If $TH(\omega_1)$ holds, $\tau<\omega_3$, $\operatorname{cf}(\tau)=\omega_1$ and $\gamma\geqq\omega_2$, then

$$\binom{\omega_2}{\omega_2^{\xi+\gamma}}\nrightarrow\begin{pmatrix}1 & \omega\\ \omega_2^{\omega_1+2} & \omega_2^\xi\end{pmatrix}^{1,1}. \tag{6.6}$$

Our second remark is that it would be sufficient for our present purposes if we could prove (6.4) with ω replaced by ω_1. We do not know if $TH(\omega_1)$ is really needed for this weaker relation. This leads to the following question.

PROBLEM 1. *Assuming GCH, does the relation*

$$\binom{\omega_2}{\omega_1}\nrightarrow\left(\begin{matrix}1\\1\end{matrix},\begin{bmatrix}\omega_1\\ \omega_1\end{bmatrix}_{\omega,<\omega_1}\right)^{1,1}$$

hold?

The relation (6.4) should be compared with the positive relation

$$\binom{\omega_1}{\omega_2^\xi}\to\begin{pmatrix}1 & \delta\\ \omega_2^\omega\omega_1 & \omega_2^\xi\end{pmatrix}^{1,1}$$

given by Theorem 4.7 ($\tau<\omega_3$, $\operatorname{cf}(\tau)=\omega_1$ and $\delta<\omega_1$). This shows that the term $\omega_2^\omega\omega_1+1$ in (6.4) cannot be decreased. Now (6.4) (obtained from $TH(\omega_1)$) implies the weaker relation

$$\binom{\omega_2}{\omega_2^\tau}\nrightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^\omega\omega_1+1 & \omega_2^\tau\end{pmatrix}^{1,1} \tag{6.7}$$

which should be compared with (3.3) of Theorem 3.2, the corresponding result for $\aleph_1$ sets. We already saw that (3.3) is best possible (Theorem 3.5) and we now show that (6.7) is also best possible, i.e.

$$\binom{\omega_2}{\omega_2^\tau}\to\begin{pmatrix}1 & \omega_1\\ \omega_2^\omega\omega_1 & \omega_2^\tau\end{pmatrix}^{1,1}\quad\text{if}\quad \tau<\omega_3,\quad \operatorname{cf}(\tau)=\omega_1. \tag{6.8}$$

PROOF. Let $\langle S_\nu:\nu<\omega_1\rangle$ be a standard decomposition of ω_2^τ. We may assume that $\operatorname{tp} S_\nu\geqq\omega_2^\omega$ for $\nu<\omega_1$. Assume $A_\alpha\subset\omega_2^\tau$ $(\alpha<\omega_2)$ and $\operatorname{tp} A_\alpha<\omega_2^\omega\omega_1$. Then for each $\alpha<\omega_2$ there are $\varrho(\alpha)<\omega_1$ and $n(\alpha)<\omega$ such that $\operatorname{tp} A_\alpha\cap S_\nu<\omega_2^{\varrho(\alpha)}$ for $\varrho(\alpha)<\nu<\omega_1$. There are $D\in[\omega_2]^{\omega_2}$, $n<\omega$, $\varrho<\omega_1$ such that $n(\alpha)=n$ and $\varrho(\alpha)=\varrho$ for $\alpha\in D$. Clearly $\operatorname{tp}(\omega_2^\xi\setminus\cup\{A_\alpha:\alpha\in D\})=\omega_2^\xi$.

This concludes our discussion of (6.3) in case $\operatorname{cf}(\tau)=\omega_1$ and we know, assuming $TH(\omega_1)$, that our results are best possible.

In case $\operatorname{cf}(\omega_2^\xi)=\omega_2$ we already know, by (3.12) of Theorem 3.3, that $2^{\aleph_1}=\aleph_2$ implies

$$\text{(6.9)}\qquad \begin{pmatrix}\omega_2\\ \omega_2^{\tau+\gamma}\end{pmatrix}\nrightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1+\gamma}+1 & \omega_2^\tau\end{pmatrix}$$

holds for $\gamma<\omega_1$.

