ON THE PRODUCT OF THE POINT AND LINE COVERING NUMBERS OF A GRAPH

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For a finite graph $G = (V, E)$, the point covering number $\alpha_0(G)$ and the line covering number $\alpha_1(G)$ are defined as follows (e.g., [3]):

$$\alpha_0(G) = \min\{|X| : X \subseteq V \text{ and every } e \in E \text{ contains some } x \in X\},$$
$$\alpha_1(G) = \min\{|Y| : Y \subseteq E \text{ and every } v \in V \text{ is contained in some } y \in Y\}.$$

We shall assume G has no isolated points so that these quantities are well defined. During his talk at this meeting, F. Harary mentioned the following two conjectures of J. Kabell and himself [1]:

(i) $\min\limits_G \alpha_0(G)\alpha_1(G) = n - 1,$

(ii) $\max\limits_G \alpha_0(G)\alpha_1(G) = (n-1)\left\lceil \dfrac{n+1}{2} \right\rceil$

where G ranges over all graphs with n points. He further noted that equality holds in (i) for the star $K_{1,n-1}$ and in (ii) for the complete graph K_n.

In this note we settle these conjectures. In particular, we show that (i) is true, and (ii), while not completely true, is nearly true. The smallest counterexample to (ii) is the graph $2K_3$ consisting of two disjoint triangles. Note that

$$\alpha_0(2K_3)\alpha_1(2K_3) = 16, \quad \alpha_0(K_6)\alpha_1(K_6) = 15.$$

However, (ii) is valid if n is odd or if G is required to be connected. We also consider the corresponding questions for hypergraphs.

THEOREM 1. For any graph G with $n \geq 3$:

(i′) $\alpha_0(G)\alpha_1(G) \geq n - 1$, with equality only for $G = K_{1,n-1}$;

(ii′) $\alpha_0(G)\alpha_1(G) \leq \begin{cases} \dfrac{n^2-1}{2} & \text{for } n \text{ odd}, \\[2ex] \dfrac{n^2-4}{2} & \text{for } n \text{ even}. \end{cases}$

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Equality in (ii′) holds only for K_n when n is odd or $n = 4$ and only for $K_a + K_b$, with a, b odd and $a + b = n$, when n is even and at least 6.

Proof: First note that

$$\alpha_1(G) \geq \frac{n}{2} \tag{1}$$

since each edge of G covers just two points of G. Now, assume $\alpha_0(G)\alpha_1(G) \leq n - 1$. If $\alpha_0(G) \geq 2$, then, by (1), $\alpha_0(G)\alpha_1(G) \geq n$, which contradicts the hypothesis. Hence, we must have $\alpha_0(G) = 1$, i.e., all edges of G contain a common point. This is exactly the definition of $K_{1,n-1}$; since

$$\alpha_0(K_{1,n-1})\,\alpha_1(K_{1,n-1}) = n - 1$$

then (i′) is proved.

To prove (ii′), assume that G satisfies

$$\alpha_0(G)\alpha_1(G) \geq \begin{cases} \dfrac{n^2 - 1}{2} & n \text{ odd,} \\[2ex] \dfrac{n^2 - 4}{2} & n \text{ even.} \end{cases} \tag{2}$$

Let $E' = \{e_1, e_2, \ldots, e_x\}$ denote a maximum set of disjoint edges of G. Thus, by a theorem of Gallai [2],

$$\alpha_1(G) = n - x \tag{3}$$

Also, $\alpha_0(G) \leq 2x$, since the $2x$ endpoints of the $e_i \in E'$ form a covering of all the edges of G (by the maximality of E'). Thus

$$\alpha_0(G)\alpha_1(G) \leq 2x(n - x). \tag{4}$$

Note also that $2x \leq n$ must always hold.

First, suppose n is odd. The right-hand side of (4) is maximized only by choosing $x = (n - 1)/2$ or $x = (n + 1)/2$ and the larger value is forbidden by the previous remark. For $x = (n - 1)/2$, (4) implies

$$\alpha_0(G)\alpha_1(G) \leq (n - 1)\left(\frac{n + 1}{2}\right)$$

By (3), $\alpha_1(G) = n - x = (n + 1)/2$ and so $\alpha_0(G) = n - 1$. However, this implies $G = K_n$. Since

$$\alpha_0(K_n)\alpha_1(K_n) = (n - 1)\left(\frac{n + 1}{2}\right) = \frac{n^2 - 1}{2}$$

then (ii′) is proved for n odd.

