OLD AND NEW PROBLEMS IN COMBINATORIAL ANALYSIS AND GRAPH THEORY

P. Erdös

Hungarian Academy of Sciences
Budapest, Hungary

I plan to discuss some of the many problems which interested me in the last few decades of my long life. My choice is essentially subjective—I shall state and discuss some of the older problems which I particularly like and which I hope can be solved if more mathematicians paid attention to them. I shall also state some new problems; here, of course, there is some risk that the problems are trivial or not stated correctly. I shall start by discussing some recent problems.

SECTION 1

Fajtlowicz and I conjectured that, for sufficiently small $\varepsilon > 0$, there is no graph $G(n)$ satisfying $v(x) < \varepsilon n$ for every vertex x of $G(n)$, where $G(n)$ has no triangle, and, for every three independent vertices x_1, x_2, x_3 of $G(n)$, there is a fourth vertex joined to all three of them. Sheehan conjectured that ε can be taken as $\frac{3}{8}$; $G(n)$ denotes a graph of n vertices and $v(x)$ the valence or degree of the vertex x.

We could not even prove the following very much weaker result: Assume that $G(n)$ satisfies the following very much stronger condition: Let $\mathscr{S}$ be any independent set of vertices. Then there is a vertex which is joined to all vertices of $\mathscr{S}$. Perhaps such a graph does not exist even if we drop the condition that G has no triangle, as long as we maintain the condition $v(x) < \varepsilon n$.

All we could prove is that no graph exists if we assume that G has no triangle and $v(x) < \varepsilon n/\log n$—but perhaps we have overlooked a simple argument.

Recently I have learned that this problem was nearly completely settled by J. Pach. The conjecture in the first paragraph is false and the one in the second, correct. However, if "three" is replaced by "four" in the first paragraph, the problem is still open.

SECTION 2

Fajtlowicz and I ask: Let $f(n)$ be the largest integer for which there is a graph $G[f(n)]$ of diameter two and $v(x) \le n$. Determine or estimate $f(n)$ as accurately as possible. This problem of course is not new. Trivially $f(n) \le n^2 + 1$, and Hoffman and Singleton [1] have proved that for $n > n_0$, $f(n) \le n^2$. We have proved that $f(n) < n^2$, for $n > n_0$, and if $n = p^\alpha$, $f(n) \ge (n-1)^2 + n + 1$.

Perhaps the reader will forgive me for telling the sad story of our third problem. Let $G(n)$ be a graph of n vertices and diameter two. We conjectured that the dominance number $d[G(n)]$ is less than $c\sqrt{n}$ (i.e., there are vertices $x_1, \ldots, x_k$, $k < c\sqrt{n}$ such that every vertex of G is joined to one of the x_i's). We only could prove $d[G(n)] < c\sqrt{n \log n}$. When I mentioned this problem in my talk at Rutgers, Máté pointed out that, in a triple paper [2], we prove that $d[G(n)] < c\sqrt{n \log n}$ is, in

0077-8923/79/0319-0177 $01.75/1 © 1979, NYAS

fact, the best possible result. There are many current jokes (usually slightly off color) about the symptoms of senility—surely a very sad symptom is that one forgets one's own theorems (the worst one is that one no longer can find new ones!).

Section 3

In a very recent paper Plesnik [3] states the following pretty conjecture. Denote by $\chi(G)$ the chromatic number of G. Decompose the complete graph K_p into n graphs $G_1, \ldots, G_n$. Then

$$\sum_{i=1}^{n} \chi(G_i) \leq p + \binom{n}{2} \tag{1}$$

Plesnik proves (1) with $2^{\binom{n}{2}}$ instead of $\binom{n}{2}$. I first thought that I would prove (1) quickly, but I was certainly wrong, since I have not yet done it, but perhaps we both have overlooked a simple argument.

Let G be a graph of chromatic number p which does not contain K_p and $\{G_i: i = 1, \ldots, n\}$ a decomposition of G into n edge-disjoint graphs. Then perhaps (1) never holds for all such decompositions, but I did not yet have time to look into this problem carefully.

Section 4

Let $|\mathscr{S}| = n; f(n; r)$ is the largest integer for which there is a family F of subsets of $\mathscr{S}$, $|F| = f(n; r)$, and, if $A_1 \in F$, $A_2 \in F$, then $|A_1 \cap A_2| \neq r$. Trivially $f(n; 0) = 2^{n-1}$. Determine or estimate $f(n; r)$. An older conjecture of mine states: To every η, $0 < \eta < \frac{1}{4}$ there is an $\varepsilon = \varepsilon(\eta) > 0$, such that, for every $\eta n < r < (\frac{1}{2} - \eta)n$,

$$f(n; r) < (2 - \varepsilon)^n \tag{2}$$

Denote by $f(n; k, r)$ the largest integer for which there is a family F of subsets of $\mathscr{S}$ satisfying $A \in F$ implies $|A| = k$ and $A_1 \in F$, $A_2 \in F$ implies $|A_1 \cap A_2| \neq r$. V. T. Sós and I proved $f(n; 3, 1) \leq n$, with equality if and only if $n \equiv 0 \pmod 4$. We conjectured that

$$f(n; k, 1) = \binom{n-2}{k-2}, \; n > n_0(k) \tag{3}$$

Equation 3 was proved by Katona for $k = 4$ (unpublished) and by Frankl [5] in the general case.