Again this can be generalized as follows.

THEOREM 6.5. *Assume* $2^{\aleph_1}=\aleph_2$. *If* $\tau<\omega_3$ *and* $\operatorname{cf}(\tau)=\omega_2$, *then* (6.9) *holds for* $\gamma<\omega_1+2$.

PROOF. To see this in case $\gamma=\omega_1$ just take cross sections. This works since $\operatorname{cf}(\omega_2^\xi)=\omega_2$. For $\gamma=\omega_1+1$ we obtain it from the statement with $\gamma=\omega_1$ using the partition relation (3.8). We omit the details.

This shows that our positive result Theorem 6.2 is again best possible for the case $\operatorname{cf}(\omega_2^\xi)=\omega_2$, at least assuming $2^{\aleph_1}=\aleph_2$. Unfortunately, we do not know if the same is true in the case $\operatorname{cf}(\tau)=\omega$. Our main unsolved problem is the following.

PROBLEM 2. *Assume* $2^{\aleph_0}=\aleph_1$. *Is the relation*

$$\begin{pmatrix}\omega_2\\ \omega_2^{\omega\cdot 2}\end{pmatrix}\nrightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^\gamma & \omega_2^{\omega\cdot 2}\end{pmatrix}^{1,1}$$

true for some γ, $\omega+2\leqq\gamma\leqq\omega\cdot 2$?

All our methods for constructing counter examples break down. We do know that (3.2) does not remain true for $\aleph_2$ sets for $\tau>\omega$, $\operatorname{cf}(\tau)=\omega$. The following partial result shows why we insist that $\gamma\geqq\omega+2$ in Problem 2.

THEOREM 6.6. *Assume* $2^{\aleph_0}=\aleph_1$. *If* $\omega<\tau<\omega_3$, $\operatorname{cf}(\tau)=\omega$ *and* $\xi<\omega_2$, *then*

$$\begin{pmatrix}\omega_2\\ \omega_2^\xi\end{pmatrix}\rightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1}\cdot\xi & \omega_2^\xi\end{pmatrix}.$$

PROOF. Let $\langle S_n:n<\omega\rangle$ be a standard decomposition of ω_2^ξ. We may assume that $\operatorname{tp} S_n=\omega_2^{\gamma_n+1}$ and $\gamma_n\geqq\omega$ for $n<\omega$. Let $\langle S_{n,\varrho}:\varrho<\omega_2\rangle$ be a standard decomposition of S_n for $n<\omega$. Assume $A_\alpha\subset\omega_2^\xi$, $\operatorname{tp} A_\alpha<\omega_2^{\omega+1}\cdot\xi$ for $\alpha<\omega_2$.

Assume now, that $\xi\leqq\alpha<\omega_2$. For each $n<\omega$ there is $\varrho(\alpha,n)<\alpha$ such that $\operatorname{tp}(A_\alpha\cap S_{n,\varrho(\alpha,n)})<\omega_2^{\omega+1}$. Put $\varrho(\alpha)=\sup\{\varrho(\alpha,n):n<\omega\}$. Since $\{\alpha<\omega_1:\xi\leqq\alpha<\omega_2,\ \operatorname{cf}(\alpha)=\omega_1\}$ is a stationary subset of ω_2 it follows that there are $D\in[\omega_2]^{\omega_2}$ and $\varrho<\omega_2$ such that $\varrho(\alpha)=\varrho$ for all $\alpha\in D$. Using $2^{\aleph_0}=\aleph_1$ it follows that there are $D'\in[D]^{\omega_2}$ and a sequence $(\varrho_n:n<\omega)$ such that $\varrho(\alpha,n)=\varrho_n$ for $\alpha\in D'$. Considering $\cup\{S_{n,\varrho_n}:n<\omega\}$ has type ω_2^ξ this shows that it is sufficient to prove only that

$$\begin{pmatrix}\omega_2\\ \omega_2^\xi\end{pmatrix}\rightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1} & \omega_2^\xi\end{pmatrix}^{1,1}$$

holds.

