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ON THE INTEGRAL OF THE LEBESGUE FUNCTION OF INTERPOLATION

By

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Let

(1) $$-1 \leqq x_{1,n} < x_{2,n} < \ldots < x_{n,n} \leqq 1$$

be the nodes of interpolation (shortly $x_k = x_{k,n}$);

$$l_k(x) = l_{k,n}(x) = \frac{\omega(x)}{\omega'(x_k)(x-x_k)} \quad \left(k = 1, \ldots, n;\ \omega(x) = \prod_{k=1}^{n} (x-x_k)\right)$$

the corresponding fundamental polynomials, and

$$\lambda_n(a, b) = \max_{a \leqq x \leqq b} \sum_{k=1}^{n} |l_k(x)| \quad \text{if} \quad -1 \leqq a < b \leqq 1.$$

The quantity $\lambda_n(-1, 1)$ called Lebesgue constant plays an important role in the theory of Lagrange interpolation; as G. FABER [1] showed [1]

(2) $$\lambda_n(-1, 1) \geqq c_1 \log n$$

for an arbitrary system of nodes (1). Moreover, S. BERNSTEIN [2] proved that

(3) $$\lambda_n(a, b) \geqq c_2 \log n \quad \left(n \geqq n_1(a, b);\ -1 \leqq a < b \leqq 1\right)$$

for all systems (1) again.

In this paper we prove a more general result from which (3) will follow as a corollary.

THEOREM. *For an arbitrary system of nodes* (1) *and subinterval* $[a, b] \subseteqq [-1, 1]$ *we have*

(4) $$\int_a^b \sum_{k=1}^{n} |l_k(x)|\, dx \geqq c_3(b-a) \log n \quad \left(n \geqq n_2(a, b)\right).$$

In the special case $a = -1$, $b = 1$, this result has been announced in [3] (with an indication of a possible method of proof). Our proof is simpler and follows a different pattern.

PROOF. According the growth rate of $\lambda_n(a, b)$ we distinguish two cases.

[1] In what follows, $c_1, c_2, \ldots$ will denote absolute positive constants.

Case 1: $\lambda_n(a,b) \geqq n^3$. Then let $y_n \in [a,b]$ be such that $\lambda_n(a,b) = \sum_{k=1}^{n} |l_k(y_n)|$, say $x_i < y_n < x_{i+1}$. On the interval $[x_i, x_{i+1}]$, $\sum_{k=1}^{n} |l_k(x)|$ is identical with a polynomial of degree less than n, and this polynomial attains its absolute maximum on $[a,b]$ also at y_n. But then by Markov's inequality, the absolute value of this polynomial is $\geqq \frac{1}{2} n^3$ in the interval $\left[y_n - \frac{b-a}{4n^2}, y_n + \frac{b-a}{4n^2}\right]$. Hence

$$\sum_{k=1}^{n} |l_k(x)| \geqq \frac{1}{2} n^3 \quad \text{if} \quad x \in [a,b] \cap \left[y_n - \frac{b-a}{4n^2},\ y_n + \frac{b-a}{4n^2}\right],$$

i.e.

$$\int_a^b \sum_{k=1}^{n} |l_k(x)|\, dx \geqq \frac{1}{2} n^3 \frac{b-a}{4n^2} = \frac{b-a}{8} n$$

which is even more than we need.

Case 2: $\lambda_n(a,b) < n^3$. Then, as we shall see from the following lemma, the intervals $[x_k, x_{k+1}] \subseteqq [a,b]$ cannot be "too long".

LEMMA. *We have*

$$\max_{a \leqq x_k < x_{k+1} \leqq b} (x_{k+1} - x_k) \leqq 25 \frac{\log \lambda_n(a,b)}{n} \quad (n \geqq n_3(a,b)) \tag{5}$$

for an arbitrary system of nodes (1).

By a slightly more complicated argument, we could replace x_k by $\arccos x_k$ in this lemma, and then (5) would be a generalization of Theorem IV from [4]. However, the given formulation will be sufficient for our purposes.

