Discrete Mathematics 19 (1977) 229–233.

ON TOTAL MATCHING NUMBERS AND TOTAL COVERING NUMBERS OF COMPLEMENTARY GRAPHS

P. ERDÖS and A. MEIR

Department of Mathematics, The University of Alberta, Edmonton, Canada

Received 11 August 1976
Revised 22 December 1976

Best upper and lower bounds, as functions of n, are obtained for the quantities $\beta_2(G)+\beta_2(\bar{G})$ and $\alpha_2(G)+\alpha_2(\bar{G})$, where $\beta_2(G)$ denotes the total matching number and $\alpha_2(G)$ the total covering number of any graph G with n vertices and with complementary graph $\bar{G}$.

The best upper bound is obtained also for $\alpha_2(G)+\beta_2(G)$, when G is a connected graph.

1.

Let G be a graph with edge set E and vertex set V. A vertex u is said to *cover* itself, all edges incident with u and all vertices joined to u. An edge (u, v) *covers* itself, the vertices u and v and all edges incident with u or v. Two elements of $E \cup V$ are independent if neither covers the other.

A Subset $\mathscr{C}$ of elements of $E \cup V$ is called a *total cover* if the elements of $\mathscr{C}$ cover G and $\mathscr{C}$ is minimal; a subset $\mathscr{T}$ of elements of $E \cup V$ is called a *total matching* if the elements of $\mathscr{T}$ are pairwise independent and T is maximal. We shall be interested in the quantities

$$\alpha_2(G) = \min |\mathscr{C}|, \ \beta_2(G) = \max |\mathscr{T}|$$

where the min is taken over all total covers of G and the max over all total matchings in G. These concepts were introduced in [2] (see also [3]), where various bounds for $\alpha_2(G)$ and $\beta_2(G)$ were obtained and exact values for particular graphs were determined.

In [1] Chartrand and Schuster have obtained lower and upper bounds for $\beta(G)+\beta(\bar{G})$ and $\beta_1(G)+\beta_1(\bar{G})$, where $\beta(G)$ denotes the vertex independence number and $\beta_1(G)$ denotes the edge independence number of a graph G having complement $\bar{G}$. Here we shall obtain bounds for the quantities $\beta_2(G)+\beta_2(\bar{G})$, $\alpha_2(G)+\alpha_2(\bar{G})$ and $\alpha_2(G)+\beta_2(G)$.

2.

We shall use the notation $\beta_2 = \beta_2(G)$, $\bar{\beta}_2 = \beta_2(\bar{G})$, $\alpha_2 = \alpha_2(G)$, $\bar{\alpha}_2 = \alpha_2(\bar{G})$. For complementary graphs we have the following results.

Theorem 2.1. *If G is a graph on n vertices, then*

$$2\left\{\frac{n}{2}\right\} \leq \beta_2 + \bar{\beta}_2 \leq \left\{\frac{3}{2}n\right\}.$$

The upper bound is best possible for all n, the lower bound is best possible for all $n \neq 2 \pmod 4$.

Proof. Let μ (resp. $\bar{\mu}$) denote the size of a smallest maximal set of independent edges in G (resp. $\bar{G}$). Then the following relations are immediate:

$$\beta_2 = n - \mu, \qquad \mu \leq [n/2], \qquad \mu + \bar{\mu} \geq (n-1)/2.$$

These imply the bounds of Theorem 1.

In order to show that the upper bound is best possible, we let $G = K_n$. Then $\bar{\beta}_2 = n$ and, as proved in [2], $\beta_2 = \{n/2\}$. For the lower bound, we set $G = K_{2m,2m}$ if $n = 4m$ and $G = K_{l,l+1}$, if $n = 2l + 1$. In these cases $\beta_2 + \bar{\beta}_2 = 2\{n/2\}$.

Remark 2.2. If n is odd then for every t such that $n + 1 \leq t \leq (3n+1)/2$, there exists a graph G on n vertices satisfying $\beta_2 + \bar{\beta}_2 = t$. It $n = 0 \pmod 4$ then for every t such that $n \leq t \leq \frac{3}{2}n$ and $t \neq n + 1$ there exists a graph G on n vertices satisfying $\beta_2 + \bar{\beta}_2 = t$. If $n = 2 \pmod 4$ then for every t such that $n + 1 \leq t \leq \frac{3}{2}n$ there exists a graph G on n vertices so that $\beta_2 + \bar{\beta}_2 = t$.