To see this consider again the S_n and $S_{n,\varrho}$ as defined in the first part of the proof and assume now that $A_\alpha \subset \omega_2^\tau$, $\operatorname{tp} A_\alpha < \omega_2^{\omega+1}$ for $\alpha < \omega_2$.

There are $\varrho(\alpha) < \omega_2$ and $k(\alpha, n) < \omega$ such that

$$\operatorname{tp}(A_\alpha \cap S_{n,\varrho}) < \omega_2^{k(\alpha,n)} \quad \text{for} \quad \varrho(\alpha) < \varrho < \omega_2 .$$

Using once again that $2^{\aleph_0} = \aleph_1$, we see that there are $D \in [\omega_2]^{\omega_2}$ and $k_n < \omega$ $(n < \omega)$ such that $k(\alpha, n) = k_n$ for $\alpha \in D$. Let $D' \in [D]^{\omega_1}$ be arbitrary and let $\varrho = \sup\{\varrho(\alpha): \alpha \in D'\}$. Then $\operatorname{tp}(A_\alpha \cap S_{n,\varrho}) < \omega_2^{k_n}$ for $\varrho < \varrho' < \omega_2$, $n < \omega$ and $\alpha \in D'$. Hence $\operatorname{tp}(S_n \setminus \bigcup \{A_\alpha : \alpha \in D\}) \geqq \omega_2$ for $n < \omega$ and the result follows.

We conclude this chapter by analyzing the analogue of (4.1) for $\aleph_2$ sets, i.e. the relation

$$(6.10) \qquad \binom{\omega_2}{\omega_2^\sigma} \to \binom{1 \quad \omega}{\omega_2^\gamma \quad \omega_2^\tau}^{1,1}$$

for $\sigma, \tau < \omega_3$ and $\gamma < \omega_1 + 2$.

In view of Theorems 6.3 and 6.4, a discussion for (6.10) will be completed by the following analogue of (4.3).

THEOREM 6.7. *Assume* $2^{\aleph_0} = \aleph_1$. *If* $\tau < \omega_3$, $\operatorname{cf}(\omega_2^\tau) = \omega$ *or* ω_2, $\delta < \omega_1$, $\gamma \leqq \tau + \delta$ *and* $\gamma < \omega_1 + 2$, *then*

$$\binom{\omega_2}{\omega_2^{\tau+\delta}} \to \binom{1 \quad \omega}{\omega_2^\gamma \quad \omega_2^{\tau+\delta}}^{1,1}.$$

PROOF. It is clearly sufficient to prove this for $\gamma = \omega_1 + 1$. Assume $A_\alpha \subset \omega_2^{\tau+\delta}$, $\operatorname{tp} A_\alpha < \omega_2^{\omega_1+1}$ for $\alpha < \omega_2$. Now $\operatorname{cf}(\omega_2^{\tau+\delta})$ is either ω or ω_2 since $\delta < \omega_1$. Hence, by the Lemma 6.2, there are $X \subset \omega_2^{\tau+\delta}$, $\gamma_0 < \omega_2$ and $D \in [\omega_2]^{\omega_1}$ such that $\operatorname{tp} X = \omega_2^{\tau+\delta}$ and $\operatorname{tp}(A_\alpha \cap X) < \omega_2^{\gamma_0}$ for all $\alpha \in D$. Hence the statement follows from

$$\binom{\omega_1}{\omega_2^{\tau+\delta}} \to \binom{1 \quad \omega}{\omega_2^{\gamma_0} \quad \omega_2^{\tau+\delta}}^{1,1}$$

which is (4.3) of Theorem 4.1.