Now, suppose $n \geq 6$ is even. If we try to use the value $x = n/2$, then because $\alpha_0(G) \leq n - 1$, we have

$$\alpha_0(G)\alpha_1(G) \leq (n - 1)(n - x) = (n - 1)\frac{n}{2} < \frac{n^2 - 4}{2}$$

since $n \geq 6$. Thus, the maximum possible value of the right-hand side of (4) occurs for the (unique) value $x = (n/2) - 1$ and yields

$$\alpha_0(G)\alpha_1(G) \leq 2\left(\frac{n}{2} - 1\right)\left(\frac{n}{2} + 1\right) = \frac{n^2 - 4}{2}$$

and so,

$$\alpha_0(G)\alpha_1(G) = \frac{n^2 - 4}{2} \tag{5}$$

Therefore,

$$\begin{aligned} \alpha_1(G) &= n - x = \frac{n}{2} + 1, \\ \alpha_0(G) &= n - 2 \end{aligned} \tag{6}$$

Write $e_i = \{a_i, b_i\}$, $1 \leq i \leq (n/2) - 1$ and let the remaining two points of G be denoted by x_1 and x_2.

(a) $\{x_1, x_2\}$ is not an edge of G by the maximality assumption on E'.

(b) Every a_i and every b_j is connected to at least one of the x_k's. For if a_1, say, is not connected to x_1 or x_2, then $V - \{a_1, x_1, x_2\}$ covers all edges of G and has only $n - 3$ points, which contradicts (6).

(c) $\{a_i, x_k\}$ is an edge of G if and only if $\{b_i, x_k\}$ is an edge of G. For suppose not, e.g., $\{a_1, x_1\} \in E$, $\{b_1, x_1\} \notin E$. By (b), $\{b_1, x_2\} \in E$; thus,

$$E' - \{a_1, b_1\} \cup \{a_1, x_1\} \cup \{b_1, x_2\}$$

is a set of $n/2$ disjoint edges of G, which contradicts the maximality of E'.

(d) x_1 is connected to $\{a_i, b_i\}$ if and only if x_2 is not connected to $\{a_i, b_i\}$. For suppose x_1 and x_2 are both connected to $\{a_i, b_i\}$. Then just as in (c), we can replace $\{a_i, b_i\}$ by two disjoint edges, forming $n/2$ disjoint edges in G.

For $i = 1, 2$, let C_i denote the set of points v such that $\{x_i, v\}$ is an edge of G.

(e) If $v_1 \in C_1$, $v_2 \in C_2$ then $\{v_1, v_2\}$ is not an edge of G. Suppose not, i.e., suppose $\{v_1, v_2\}$ is an edge of G. Let w_1, w_2 be the vertices adjacent to v_1, v_2 in E'. If the edges of E' containing v_1 and v_2 are removed from E' and the edges $\{x_1, w_1\}$, $\{x_2, w_2\}$ and $\{v_1, v_2\}$ are added, then we have a set of $n/2$ disjoint edges in G, which is impossible.

Thus, G consists of two connected components $C_1 \cup \{x_1\}$ and $C_2 \cup \{x_2\}$.

(f) If x_i is connected to $\{a_j, b_j\}$ and $\{a_k, b_k\}$ then both points in $\{a_j, b_j\}$ are connected to both points in $\{a_k, b_k\}$. For suppose (without loss of generality) that $\{a_j, a_k\}$ is not an edge of G. Then

$$\alpha_0(C_i \cup \{x_i\}) \leq |C_i| - 1$$

where $|C_i|$ denotes the number of points in C_i. But this implies

$$\alpha_0(G) = \sum_{j=1}^{2} \alpha_0(C_j \cup \{x_j\}) \leq n - 3$$

which contradicts (6).

Therefore, we conclude that G is made up of two components which are (disjoint) complete graphs, each of odd order. It is easily checked that in this case

$$\alpha_0(G)\alpha_1(G) = \frac{n^2 - 4}{2}$$

and (ii') holds for even $n \geq 6$.