Proof. If n is odd we let $G = K_{x,n-x}$ with $0 \leq x < n/2$. If n is even, we let $G = K_{x,n-x}$ with even values of $x, 0 \leq x \leq n/2$; further we let G be the graphs obtained from $K_{x,n-x}$ with odd values of $x, 3 \leq x \leq n/2$, when joining two vertices among the x vertices by an edge. Easy calculation shows that these examples yield the result.

Remark 2.3. We can show that if $n = 2 \pmod 4$, the lower bound in Theorem 1 is in fact $n+1$. Also, a result of Galvin implies that if $n = 0 \pmod 4$, then $\beta_2(G) + \beta_2(\bar{G}) \neq n + 1$.

Theorem 2.4. *If G is a graph on n vertices then*

$$\left\{\frac{n}{2}\right\} + 1 \leq \alpha_2 + \bar{\alpha}_2 \leq \left\{\frac{3n}{2}\right\}.$$

The upper bound is best possible for all n, the lower bound is best possible for odd n.

Proof. Let $\mathscr{C}$ be a total cover of G consisting of x edges and y vertices such that $\alpha_2 = x + y$. We may assume that the x edges are pairwise disjoint and that none of the y vertices is joined to any of the x edges. If $n = 2x + y + z$, then there are z vertices each of which must be joined in G to some of the y vertices in $\mathscr{C}$. It is easy to see that no two of these z vertices can be joined in G and therefore $\bar{G}$ contains

K_z as a subgraph. It follows then by [2], that $\bar{\alpha}_2 \geq \{z/2\}$. Thus, we have $\alpha_2 + \bar{\alpha}_2 \geq x + y + \frac{1}{2}z = \frac{1}{2}(n + y)$. This proves our statement if $y \geq 2$. If $y = 1$, let vertex $v_0 \in \mathscr{C}$. In order to cover v_0 in $\bar{G}$, we must have $\bar{\alpha}_2 \geq \{z/2\} + 1$, since v_0 is not joined in $\bar{G}$ to any of the z vertices of K_z. Thus in this case $\alpha_2 + \bar{\alpha}_2 \geq \frac{1}{2}n + \frac{3}{2}$ which is stronger than needed. Finally, if $y = 0$ then $z = 0$, so that $\alpha_2 = x = n/2$. Since $\bar{\alpha}_2 \geq 1$ in any case, we get the desired lower bound in this case as well. The upper bound in Theorem 3 is a consequence of the inequalities $\alpha_2 \leq \beta_2$, $\bar{\alpha}_2 \leq \bar{\beta}_2$ and Theorem 1. The upper bound is best possible if $G = K_n$. To show that the lower bound is best possible if $n = 2l + 1$, we let G be the star graph on n vertices. We have $\alpha_2 = 1$, $\bar{\alpha}_2 = l + 1$.

Remark 2.5. If n is odd then for every t such that $\frac{1}{2}(n + 1) + 1 \leq t \leq \frac{1}{2}(3n + 1)$ and $t \neq \frac{1}{2}(3n - 1)$ there exists a graph on n vertices satisfying $\alpha_2 + \bar{\alpha}_2 = t$. If n is even then for every t such that $\frac{1}{2}n + 2 \leq t \leq \frac{3}{2}n$ there exists a graph on n vertices satisfying $\alpha_2 + \bar{\alpha}_2 = t$.

Proof. If n is even, we let G be the graph consisting of K_x, $1 \leq x \leq n$, and of $n - x$ vertices joined to all vertices of K_x. If n is odd, we first let G be graphs as described above, allowing odd values of x, $1 \leq x \leq n$; further we let G be the same graphs with one edge of K_x omitted. Simple calculations show that Remark 2.5 is valid.

Remark 2.6. By a bit more complicated argument we can prove that if n is even, then the lower bound in Theorem 3.1 is in fact $n/2 + 2$ and if n is odd, then $\alpha_2 + \bar{\alpha}_2 \neq (3n - 1)/2$.

3.

It was proved in [3] that if G is a *connected* graph on n vertices *without triangles* then $\alpha_2 + \beta_2 \leq 5n/4$, but that for infinitely many connected graphs $\alpha_2 + \beta_2 > 5n/4$ holds. In the following result the restriction concerning triangles is absent.