PROOF OF THE LEMMA. Assume the contrary; then there exists a subinterval $[c_n, d_n] \subset [a,b]$ of length

$$d_n - c_n = 25 \frac{\log \lambda_n(a,b)}{n}$$

which does not contain any of the nodes x_k, $k = 1, 2, \dots, n$.[2] Let

$$\gamma_n = \frac{3c_n + 2d_n}{5}, \quad \delta_n = \frac{2c_n + 3d_n}{5}$$

and z_k, $k = 1, \dots, n$, the roots of the Chebyshev polynomial $T_n(x)$ of degree n. The polynomial

$$p_n(x) = \prod_{z_k \notin [\gamma_n, \delta_n]} (x - z_k)$$

is of degree less than n. Let $x_0 \in [\gamma_n, \delta_n]$ be a point such that $|T_n(x)|$ attains its local

[2] We may assume that $25 \frac{\log \lambda_n(a,b)}{n} < b - a$; otherwise there is nothing to prove.

maximum at x_0. Such a point exists because by the Bernstein's result (3)

$$\delta_n-\gamma_n=\frac{d_n-c_n}{5}=5\,\frac{\log\lambda_n(a,b)}{n}>\frac{\pi}{n}>\max_{1\leqq k\leqq n-1}|z_{k+1}-z_k|\quad (n\geqq n_4(a,b))$$

holds. Thus we obtain for $x\in[-1,1]\setminus[c_n,d_n]$

$$|p_n(x)|=\left|\frac{T_n(x)}{\prod\limits_{z_k\in[\gamma_n,\delta_n]}(x-z_k)}\right|=|p_n(x_0)|\cdot\left|\frac{T_n(x)}{T_n(x_0)}\right|\cdot\prod_{z_k\in[\gamma_n,\delta_n]}\left|\frac{x_0-z_k}{x-z_k}\right|\leqq$$

$$\leqq|p_n(x_0)|\cdot\prod_{z_k\in[\gamma_n,\delta_n]}\frac{1}{2}\leqq|p_n(x_0)|\cdot 2^{-\left[\frac{\delta_n-\gamma_n}{\pi}n\right]}<|p_n(x_0)|\,\lambda_n(a,b)^{-1.1}\quad (n\geqq n_5(a,b)).$$

Hence, by the Lagrange interpolation formula

$$|p_n(x_0)|\leqq\sum_{k=1}^{n}|p_n(x_k)|\cdot|l_k(x_0)|<|p_n(x_0)|\,\lambda_n(a,b)^{-1.1}\sum_{k=1}^{n}|l_k(x_0)|\leqq|p_n(x_0)|\,\lambda_n(a,b)^{-0.1},$$

i.e. $\lambda_n(a,b)<1$, a contradiction. The lemma is proved.

Returning to the proof of our theorem, (5) implies that in case $\lambda_n(a,b)<n^3$ we have

$$\max_{a\leqq x_k<x_{k+1}\leqq b}(x_{k+1}-x_k)\leqq 75\,\frac{\log n}{n}\quad (n\geqq n_3(a,b)). \tag{6}$$

Let

$$(a\leqq)x_i<x_{i+1}<\ldots<x_j(\leqq b)$$

be all the nodes lying in the interval $[a,b]$, then

$$x_i\to a,\quad x_j\to b\quad\text{as}\quad n\to\infty$$

(otherwise even $|l_i(a)|$ or $|l_j(b)|$ would increase at least as a geometric progression). Further

(7)

$$\int_a^b\sum_{k=1}^{n}|l_k(x)|\,dx\geqq\sum_{m=i}^{j-1}\int_{x_m}^{x_{m+1}}\sum_{k=i}^{j}|l_k(x)|\,dx\geqq\frac{1}{2}\sum_{k,m=i}^{j-1}\int_{x_m}^{x_{m+1}}\{|l_k(x)|+|l_{k+1}(x)|\}\,dx>$$

$$>\frac{1}{2}\sum_{m=i}^{j-1}\sum_{k=m}^{j-1}\left\{\int_{x_m}^{x_{m+1}}\big(|l_k(x)|+|l_{k+1}(x)|\big)\,dx+\int_{x_k}^{x_{k+1}}\big(|l_m(x)|+|l_{m+1}(x)|\big)\,dx\right\}.$$

Let $\Delta x_k=x_{k+1}-x_k$ and

$$y=\frac{\Delta x_k}{\Delta x_m}(x-x_m)+x_k\quad (i\leqq m\leqq k\leqq j-1),$$

then using the inequality

$$l_k(y)+l_{k+1}(y)\geqq 1\quad (x_k\leqq y\leqq x_{k+1})$$

(cf. [5, Lemma IV]) we get

$$|l_k(x)|+|l_{k+1}(x)| = \left|\frac{\omega(x)}{\omega(y)}\right| \left\{ l_k(y)\frac{y-x_k}{x_k-x} + l_{k+1}(y)\frac{x_{k+1}-y}{x_{k+1}-x} \right\} \geqq$$

$$\geqq \left|\frac{\omega(x)}{\omega(y)}\right| \frac{\Delta x_k}{4(x_{k+1}-x_m)} \{l_k(y)+l_{k+1}(y)\} \geqq \left|\frac{\omega(x)}{\omega(y)}\right| \frac{\Delta x_k}{4(x_{k+1}-x_m)}$$

$$\left(x_m+\frac{\Delta x_m}{4} \leqq x \leqq x_{m+1}-\frac{\Delta x_m}{4}\right).$$