Inequality (2) would have an interesting application in geometry and graph theory. Hadwiger, and independently Nelson have raised the following problem: Join two points in n-dimensional Euclidean space if their distance is 1. Denote by $f(n)$ the chromatic number of this graph. It is well-known that $4 \leq f(2) \leq 7$. It is now generally believed (?) that $f(2) > 4$. Let $x_1, \ldots, x_t$ be points in n-dimensional space. Join two of them if their distance is 1. Thus we obtain the graph $G(x_1, \ldots, x_t)$. By a well-known theorem of de Bruijn and myself there is a graph $G(x_1, \ldots, x_t)$ that has chromatic number $f(n)$—or the determination of the "chromatic number of n-dimensional space" is a finite problem. Unfortunately (even for the plane) we can not give an upper bound for t.

Inequality (2) would imply that $f(n) > (1 + \eta)^n \cdot f(n)$ was studied by Larman and Rogers [4] and Frankl [5]. Currently the best result is that $f(n) > n^k$ for $n > n_0(k)$.

That $f(2) > 3$ was shown by the Moser brothers. I conjectured that if the x_i's are points in the plane and $G(x_1, \ldots, x_t)$ contains no triangle, then its chromatic number is ≤ 3. For further problems and results in combinatorial problems in geometry, see [6] and my paper, Combinatorial Problems in Geometry and Number Theory, given at the Columbus meeting in March 1978, which will soon be published.

Section 5

Here are two recent conjectures of Frankl [7] and Szemerédi: Let L be a set of nonnegative integers not exceeding $n - 1$. Let $\mathscr{S}$ be a set of n elements and $F(L, n)$ a family of subsets of $\mathscr{S}$ for which $A_\mu \in F(L, n)$, $A_\nu \in F(L, n)$ implies that $|A_\mu \cap A_\nu|$ is in L. They, first of all, conjecture that to every $\varepsilon > 0$ there is a $\delta > 0$ for which $|L| < \delta n$ implies

$$|F(L, n)| < 2^{\varepsilon n}. \tag{4}$$

They further conjecture that (4) holds if $k > k_0(\varepsilon)$, $n = 2k^2$, and L is the set of integers $2ak + b$, $0 \leq a < k$, $0 \leq b < k$.

Both conjectures seem very interesting. Murty and I have conjectured and Berlekamp [8] has proved that if $n = 2k$ and L is the set of even numbers, then $F(L, n) = 2^{n/2}$, but this does not seem to help.

Section 6

Now I state some problems and results which Galvin and I have obtained some time ago and which I hope will soon appear in detailed form, but since I am not sure of this (and, in fact, they may appear only posthumously), I shall state some of them here without proof.

There is an absolute constant, $c_{r,k}$ so that if we color the r-tuples of the integers by k colors in an arbitrary way, there always is an infinite sequence $1 \leq a_1 < a_2 < \ldots$, the set of r-tuples of which get at most 2^{r-1} colors and for infinitely many n's:

$$\sum_{a_i < n} 1 > c_{r,k} \log_{r-1} n \tag{5}$$

where $\log_{r-1} n$ denotes the $(r-1)$-times iterated logarithm.

We further show that 2^{r-1} can not be replaced by a smaller number. It is almost certain that $\log_{r-1} n$ in (5) is the best possible result, but this depends on the finite version of Ramsey's theorem, which is not yet completely explored. Máté found a different proof than our original one.

We also investigated infinite paths—here are some of our results: Given any coloring of the pairs (i,j) of positive integers by $k \geq 3$ colors, there is always an infinite sequence $1 \leq a_1 < a_2 < \cdots$ for which the edges (a_i, a_{i+1}), $i = 1, 2, \ldots$, have only two colors and for infinitely many n,

$$\sum_{a_i < n} 1 \geq n^{1/k} \tag{6}$$

We do not know if $n^{1/k}$ is the best possible here.

If we do not insist on a monotone path, we obtain very much sharper results. We can easily show that if we color the edges of an infinite complete graph with two colors, there always are two edge-disjoint monochromatic paths which contain all the vertices. If the vertices are the integers, this immediately gives a monochromatic infinite path where the upper density of the vertices is $\geq \frac{1}{2}$. We could not decide if $\frac{1}{2}$ was the best possible value. We have many further results on paths which I hope will all appear in a finite time.