Theorem 3.1. *If G is a connected graph on n vertices $(n \geq 2)$, then*

$$\alpha_2 + \beta_2 \leq n + \frac{1}{2}\left\{\frac{n}{2}\right\}.$$

Proof. It was proved in [2] that $\alpha_2 \leq \{n/2\}$ for a connected G. Clearly we also have $\alpha_2 \leq 2\mu$ in this case. Combining these with $\beta_2 = n - \mu$, we obtain the result.

The examples given in [3] (subsequent to the proof of (1)) show that the bound given in Theorem 3.1 is best possible if $n \equiv 0$ or $3 \pmod 4$. It is easy to construct examples showing that it is also best possible if $n \equiv 1$ or $2 \pmod 4$.

Theorem 3.2. *Every connected graph on n vertices contains a total matching of size at most $n-2\sqrt{n}+2$. This bound is best possible.*

Proof. Let $\mathscr{H}$ be a largest independent set of *edges* in G. We denote by $(v_1, v_2), (v_3, v_4), \dots, (v_{2x-1}, v_{2x})$ the edges in $\mathscr{H}$, and by $v_{2x+1}, \dots, v_n$ the rest of the vertices of G. Then:

(i) because of the maximality of $\mathscr{H}$, no two of $v_{2x+1}, \dots, v_n$ are joined;

(ii) because of the connectedness of G, each of $v_{2x+1}, \dots, v_n$ is joined to at least one of the vertices $v_1, \dots, v_{2x}$;

(iii) since $\mathscr{H}$ is the *largest* independent set of edges, it is not possible that one of $v_{2x+1}, \dots, v_n$ be joined to one end vertex and another to another end vertex of the same edge.

Therefore, we may assume that each of the vertices $v_{2x+1}, \dots, v_n$ is joined to at least one of the vertices $v_2, v_4, \dots, v_{2x}$. Thus there exists a vertex, say v_2, with at least $k = \{(n-2x)/x\}$ vertices, say $v_{2x+1}, \dots, v_{2x+k}$ joined to it. Let now $\mathscr{I}$ consist of the edges $(v_3, v_4), \dots, (v_{2x-1}, v_{2x})$ and of the vertices $v_{2x+k+1}, \dots, v_n$ and v_2. Then $|\mathscr{I}| = (x-1)+(n-2x-k)+1 = n-x-k \leq n-x-n/x+2$ and $\mathscr{I}$ is clearly maximal independent.

Now, $x+n/x \geq 2\sqrt{n}$ for all x, so $|\mathscr{I}| \leq n-2\sqrt{n}+2$ as required. In order to show that this estimate is best possible, we consider (see [3], proof of (3)) the graph G of order $n = m^2$ consisting of K_m with $m-1$ end vertices joined to each vertex of K_m. As shown in [3], for every maximal independent set $\mathscr{I}, |\mathscr{I}| \geq 1+(m-1)^2 = m^2-2m+2$. This proves our claim.

4.

The bounds given by Theorem 2.1 yield estimates for the product $\beta_2 \cdot \bar{\beta}_2$. For example: If $n \equiv 0 \pmod 4$ then $n^2/4 \leq \beta_2 \cdot \bar{\beta}_2 \leq 9n^2/16$. Both bounds are best possible.

For the product of the covering numbers Theorem 2.4 does not yield best possible estimates. Indeed we have the following result: If G is a graph on n vertices, then $\alpha_2 \cdot \bar{\alpha}_2 \leq n \cdot \{n/2\}$. This estimate is best possible.

Proof. For every graph G, either G or $\bar{G}$ is connected. Hence, by [2], either $\alpha_2 \leq \{n/2\}$ or $\bar{\alpha}_2 \leq \{n/2\}$. The choice $G = K_n$ shows that the estimate is best possible.

Acknowledgement

The authors are indebted to the referee for supplying alternative (much simpler) proofs of Theorems 2.1 and 3.1.

References

[1] Y. Alavi, M. Behzad, L.M. Lesniak and E.A. Nordhaus, Total matchings and total coverings of graphs, J. Graph Theory (to appear).

[2] G. Chartrand and S. Schuster, On the independence number of complementary graphs, Trans. New York Acad. Sci., Ser. II 36 (1974) 247–251.

[3] A. Meir, On total covering and matching of graphs, J. Combinatorial Theory Ser. B (to appear).