Thus

$$\int_{x_m}^{x_{m+1}} \{|l_k(x)|+|l_{k+1}(x)|\}\,dx \geqq \int_{x_m+\frac{\Delta x_m}{4}}^{x_{m+1}-\frac{\Delta x_m}{4}} \left|\frac{\omega(x)}{\omega(y)}\right| dx \cdot \frac{\Delta x_k}{4(x_{k+1}-x_m)} =$$

$$= \frac{\Delta x_m}{4(x_{k+1}-x_m)} \int_{x_k+\frac{\Delta x_k}{4}}^{x_{k+1}-\frac{\Delta x_k}{4}} \left|\frac{\omega(x)}{\omega(y)}\right| dy$$

Similarly, by changing the roles of k and m, x and y,

$$\int_{x_k}^{x_{k+1}} \{|l_m(x)|+|l_{m+1}(x)|\}\,dx \geqq \frac{\Delta x_m}{4(x_{k+1}-x_m)} \int_{x_k+\frac{\Delta x_k}{4}}^{x_{k+1}-\frac{\Delta x_k}{4}} \left|\frac{\omega(y)}{\omega(x)}\right| dy.$$

Hence and from (7)

$$(8) \qquad \int_a^b \sum_{k=1}^n |l_k(x)|\,dx > \frac{1}{8} \sum_{m=i}^{j-1} \sum_{k=m}^{j-1} \frac{\Delta x_m}{x_{k+1}-x_m} \int_{x_k+\frac{\Delta x_k}{4}}^{x_{k+1}-\frac{\Delta x_k}{4}} \left\{ \left|\frac{\omega(x)}{\omega(y)}\right| + \left|\frac{\omega(y)}{\omega(x)}\right| \right\} dy \geqq$$

$$\geqq \frac{1}{8} \sum_{a \leqq x_m \leqq \frac{a+b}{2}} \Delta x_m \sum_{k=m}^{j-1} \frac{\Delta x_k}{x_{k+1}-x_m}.$$

In order to estimate the inner sum, let

$$I_{t,m} = \left[x_m + \frac{75\log n}{n}t,\; x_m + \frac{75\log n}{n}(t+1)\right] \quad (t=0,1,\ldots,s_n)$$

where

$$s_n = \left[\frac{(b-a)n}{150\log n}\right].$$

Then by (6), $I_{t,m}\subset[a,b]$, and each $I_{t,m}$ contains at least one of the nodes x_k. Hence

$$\sum_{k=m}^{j-1}\frac{\Delta x_k}{x_{k+1}-x_m}\geqq\sum_{t=0}^{s_n}\sum_{x_k\in I_{t,m}}\frac{\Delta x_k}{x_{k+1}-x_m}\geqq\sum_{t=0}^{s_n}\frac{n}{75(t+1)\log n}\sum_{x_k\in I_{t,m}}\Delta x_k\geqq$$

$$\geqq\frac{n}{75\log n}\sum_{t=1}^{\left[\frac{s_n}{3}\right]}\frac{1}{t}\sum_{x_k\in I_{t,m}\cup I_{t+1,m}\cup I_{t+2,m}}\Delta x_k\geqq\sum_{t=1}^{\left[\frac{s_n}{3}\right]}\frac{1}{t}\geqq\frac{1}{2}\log n\quad(n\geqq n_8(a,b)).$$

Thus (8) yields

$$\int_a^b\sum_{k=1}^{n}|l_k(x)|\,dx\geqq\frac{\log n}{16}\sum_{a\leqq x_m\leqq\frac{a+b}{2}}\Delta x_m\geqq\frac{b-a}{40}\log n\quad(n\geqq n_2(a,b)).$$

Q.E.D.

The best constants in (2) and (3) are (roughly speaking) $2/\pi$. Apparently, our c_3 in (4) is far from being best possible, and our method does not seem to be applicable to finding of the largest c_3.

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