7. Pointwise-finite systems. A system $\langle A_\alpha : \alpha < \varkappa \rangle$ of subsets of S is said to be *pointwise-finite* if each point of the underlying set S is a member of only a finite number of the A_α $(\alpha < \varkappa)$. In this chapter we investigate analogues of some of our earlier results for pointwise-finite systems. This amounts to an investigation of relations of the form

$$(7.1) \qquad \binom{\varkappa}{\omega_2^\varrho} \to \left(\frac{1}{\omega_2^\sigma} \vee \frac{\omega \quad \omega}{1 \quad \omega_2^\tau}\right)^{1,1}$$

for $\varrho, \sigma, \tau < \omega_3$ and $\varkappa = \omega, \omega_1$ or ω_2. While this leads to several quite interesting new problems we shall not discuss this in the same detail as we did for the case when the pointwise-finite condition is left out. The results we prove below give the analogues for (7.1) in the cases $\varkappa = \omega, \omega_1$ and ω_2 which correspond respectively to the negative relations of (1.2), Theorem 1.1 and Theorem 6.1.

The following theorem is related to our earlier result ([7], Theorem 8) and so is the method of proof, but we believe it is worth giving the details.

THEOREM 7.1. *Assume* $2^{\aleph_0}=\aleph_1$. *Let* $\gamma<\omega_2$, $\tau\leqq\omega_2$. *Then*

$$\binom{\omega}{\omega_2^\gamma} \nrightarrow \left(\begin{matrix}1\\ \omega_2^\omega\end{matrix} \vee \begin{matrix}\omega & \omega\\ 1 & \omega_2^{\omega_1}\end{matrix}\right)^{1,1}, \tag{7.2}$$

$$\binom{\omega_1}{\omega_2^\gamma} \nrightarrow \left(\begin{matrix}1\\ \omega_2^{\omega_1}\end{matrix} \vee \begin{matrix}\omega & \omega\\ 1 & \omega_2^{\omega_1}\end{matrix}\right)^{1,1}, \tag{7.3}$$

and

$$\binom{\omega_2}{\omega_2^\tau} \to \left(\begin{matrix}1\\ \omega_2^{\omega_1+2}\end{matrix} \vee \begin{matrix}\omega & \omega\\ 1 & \omega_2^{\omega_1+2}\end{matrix}\right)^{1,1}. \tag{7.4}$$

We need the following lemma which was also used in [7] (Lemma 5). For the convenience of the reader we give the short proof.

LEMMA 7.2. *Let* $\langle T_n:n<\omega\rangle$ *be a sequence of denumerable subsets of* ω. *Then there is a pointwise-finite sequence* $\langle C_k:k<\omega\rangle$ *of finite subsets of* ω *such that*

$$C_k\subset k \tag{7.5}$$

and

$$\omega\setminus\bigcup\{C_k:k\in T_n\} \quad \text{is finite for all} \quad n<\omega. \tag{7.6}$$

PROOF. We can assume that the sets T_n are pairwise disjoint (since we can replace them by infinite disjoint subsets). Let t_n denote the least member of T_n and let $T_n'=T_n\setminus\{t_n\}$. We define the C_k $(k<\omega)$ as follows. If $k\in\omega\setminus\bigcup\{T_n':n<\omega\}$ put $C_k=\emptyset$. If $k\in\bigcup\{T_n':n<\omega\}$ then there are unique integers $m(k)<k$ and $n(k)$ such that $k\in T_{n(k)}'$, $m(k)\in T_{n(k)}$ and $i\notin T_{n(k)}$ for $m(k)<i<k$. In this case put $C_k=[m(k),k)$. It is easy to verify that the C_k are pointwise-finite and that (7.5) and (7.6) hold.

PROOF OF (7.2). We prove slightly more. We show that there is a pointwise-finite system $\langle A_k:k<\omega\rangle$ of subsets of ω_2^γ which establishes (7.2) and satisfies the stronger condition that

$$\operatorname{tp} A_k<\omega_2^k \quad \text{for} \quad k<\omega. \tag{7.7}$$

The proof is by induction on γ. For $\gamma=0$ the result is trivial. Now assume that $\gamma>0$.