SECTION 7

F. Galvin, M. Krieger, and I have several results which I also hope will be published in a finite time; here I state some of them without proof. Let $G(n)$ be a graph of n vertices and $\bar{G}(n)$ the complementary graph of $G(n)$; let $c(G)$ denote the number of cliques (i.e., maximal complete graphs contained in G). Galvin and Krieger [9] have proved that $c(G(n)) + c(\overline{G(n)}) \geq n + 1$ (this has also been proved by Hoffman and Milner). First of all, we obtain the following improvement. Denote by $d(G)$ the largest integer so that every vertex x of G is contained in at least $d(G)$ cliques. Then we have $c(G(n)) + c(\overline{G(n)}) \geq n + d(G(n)) + d(\overline{G(n)}) - 1$. Let $f_2(k, n)$ be the minimum of $c(G_1) + \cdots + c(G_k)$, where $G_1, \ldots, G_k$ are edge-disjoint spanning subgraphs of K_n (the complete graph on n vertices) whose union is K_n. Thus the theorem of Galvin and Krieger states that $f_2(2, n) = n + 1$.

The exact determination of $f_2(k, n)$ for all k and n seems to us to be a difficult and interesting problem. Here are some of our results: $f_2(n + 1, n^2) \geq n^2 + n$ with equality if and only if $n = 1$ or there is a projective plane of order n. In fact, if there is a projective plane of order n then for every m we have

$$f_2(n + 1, m + n^2 - 1) \leq f_2(n + 1, m) + n^2 - 1 \tag{7}$$

We cannot decide if, for every k, there is a c_k such that

$$f_2(k, n) \leq n + c_k \tag{8}$$

Inequality (6) follows from (7) if $k = p^{\alpha} + 1$ and Graham and Van Lint have proved (8) for $k = p^{\alpha}$. We further have

$$f_2(3, n) \leq \begin{cases} n + 2 \text{ if } n \equiv 1 \pmod{3} \\ n + 3 \text{ otherwise.} \end{cases} \tag{9}$$

Equality holds in (9) for $n \leq 10$, perhaps it holds for all n, but we cannot even prove $f_2(3, n) > cn$; $c(G)$ can be extended for hypergraphs and we have some results and many interesting problems. I hope all this will appear soon.

SECTION 8

Now I shall discuss some older problems on chromatic numbers of hypergraphs. A hypergraph is a family of sets $\{A_k\}$ called the *edges*, the elements of $\bigcup_k A_k$ are called the *vertices* of the hypergraph. If all the $|A_k|$ are equal, the hypergraph is said to be *uniform*—if all the $|A_k|$ are two, we obtain the ordinary graphs. If $|A_i \cap A_j| \leq 1$, the hypergraph is called *simple*, and if $|A_i \cap A_j| > 0$, for all i and j, the hypergraph is called a *clique*. A hypergraph is said to be *r-chromatic* if the vertex set $\mathscr{S}$ can be decomposed into r sets $\mathscr{S} = \bigcup_{i=1}^{r} \mathscr{S}_i$, such that none of the $\mathscr{S}_i$ contain an edge of the hypergraph, but such a decomposition is impossible for $r - 1$. Hajnal and I [10] considered the following problem: Let $m(n)$ be the smallest integer for which there is an n-uniform 3-chromatic hypergraph. Determine or estimate $m(n)$; $m(2)=3$ is trivial, $m(3)=7$ is easy, and $m(4)$ is not known. If I remember correctly the current best results are $19 \leq m(4) \leq 23$. The current best results for $m(n)$ are

$$c_1 n^{1/3} 2^n < m(n) < n^2 2^n \tag{10}$$

The lower bound in (10) is due to Beck and the upper bound to me. Beck has also proved the following conjecture of Lovász and myself: Let G be a 3-chromatic (not necessarily uniform) hypergraph. Assume that $|A_i| \geq t$ for all the edges of G. Then

$$\sum_i \frac{1}{2^{|A_i|}} \geq f(t) \tag{11}$$

where $f(t)$ tends to infinity with t (the summation in (11) is extended over all the edges of G). Probably $f(t) > t^c$, but Beck's proof only gives a much weaker result.

Let $m^*(n)$ be the smallest integer for which there is a simple n-uniform 3-chromatic hypergraph of $m^*(n)$ edges. Lovász and I [11] have proved that

$$c_2 4^n/n^4 < m^*(n) < c_3 4^n n^3 \tag{12}$$

In the same paper we estimate $m^{**}(n)$, the smallest integer for which there is an n-uniform 3-chromatic clique with $m^{**}(n)$ edges, but here our results are not as accurate as (12). We do not even prove that $\lim_{n\to\infty} (m^{**}(n))^{1/n}$ exists; $m^{**}(n) \leq 7^{(n-1)/2}$ is our sharpest result.

We further prove that a 3-chromatic n-uniform hypergraph always has a vertex which is contained in at least $2^{n-1}/4n$ edges. In view of (10), this is not very far from being the best possible result. Perhaps it is not quite hopeless to try to obtain a best possible result.

Lovász, Shelah, and I [12] have investigated the following question: Is it true that a 3-chromatic n-uniform clique contains two edges having $n-3$ elements in common? We have only proved this with $cn/\log n$ instead of $n-3$; $n-3$ is perhaps too optimistic, but cn certainly should hold.

Let G be a 3-chromatic n-uniform clique and $\{A_i\}$ its set of edges. We conjectured that the number of distinct values of $|A_i \cap A_j|$ tends to infinity with n, but only show that it is at least 3.

Denote by $M(n)$ the maximum number of edges in an n-uniform 3-chromatic clique. Lovász and I proved that

$$n!\,(e-1) < M(n) < n^n \tag{13}$$

Perhaps the lower bound is best possible in (13).

Lovász and Woodall have proved that if G is a hypergraph with the property that every $G' \subset G$ has fewer edges than vertices, then G is 2-chromatic. A surprising example of Woodall [13] shows that this is the best possible result in the following strong sense: For every n there is an n-uniform 3-chromatic hypergraph G so that every proper subgraph of it has fewer edges than vertices. Woodall's example has $(1+o(1))n!$ edges. Lovász and I conjectured that the number of edges in the example of Woodall is minimal (or nearly so).

Shelah and I have investigated the following problem: Let G be an n-uniform hypergraph of chromatic number 3. Assume that $|A_i \cap A_j| \leq m$ for any two edges of G; $f(n; m)$ is the largest integer so that every such G has at least $f(n; m)$ independent (i.e., pairwise disjoint) edges. We conjectured that $f(n; 1) > 2^{cn}$, for every $c < \frac{1}{2}$, but only could prove a very much weaker result.

We have proved the following conjecture of Hechler: Let $\{A_\alpha\}$, $|A_\alpha| = \aleph_0$ be the edges of a 3-chromatic hypergraph satisfying $|A_{\alpha_1} \cap A_{\alpha_2}| < \aleph_0$. Then our graph contains two disjoint edges, but we show that it does not have to contain three disjoint edges. For some of the older results on the chromatic numbers of graphs and hypergraphs, see [15, 16, 17].

Section 9

Now I state four of my favorite older conjectures. First of all an old conjecture of Rado and myself [18]: Define $f_k(n)$ to be the smallest integer with the following property: Let $|A_i| = n$, $1 \leq i \leq f_k(n)$. Then there are always k A's, say $A_{i_1}, \ldots, A_{i_k}$, which pairwise have the same intersection. Our old conjecture states:

$$f_k(n) < c^n k^n \tag{14}$$

for some absolute constant c. I offer 500 dollars for a proof or disproof of (14)—in fact, I would be satisfied with a proof of (14) for $k = 3$ (this almost certainly contains the main difficulty).

A more recent problem of Rado and myself is: Determine or estimate the largest integer $g(n)$ for which there are sets $|A_k| = n$, $1 \leq k \leq g(n)$, such that, for every three of them, there are two whose union contains the third. I have observed $g(n)^{1/n} \to 1$ and P. Frankl has improved this to $g(n) < c^{c\sqrt{n}}$. J. Larson has shown that $g(2n) \geq (n+1)^2$. Perhaps for every k, $\lim_{n\to\infty} g(n)/n^k = \infty$ (see [19, 20]).

Faber, Lovász, and I conjectured that if $|A_k| = 1$, $1 \leq k \leq n$ and $|A_{k_1} \cap A_{k_2}| \leq 1$, $1 \leq k_1 < k_2 \leq n$, then the elements of $\bigcup_{k=1}^n A_k$ can be colored by n colors such that every A_k, $k = 1, \ldots, n$ contains an element of each color. An equivalent graph theoretic formulation states: Define a graph $G(A_1, \ldots, A_n)$ as follows: The vertices are the elements of $\bigcup_{k=1}^n A_k$. Two vertices are joined if they belong to the same A_k. Prove that the chromatic number of $G(A_1, \ldots, A_n)$ is n. I offer 250 dollars for a proof or disproof.

Lovász and I [11] have investigated the following problem: Let t_n be the smallest integer for which there is a family of sets A_k, $1 \leq k \leq t_n$, $|A_k| = n$, $|A_{k_1} \cap A_{k_2}| \geq 1$, for $1 \leq k_1 < k_2 \leq n$, and for every $|\mathscr{S}| \leq n - 1$ there is an A_k for which $A_k \cap \mathscr{S} = \phi$. In other words, t_n is the smallest integer for which there is an n-uniform hypergraph which is a clique and which can not be represented by $n - 1$ elements. We prove that $t_n < n^{3/2+\epsilon}$, and very likely our proof can be improved to give $t_n < cn \log n$. The real mystery is that we cannot decide if $t_n < cn$ is true. I offer 100 dollars for a proof or disproof and 250 dollars for an asymptotic formula for t_n.

Frankl has the following very interesting problem: Let $\{A_k\}$, $|A_k| = n$ be a 2-chromatic clique and let $f(n)$ be the smallest integer for which there always is a subset $S \subseteq \bigcup_k A_k$, $|S| \leq f(n)$, $S \cap A_k \neq \phi$, $S \not\supset A_k$ for every k. Frankl has proved that $f(n) < cn^2 \log n$, but only has $f(n) > n + c\sqrt{n}$ as a lower bound. Determine or estimate $f(n)$ as well as possible. In particular, does $f(n) < cn$ hold?