Let $\langle S_\nu:\nu<\operatorname{cf}(\omega_2^\gamma)\rangle$ be a standard decomposition of ω_2^γ. By the induction hypothesis, for each $\nu<\operatorname{cf}(\omega_2^\gamma)$ there is a pointwise-finite sequence $\langle A_k^\nu:k<\omega\rangle$ of subsets of S_ν establishing the corresponding result for $\operatorname{tp}(S_\nu)$. We now distinguish the cases (1) $\operatorname{cf}(\omega_2^\gamma)=\omega$ or ω_2 and (2) $\operatorname{cf}(\omega_2^\gamma)=\omega_1$.

Case 1. Take cross sections in the natural way, i.e. put $A_0=\emptyset$ and $A_{k+1}=\bigcup\{A_k^\nu:\nu<\operatorname{cf}(\omega_2^\gamma)\}$. (Note that, in the case $\operatorname{cf}(\omega_2^\gamma)=\omega_2$ we have that γ is a successor and for each k the ω_2 sets A_k^ν $(\nu<\omega_2)$ all have the same type.)

Case 2. By the hypothesis $2^{\aleph_0}=\aleph_1$ we can assume that $[\omega]^\omega=\{T_\nu:\nu<\omega_1\}$. Let $F_\nu=\{T_\mu:\mu<\nu\}$ for $\nu<\omega_1$.

For each $\nu<\omega_1$, let $\langle C_k^\nu:k<\omega\rangle$ be a pointwise-finite system of finite subsets of ω satisfying the requirements of Lemma 7.2 for the countable system F_ν of denumerable subsets of ω. Also, let $\langle B_k^\nu : k<\omega\rangle$ be a disjoint paradoxical decomposition of S_ν $(\nu<\omega_1)$ such that $\operatorname{tp} B_k^\nu<\omega_2^k$ and $S_\nu=\bigcup\{B_k^\nu : k<\omega\}$ (see (1.1)). Now put

$$\tilde{A}_k^\nu=\bigcup\{B_i^\nu:\ i\in C_k^\nu\}\quad\text{for}\quad k<\omega\quad \textit{and}\quad \nu<\omega_1.$$

We know that $\operatorname{tp}\tilde{A}_k^\nu<\omega_2^k$ since $C_k^\nu\subset k$. Moreover, $\langle\tilde{A}_k^\nu : k<\omega\rangle$ is pointwise-finite since $\langle C_k^\nu : k<\omega\rangle$ is and the sets B_i^ν are pairwise disjoint. Finally, by (7.6) of the lemma, we also know that

(7.8) $$\operatorname{tp}(S_\nu\setminus\bigcup\{\tilde{A}_k^\nu:\ k\in T\})<\omega_2^\omega$$

for any $T\in F_\nu$.

Now put $A_k'=\bigcup\{A_k^\nu:\nu<\omega_1\}$, $A_k''=\bigcup\{\tilde{A}_k^\nu:\nu<\omega_1\}$ and $A_k=A_k'\cup A_k''$ for $k<\omega$. Clearly $\langle A_k : k<\omega\rangle$ is pointwise-finite and $\operatorname{tp} A_k<\omega_2^k$.

Suppose $D\in[\omega]^\omega$. Then $D=T_\mu$ for some $\mu<\omega_1$ and hence $D\in F_\nu$ for $\mu<\nu<\omega_1$. Therefore, by (7.8) and the fidefinition of A_k, we have

$$\operatorname{tp}(S_\nu\setminus\bigcup\{A_\alpha:\ \alpha\in D\})<\omega_2^\omega\quad\text{for}\quad \mu<\nu<\omega_1.$$

Further, by the inductive property of the A_α^ν, we also have that

$$\lambda_\nu=\operatorname{tp}(S_\nu\setminus\bigcup\{A_\alpha:\ \alpha\in D\})<\omega_2^{\omega_1}$$

for any $\nu<\omega_1$. Combining these we see that

$$\operatorname{tp}(\omega_2^\gamma\setminus\bigcup\{A_\alpha:\ \alpha\in D\})\leqq\sum\{\lambda_\nu:\ \nu\leqq\mu\}+\omega_2^\omega\omega_1<\omega_2^{\omega_1}.$$

Proof of (7.3). Again the proof is by induction on $\gamma<\omega_2$. For $\gamma=0$ the result is trivial. Also, the induction step in the cases when γ is a successor ordinal or an ω-limit is very easy — simply take identical cross sections or cross sections. The main difficulty in the induction is for the case $\operatorname{cf}(\gamma)=\omega_1$ which we now consider in detail.