A special case of an old problem of Hajnal and myself [21] is stated as follows: Prove that for every cardinal number n there is a graph G_n which contains no K_4, but if you color the edges by n colors in an arbitrary way, there is always a monochromatic triangle.

Nothing seems to be known if $n \geq \aleph_0$, but for finite n the problem has been solved by Folkman and Nešetřil-Rodl. The only trouble is that the graph G_n given by them is of exorbitant size, and both Hajnal and I believe that graphs G_n of reasonable size exist.

To fix our ideas I offer 100 dollars for a graph G_4 having fewer than 10^9 vertices which contains no K_4, and for every coloring of the edges by two colors there is a monochromatic triangle. I also offer 100 dollars for a proof that such a graph does not exist.

I also offer 100 dollars for a proof or disproof that for every $n \geq \aleph_0$, an appropriate G_n exists. Perhaps such a G_n exists for which the power of the vertex set is $(2^n)^{!}$.

Section 10

Finally I state a few older problems which seem interesting to me and which perhaps were neglected.

Let $G(n; l)$ denote a graph of n vertices and l edges. Sauer and I have the following problem: Let $f_k(n)$ be the smallest integer for which every $G(n; f_k(n))$ contains a regular subgraph of degree k (i.e., each vertex has degree or valence k). Trivially $f_2(n) = n_1$, but the determination or estimation of $f_3(n)$ seems difficult. Is it true that $f_3(n) < cn$? We cannot even prove that $f_3(n) < n^{1+\varepsilon}$ for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$. Is it true that every $G(n; [n^{1+\varepsilon}])$ contains a maximal planar graph on $\leq C_\varepsilon$ vertices (i.e., a triangulation of $\leq C_\varepsilon$ vertices)?

Assume that every induced (or spanned) subgraph of $G(10n)$ having $5n$ vertices has more than $2n^2$ edges. Then I conjecture that our $G(10n)$ contains a triangle. It is easy to see that this, if true, is the best possible conjecture. Clearly many generalizations and extensions are possible.

Let G be a graph of $5n$ vertices that has no triangle. Is it true that one can always omit $\leq n^2$ edges so that the resulting graph should be bipartite? Again, if true, it is easy to see that it is the best possible case and that many generalizations are possible.

Another old conjecture of mine states: If $G(5n)$ contains more than n^5 pentagons, then it contains a triangle. Here too it is easy to see that, if true, this is the best possible conjecture and, again, many generalizations are possible.

Bruce Rothschild and I have the following problem: Let $G(n)$ be a graph of n labeled vertices. Denote by $F(n)$ the number of distinct colorings by two colors such that neither color contains a monochromatic triangle. We conjecture that

$$F(n) = 2^{[n^2/4]}, \text{ for } n > n_0$$

with equality only for the complete bipartite graph having $[n/2]$ black and $[(n+1)/2]$ white vertices. Let $|\mathscr{S}| = 2n$, $A_k \subset \mathscr{S}$, $1 \leq k \leq T_n$. Assume that the number of pairs (i, j) satisfying $A_i \cap A_j = \phi$ is $\geq 2^{2n}$. Then $T_n \geq (1 + o(1))2^{n+1}$. Observe that the order of magnitude is the best possible. Let the A's be the $2^{n+1} - 1$ subsets of $\mathscr{S}_1$ and $\mathscr{S}_2$, where $|\mathscr{S}_1| = |\mathscr{S}_2| = n$, $\mathscr{S}_1 \cup \mathscr{S}_2 = \mathscr{S}$. Clearly there are more than 2^{2n} pairs $A_i \cap A_j = \phi$.

Perhaps the proof of $T_n \geq (1 + o(1))2^{n+1}$ will not be difficult, but at the moment I do not see how to do it. Again, many generalizations are possible.

The following problem is due to Graham and myself: Decompose a K_{2^n+1} into n graphs. Clearly at least one of them must have chromatic number ≥ 3. Determine or estimate the length of the least monochromatic cycle one of these graphs must contain. More precisely: Denote by $c(n)$ the smallest integer such that if we color the edges of a K_{2^n+1} with $n + 1$ colors there always is a monochromatic odd cycle of length $\leq c(n)$. Determine or estimate $c(n)$.

The following further questions can be posed: Let $f_k(n)$ be the largest integer for which $K_{f_k(n)}$ can be decomposed into n graphs, all of whose odd cycles are longer than k, and $F_k(n)$ is the largest integer for which $K_{F_k(n)}$ can be decomposed into n

graphs, all of whose odd cycles are shorter than k. Determine or estimate $f_k(n)$ and $F_k(n)$.

Clearly K_{3^n} is the union of n 3-chromatic graphs. One could try to study the 4-chromatic graphs which must occur in the decomposition of the edges of K_{3^n+1} into n graphs, but we have no results.

Let there be given n^2 points in the plane. Is it true that one can always find $2n-2$ of them which do not determine a right angle? The lattice points (x, y) $0 \leq x, y \leq n-1$ show that if true this conjecture is the best possible. In 3-space I do not even know the maximum number of lattice points (x_i, y_i, z_i), $0 \leq x_i, y_i, z_i < n$, which do not determine a right-angled triangle.

Define a graph whose vertices are the integers as follows: i and j are joined if $i+j$ is a square. Silverman and I conjecture that this graph has chromatic number $\aleph_0$. The same seems to hold if the square is replaced by the kth power. We have not succeeded in formulating a reasonable general conjecture. The following seems to us to be true: Let $1 \leq a_1 < \cdots < a_k \leq n$ be a sequence of integers so that $a_i + a_j$ is never a square. Then max $k = n/3 + o(n)$. The integers $\equiv 1 \pmod 3$ show that, if true, this is the best possible result.

The following conjecture is due to Woodall: Let $G(n)$ be a graph of n vertices. Let $|\mathscr{S}| = t$ be a subset of the set of vertices; $\mathscr{S}_1$ is the set of those vertices of $G(n)$ which are joined (by an edge) to some vertex of $\mathscr{S}$. Assume $|\mathscr{S}_1| \geq \min(3|\mathscr{S}|/2, n)$, for every t and every choice of the vertices of $\mathscr{S}$. Then G contains a triangle. I first thought that this must be easy but certainly did not succeed in settling it. I find this conjecture very attractive and challenging.

Chvátal has the following beautiful conjecture: Let $\mathscr{S}$ be a finite family of finite sets having the property that if $A \in \mathscr{S}$ and $B \subset A$, then $B \in \mathscr{S}$. Then there is a subfamily $A_1, \ldots, A_k$ satisfying $A_i \cap A_j \neq \phi$; for all $1 \leq i \leq j \leq k$, with k maximal which also has $\bigcap_{i=1}^k A_i \neq \phi$.

Another conjecture of Chvátal which generalizes and sharpens one of my conjectures states as follows: Let $|\mathscr{S}| = n$. An (n, k)-set is a set F of distinct subsets of $\mathscr{S}$. It is said to be m-interesting if $X_1 \ldots X_m$ is nonempty whenever $X_i \in F$. An (n, k, m)-set is an (n, k)-set F such that every m-intersecting subset of F is necessarily $(m+1)$-intersecting. $f(n, k, m)$ is the largest cardinality of an (n, k, m)-set. A well-known theorem of Ko, Rado, and myself states: $f(n, k, 1) = \binom{n-1}{k-1}$ whenever $n \geq 2k$. I conjectured $f(n, k, 2) = \binom{n-1}{k-1}$ for $k \geq 3$, $n \geq 3k/2$. Chvátal [24] proves this for $k = 3$. More precisely he proves $f(n, k, k-1) = \binom{n-1}{k-1}$ if $n \geq k+2$ and $k \geq 3$. He conjectures: $f(n, k, m) = \binom{n-1}{k-1}$, whenever $1 \leq m < k$ and $n \geq (m+1)k/m$.

Denote by $r(n, m)$ the smallest integer for which every graph of $r(n, m)$ vertices contains either a K_n or an independent set of m points:

$$c_1 n 2^{n/2} < r(n, n) < \binom{2n-2}{n-1} \frac{c_2 \log \log n}{\log n}$$

are well-known. I offered (and still offer) 100 dollars for the following problems: Prove that $\lim_{n \to \infty} r(n, n)^{1/n}$ exists. This is perhaps not difficult. Evaluate its value, and finally give a constructive proof for $r(n, n) > (1+c)^n$.

Burr and I conjectured

$$r(n+1, n+1) > (1+c)r(n, n), \; r(n+1, n) > (1+c)r(n, n) \tag{15}$$

The second conjecture is of course stronger. Trivially,

$$r(n+1, n) - r(n, n) \geq n - 1 \tag{16}$$

We noticed to our annoyance that we cannot improve **(16)**. I offer 25 dollars for any improvement of **(16)** (we expect that this will not be difficult) and 100 dollars for a proof of **(15)** (we feel that this may not be quite simple).

Faudree, Rousseau, Schelp, and I have several papers (most of which are not yet published and some of which are joint papers with Burr); here I only state one of our favorite problems: Denote by $\hat{r}(p_n)$ the smallest integer for which there is a graph of $\hat{r}(p_n)$ edges so that, if we color the edges with two colors, there always is a monochromatic path of length n. Is it true that

$$\frac{\hat{r}(p_n)}{n} \to \infty, \ \frac{\hat{r}(p_n)}{n^2} \to 0? \tag{17}$$

I offer 100 dollars for a proof or disproof of **(17)**.

Let $G^{(r)}(n; k)$ be an r-uniform hypergraph of n vertices and k edges (i.e., r-tuples). An old result of mine states that, for every $\varepsilon > 0$ and any t, if $n > n_0(\varepsilon, t)$, then every $G^{(r)}(n; [\varepsilon, n^r])$ contains $K_r^{(r)}(t, \ldots, t)$; i.e., a complete r-chromatic graph with t vertices. An even older result of Stone, Simonovits, and myself states that, for every $\varepsilon > 0$ and for any t, if $n > n_0(\varepsilon, t)$, then every $G^{(2)}(n; \frac{1}{2}(1 - (1/t - 1) + \varepsilon))$ contains a $K_r^{(2)}(t, \ldots, t)$, i.e., a complete r-chromatic ordinary $(r = 2)$ graph with t vertices of each color.