Let $\langle S_\nu:\nu<\omega_1\rangle$ be a standard decomposition of ω_2^γ and let $\langle A_\alpha^\nu:\alpha<\omega_1\rangle$ be a pointwise-finite system of subsets of S_ν for $\nu<\omega_1$ which establishes the result in S_ν, i.e. for $\nu<\omega_1$ we have

(7.9) $$\operatorname{tp} A_\alpha^\nu<\omega_2^{\omega_1}\quad(\alpha<\omega_1),$$

(7.10) $$\operatorname{tp}(S_\nu\setminus\bigcup\{A_\alpha^\nu:\ \alpha\in D\})<\omega_2^{\omega_1}\quad\text{for all}\quad D\in[\omega_1]^\omega.$$

Let $\omega\leqq\nu<\omega_1$. By (7.2) just proved, there is a pointwise-finite system $\langle\hat{A}_\alpha^\nu:\alpha<\nu\rangle$ of $\aleph_0$ subsets of S_ν such that

(7.11) $$\operatorname{tp}\hat{A}_\alpha^\nu<\omega_2^\omega,$$

(7.12) $$\operatorname{tp}(S_\nu\setminus\bigcup\{\hat{A}_\alpha^\nu:\ \alpha\in D\})<\omega_2^{\omega_1}\quad\text{for all}\quad D\in[\nu]^\omega.$$

By the hypothesis $2^{\aleph_0}=\aleph_1$ we may write $[\omega_1]^\omega=\{T_\mu:\mu<\omega_1\}$. For $\nu<\omega_1$ define $F_\nu=\{T_\mu:\mu<\nu,\ T_\mu\subset\nu\}$.

Again, let $\omega\leqq\nu<\omega_1$. We define another pointwise-finite system $\langle\tilde{A}_\alpha^\nu:\alpha<\nu\rangle$ of $\aleph_0$ subsets of S_ν as follows. By Lemma 7.2 there is a pointwise-finite system $\langle C_\alpha^\nu:\alpha<\nu\rangle$ of $\aleph_0$ finite subsets of ν such that

(7.13) $$\nu\setminus\bigcup\{C_\alpha^\nu:\alpha\in D\}\ \textit{is finite for all}\ D\in F_\nu.$$

Let $\langle B_\alpha^\nu : \alpha < \nu\rangle$ be $\aleph_0$ disjoint subsets of S_ν such that $S_\nu = \bigcup\{B_\alpha^\nu : \alpha < \nu\rangle$ and $\mathrm{tp}\, B_\alpha^\nu < \omega_2^\omega$. Such sets exist by the ordinary paradox (1.1). Now define

$$\tilde{A}_\alpha^\nu = \bigcup\{B_\alpha^\nu : \alpha \in C_\alpha^\nu\} \quad (\alpha < \nu).$$

Clearly the system $\langle \tilde{A}_\alpha^\nu : \alpha < \nu\rangle$ is pointwise-finite since $\langle C_\alpha^\nu : \alpha < \nu\rangle$ is and the B_α^ν $(\alpha < \omega_1)$ are pairwise disjoint. Moreover, since $\tilde{A}_\alpha^\nu$ is the union of a finite number of sets of type $< \omega_2^\omega$, we have

(7.14) $$\mathrm{tp}\, \tilde{A}_\alpha^\nu < \omega_2^\omega \quad (\alpha < \nu < \omega_1).$$