These theorems lead me to the following problem (I offer 500 dollars for a solution or clarification): Let $G^{(r)}(n_i; k_i)$ be a sequence of r-uniform hypergraphs. We define its edge density to be α if α is the largest number for which there is a sequence $m_i \to \infty$ such that for infinitely many i our $G^{(r)}(n_i; k_i)$ has a subgraph $G^{(r)}(m_i; (\alpha + o(1))\binom{m_i}{r})$. My conjecture states that for each r the set of possible edge densities is a nowhere dense set S_r. For $r = 2$, S_2 is, by the theorem of Stone, Simonovits, and myself, the set $\{1 - 1/r\}$, $r = 1, 2, \ldots, \infty$. For the general case, my theorem gives the smallest possible positive value of the edge density of n r-graphs which is $r!/r^r$.

The first unsolved problem states that if the edge density is greater than $r!/r^r$, then it is greater than $(r!/r^r) + \varepsilon_r$, for some positive ε_r independent of the n_i. I would guess that our graphs must contain some canonical subgraphs and there will only be countably many choices for these graphs and their densities will form a discrete set. Brown, Simonovits, and I have some more general conjectures for multiple r-graphs, but here even the case $r = 2$ presents enormous difficulties. I tried to obtain a theorem of this type for the set of all subsets of the integers, but there is a trivial and devastating counterexample. Let $0 \le \alpha \le 1$ and $1 \le a_1 < a_2 < \cdots$ be a sequence of integers of density α. Let $\mathscr{S}$ be the family of subsets having a_i elements, $i = 1, 2, \ldots$. Clearly on every set of m integers $\mathscr{S}$ induces a hypergraph of $(\alpha + o(1))2^m$ edges.

It is easy to see that if $G(n)$ has no circuit of length $< C \log n$ (C sufficiently large), then the chromatic number of $G(n)$ is at most three. Gallai constructed a graph of chromatic number 4 whose smallest odd circuit has length $\ge n^{1/2}$. Lovász and I, by a slight modification of Gallai's construction, gave a $G(n)$ of chromatic number $k + 2$ whose smallest odd circuit has length $\ge n^{1/k}$. Gallai and I conjectured that these results are best possible. In other words, if $G(n)$ has no odd circuit of length $< Cn^{1/k}$, then the chromatic number of $G(n)$ is less than $k + 2$.

I claimed that I could prove this but could never reconstruct my proof, not even for $k = 2$—perhaps my "proof" was not correct. In any case the conjecture has to be considered as open, and I offer 50 dollars for a proof or disproof.

Let K_n be a complete graph of n vertices. Two players color the edges alternately white and black. The first player wins if the largest white clique is larger than the largest black clique. The second player wins if this does not happen. I guess that for all (or most) n, the second player has a win. What is the largest $f(n)$ so that the first player can be sure to get a white clique of size $f(n)$? I am sure this problem is not new. It would be nice to know the relation of $f(n)$ to the Ramsey function. Many generalizations are possible.

Finally I state an old and forgotten problem of Sárközi, Szemerédi, and myself. Let $|\mathcal{S}| = n$, $A_i \subset \mathcal{S}$, $1 \le i \le r$, $B_j \subset \mathcal{S}$, $1 \le j \le s$, where the A's and B's are all distinct. Assume that, for all i and j,

$$|A_i \cap A_j| \ge 2, \; |B_i \cap B_j| \ge 2, \; |A_i \cap B_j| \ge 1$$

We conjecture that

$$r + s \le 2^{n-1} - \frac{c2^n}{n} \tag{18}$$

It is easy to see that, apart from the value of c, (**18**), if true, is best possible choice. We observed that (**18**) would have the following consequence: Let $|z_i| \le 1$, $1 \le i \le n$, be complex numbers. Then the number of sums

$$\sum_{i=1}^{n} \varepsilon_i z_i \le \sqrt{2}, \; e_i = \pm 1$$

is at least $c(2^n)/n$.

For a large collection of unsolved problems, see my papers presented at various combinatorial conferences, in particular the Southeastern Conferences on Combinatorics held at Baton Rouge and Boca Raton, and also the meetings in Hungary, Prague, and Aberdeen.

For a survey of the old papers (old here means less than five years) on generalized Ramsey theory, see S. A. Burr [33]. A second survey paper of Burr will appear soon.