For the same reason we also know by (7.13) that

(7.15) $$\mathrm{tp}\, (S_\nu \setminus \bigcup\{\tilde{A}_\alpha^\nu : \alpha \in D\}) < \omega_2^\omega \quad \text{for} \quad D \in F_\nu.$$

Now define sets $A_\alpha \subset \omega_2^\xi$ for $\alpha < \omega_1$ by putting $A_\alpha = A'_\alpha \cup A''_\alpha$, where

$$A'_\alpha = \bigcup\{A_\alpha^\nu : \nu < \alpha\} \cup \bigcup\{\hat{A}_\alpha^\nu : \omega \leqq \nu < \omega_1, \alpha < \nu\},$$
$$A''_\alpha = \bigcup\{\tilde{A}_\alpha^\nu : \omega \leqq \nu < \omega_1, \alpha < \nu\}.$$

By (7.9), (7.11) and (7.14) we easily see that $\mathrm{tp}\, A_\alpha < \omega_2^{\omega_1}$ $(\alpha < \omega_1)$. Also the system $\langle A_\alpha : \alpha < \omega_1\rangle$ is a pointwise-finite since the sets S_ν are pairwise disjoint and the $\langle A_\alpha^\nu : \alpha < \omega_1\rangle$, $\langle \hat{A}_\alpha^\nu : \alpha < \omega_1\rangle$ and $\langle \tilde{A}_\alpha^\nu ; \alpha < \omega_1\rangle$ are pointwise-finite. To complete the proof we must verify that

(7.18) $$\mathrm{tp}\, (\omega_2^\xi \setminus \bigcup\{A_\alpha : \alpha \in D\}) < \omega_2^{\omega_1}$$

holds for any $D \in [\omega_1]^\omega$.

Suppose $D \in [\omega_1]^\omega$ and $\nu < \omega_1$. Either $D \setminus \nu$ is infinite or $\omega \leqq \nu < \omega_1$ and $D \cap \nu$ is infinite. In either case, by the definition of A'_α, we have that

(7.19) $$\mathrm{tp}\, (S_\nu \setminus \bigcup\{A'_\alpha : \alpha \in D\}) < \omega_2^{\omega_1}$$

by (7.10) or by (7.12). Also $D = T_\mu$ for some $\mu < \omega_1$. Let $\nu_0 = \sup(\{\mu\} \cup T_\mu)$. Then for $\nu_0 < \nu < \omega_1$ we have $D \in F_\nu$ and hence

(7.20) $$\mathrm{tp}\, (S_\nu \setminus \bigcup\{A''_\alpha : \alpha \in D\}) < \omega_2^\omega \quad (\nu_0 < \nu < \omega_1)$$

by (7.15) (7.18) easily follows from (7.19) and (7.20).

Proof of (7.4). Again this is trivial for $\tau < \omega_1 + 2$ and we use induction on τ. For the case when τ is an ω or ω_1 limit there is no problem, we simply take cross sections. We have only to prove the induction step for the case when $\mathrm{cf}\,(\omega_2^\xi) = \omega_2$, $\tau \leqq \omega_2$.

As usual, let $\langle S_\nu : \nu < \omega_2\rangle$ be a standard decomposition for ω_2^ξ. By the induction hypothesis there is a pointwise-finite system $\langle A_\alpha^\nu : \alpha < \omega_2\rangle$ of subsets of S_ν such that

$$\mathrm{tp}\, A_\alpha^\nu < \omega_2^{\omega_1+2} \quad \text{for} \quad \alpha < \omega_2, \quad \nu < \omega_2,$$

(7.21) $$\mathrm{tp}\, (S_\nu \setminus \bigcup\{A_\alpha^\nu : \alpha \in D\}) < \omega_2^{\omega_1+2} \quad \text{for} \quad D \in [\omega_2]^\omega \quad \text{and} \quad \nu < \omega_2.$$

By (7.3) already proved, for $\omega_1 \leqq \nu < \omega_2$ there is a pointwise-finite system $\langle \hat{A}_\alpha^\nu : \alpha < \nu\rangle$ of $\aleph_1$ subsets of S_ν such that $\mathrm{tp}\, \hat{A}_\alpha^\nu < \omega_2^{\omega_1}$ $(\alpha < \nu)$,

(7.22) $$\mathrm{tp}\, (S_\nu \setminus \bigcup\{\hat{A}_\alpha^\nu : \alpha \in D\}) < \omega_2^{\omega_1} \quad \text{for all} \quad D \in [\nu]^\omega.$$

Now put $A_\alpha = \bigcup\{A_\alpha^\nu : \nu < \alpha\} \cup \bigcup\{\hat{A}_\alpha^\nu : \omega_1 \leqq \nu < \omega_2,\ \alpha < \nu\}$ $(\alpha < \omega_2)$. Clearly the system $\langle A_\alpha : \alpha < \omega_2 \rangle$ is pointwise finite and $\operatorname{tp} A_\alpha < \omega_2^{\omega_1+2}$ $(\alpha < \omega_2)$.

Suppose $D \in [\omega_2]^\omega$. Put $\nu_0 = \sup(\omega_1 \cup D)$. Then for $\nu_0 < \nu < \omega_2$ we have $D \in [\nu]^\omega$ and so by (7.22)

$$\operatorname{tp}(S_\nu \setminus \bigcup\{A_\alpha : \alpha \in D\}) < \omega_2^{\omega_1} \quad (\nu_0 < \nu < \omega_2).$$

It follows from this and (7.21) that

$$\operatorname{tp}(\omega_2^\nu \setminus \bigcup\{A_\alpha : \alpha \in D\}) < \omega_2^{\omega_1+2}.$$

Thus $\langle A_\alpha : \alpha < \omega_2 \rangle$ establishes (7.4).

References

[1] J. BAUMGARTNER, Ineffability properties of cardinals II, *Proceedings of the Fifth International Congress of Logic, Methodology and Philosophy of Science* (to appear).

[2] J. BAUMGARTNER, *Generalizing Martin's axiom* (unpublished).

[3] J. BAUMGARTNER and A. HAJNAL, A proof (involving Martin's axiom) of a partition relation, *Fund. Math.*, **78** (1973), 193—203.

[4] M. BENDA, preprint (Seattle).

[5] P. ERDŐS and A. HAJNAL, Unsolved problems in set theory, *Proc. Sympos. Pure Math.*, **13** (**1**), Amer. Math. Soc., Providence, R. I., (1971), 17—48.

[6] P. ERDŐS, A. HAJNAL and R. RADO, Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hungar.*, **16** (1965), 93—196.

[7] P. ERDŐS, A. HAJNAL and E. C. MILNER, Polarized partition relations for ordinal numbers, *Studies in Pure Mathematics* (1971), Academic Press, 63—87.

[8] F. GALVIN, On a partition theorem of Baumgartner and Hajnal, *Colloquia Math. Soc. János Bolyai*, **10**, *Infinite and Finite Sets*, Keszthely (Hungary), (1973), 711—729.

[9] F. GALVIN and A. HAJNAL, Inequalities for cardinal powers, *Annals of Math.*, **101** (1975), 491—498.

[10] R. B. JENSEN, The fine structure of the constructible hierarchy, *Annals of Math. Logic*, **4** (1972), 229—308.

[11] R. LAVER, *A generalization of M. A.*, preprint.

[12] A. MARTIN and R. M. SOLOVAY, Internal Cohen extensions, *Annals of Math. Logic*, **2** (1970), 143—178.

[13] E. C. MILNER and R. RADO, The pigeon-hole principle for ordinal numbers, *Proc. London Math. Soc.*, (**3**) **15** (1965), 750—768.

[14] K. PRIKRY, On a problem of Erdős, Hajnal, and Rado, *Discrete Math.*, **2** (1972), 51—59.

[15] P. ERDŐS and E. SPECKER, On a theorem in the theory of relations and a solution of a problem of Knaster, *Colloq. Math.*, **8** (1961), 19—21.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13–15.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALGARY
CALGARY, ALBERTA
CANADA