References

1. Hoffman, A. J. & R. R. Singleton. 1960. On Moore graphs with diameters 2 and 3. IBM J. Res. Dev. **4:** 497–504.
2. Erdös, P., L. Gerencsér, & A. Maté. 1970. Problems of graph theory concerning optimal designs. Combinatorial Theory and its Applications (Colloq. Budapest, 1969) Colloq. Math. Soc. J. Bolyai, North-Holland, Amsterdam. **4:** 317–327.
3. Plesnik, J. 1978. Bounds on chromatic numbers of multiple factors of a complete graph. J. Graph Theory. **2:** 9–17.
4. Larman, D. E. & C. A. Rogers. 1972. The realisation of distances within sets in Euclidean space. Mathematica. **19:** 1–24.
5. Frankl, P. To appear.
6. Erdös, P., 1975. On some problems of elementary and combinatorial geometry. Ann. Mat. Pura Appl. **103:** 99–108.
7. Frankl, P. 1977. A constructive lower bound for some Ramsey numbers. Ars Combinatoria. **3:** 297–302.
8. Berlekamp, E. R. 1969. On subsets with intersections of even cardinality, Can. Math. Bull. **12**: 471–474.
9. Galvin, F. & M. M. Krieger. 1971. The minimum number of cliques in a graph and its complement. Proc. Second Louisiana Conf. Combinatories, Graph Theory and Computing. Louisiana State University, Baton Rouge. pp. 345–352.
10. Erdös, P. 1964. On a combinatorial problem II. Acta Math. Acad. Sci. Hung. **15**: 445–447.
11. Erdös, P. & L. Lovász. 1975. Problems and results on three-chromatic hypergraphs and some related questions. Infinite and Finite Sets (Colloq. Keszthely, 1973) Colloq. Math. Soc. J. Bolyai, North-Holland, Amsterdam. **10:** 609–627.

12. ERDÖS, P. & S. SHELAH. 1972. Separability properties of almost disjoint families of sets. Isr. J. Math. **12**: 207–274.
13. WOODALL, D. R. 1973. Property B and the four colour problem. Combinatorics. (Proc. Conf. Oxford 1972.) pp. 322–340.
14. LOVÁSZ, L. 1973. Coverings and colorings of hypergraphs. Proc. Fourth Southeastern Conf. Combinatorics, Graph Theory and Computing, Boca Raton, Florida. pp. 3–12.
15. ERDÖS, P. & A. HAJNAL. 1966. On chromatic numbers of graphs and set systems. Acta Math. Acad. Sci. Hung. **17**: 61–99.
16. ERDÖS, P. 1969. Problems and results in chromatic graph theory. *In* Proof Techniques in Graph Theory. F. Harary, ed. Academic Press, New York. pp. 27–35.
17. DEZA, M. 1974. Solution d'un problème de ERDÖS-LOVÁSZ. J. Comb. Theory. (B) **16:** 166–167.
18. ERDÖS, P. & R. RADO. 1960. Intersection theorems for systems of sets I and II. J. London Math. Soc. **35:** 85–90; **44** (1969): 467–479.
19. ERDÖS, P., E. MILNER & R. RADO. 1969. Intersection theorems for systems of sets III. J. Aust. Math. Soc. **18**: 22–41.
20. ERDÖS, P. & E. SZMERÉDI. J. Comb. Theory. Submitted for publication.
21. ERDÖS, P. & A. HAJNL. 1966. On chromatic number of graphs and set systems. Acta Math. Acad. Sci. Hung. **17:** 67–99.
22. ERDÖS, P. & A. HAJNAL. 1967. On the decomposition of graphs. Acta Math. Acad. Sci. Hung. **18:** 359–376.
23. ERDÖS, P. & A. HAJNAL. 1970. Problems and results in finite and infinite combinatorial analysis. Ann. N.Y. Acad. Sci. **175:** 115–124.
24. CHVÁTAL, V. 1974/75. An extremal set intersection theorem. J. London Math. Soc. Ser. 2. **9:** 355–359.
25. ERDÖS, P. & A. H. STONE. 1946. On the structure of linear graphs. Bull. Am. Math. Soc. **52:** 1087–1091.
26. KÖVÁRI, T., V. T. SÓS & P. TURAN. 1954. On a problem of K. Zarankiewicz. Coll. Math. **3:** 50–57.
27. ERDÖS, P. & M. SIMONOVITS. 1966. A limit theorem in graph theory. Studia Sci. Mat. Hung. Acad. **1:** 51–57.
28. ERDÖS, P. 1964. On external problems of graphs and generalized graphs. Isr. J. Math. **2:** 183–190, 251–261.
29. ERDÖS, P. 1971. On some extremal properties of r-graphs. Discrete Math. **1:** 1–6.
30. BROWN, W. G., P. ERDÖS & M. SIMONOVITS. 1973. Extremal problems for directed graphs. J. Comb. Theory. B. **15:** 77–93.
31. ERDÖS, P. 1962. On circuits and subgraphs of chromatic graphs. Mathematika. **9:** 170–175.
32. BAUMGARTNER, J., F. GALVIN, R. MCKENRIE & R. LAVER. 1975. Game theoretic versions of partition relations. Infinite and Finite Sets (Colloq. Keszthely, 1973) Colloq. Math. Soc. J. Bolyai North-Holland, Amsterdam **10:** 131–135.
33. BURR, S. A. 1974. Generalized ramsey theory for graphs. *In* Graphs and Combinatorics. R. Bari & F. Harary, Eds. Springer-Verlag, Berlin. pp. 52–75.