For the final case $n = 4$, the bound in (4) (choosing $x = 2$) implies $\alpha_0(G)\alpha_1(G) \le 8$. It is easily seen that this implies $\alpha_0(G)\alpha_1(G) \le 6$, which can only occur when $\alpha_0(G) = 3$, $\alpha_1(G) = 2$, i.e., G must be K_4. This completes the proof of the theorem. $\square$

We note here that if we require that G be connected, then it can be shown, using similar arguments, that the original conjecture (ii) is valid with K_n always being the unique graph achieving max $\alpha_0(G)\alpha_1(G)$.

An Extension to Hypergraphs

We now consider an r-uniform hypergraph $H = (V, E)$, where, as usual, E consists of certain r-element subsets of V for some fixed $r \ge 2$. We define $\alpha_0(H)$ and $\alpha_1(H)$ in the obvious way, i.e., $\alpha_0(H)$ denotes the minimum number of points of H hitting all edges of H and $\alpha_1(H)$ denotes the minimum number of edges of H hitting all points of H. Also, we assume H has no isolated points.

Theorem 2. For any r-uniform hypergraph H on n points,

$$\frac{n-1}{r-1} \le \alpha_0(H)\alpha_1(H) \le \frac{r}{4(r-1)} n^2 \tag{7}$$

Proof: Observe that $\alpha_1(H) \ge n/r$. Hence, if $\alpha_0(H) \ge 2$ then

$$\alpha_0(H)\alpha_1(H) \ge 2 \cdot \frac{n}{r} > \frac{n-1}{r-1}$$

On the other hand, if $\alpha_0(H) = 1$ then all edges $e \in E$ contain a common point, and so

$$\alpha_1(H) \ge \frac{n-1}{r-1}$$

i.e.,

$$\alpha_0(H)\alpha_1(H) \ge \frac{n-1}{r-1}$$

which is the left-hand side of (7).

To prove the right-hand side of (7), let $E' = \{e_1, \ldots, e_x\}$ denote a maximum set of disjoint edges of E. Then

$$\alpha_1(H) \le x + n - rx = n - (r-1)x \tag{8}$$

since the $n - rx$ points not in E' can be covered by at most $n - rx$ additional edges; also,

$$\alpha_0(H) \le rx \tag{9}$$

since by the maximality of E', the rx points of E' hit every edge of E; therefore,

$$\alpha_0(H)\alpha_1(H) \le rx(n - (r-1)x) \tag{10}$$

The right-hand side of (10) is maximized by taking

$$x = \frac{n}{2(r-1)}$$

which yields

$$\alpha_0(H)\alpha_1(H) \leq \frac{r}{4(r-1)} n^2$$

This completes the proof of (7) and the theorem is proved. $\square$

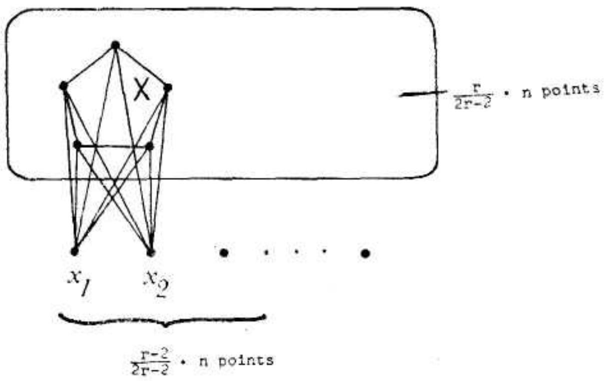
The lower bound in THEOREM 2 can be achieved whenever $n \equiv 1 (\text{mod } r-1)$ by taking V to be $\{0, 1, 2, \ldots, n-1\}$ and $E = \{e_1, e_2, \ldots, e_{(n-1)/(r-1)}\}$ given by

$$e_i = \{0\} \cup \{(r-1)(i-1)+1, (r-1)(i-1)+2, \ldots, (r-1)i\}$$

for

$$1 \leq i \leq \frac{n-1}{r-1}.$$

We have not analyzed the fine structure of the exact upper bound for $\alpha_0(H)\alpha_1(H)$. The bound of THEOREM 2 is asymptotically best possible. This can be seen by considering the hypergraph H_0 formed as shown in the figure. The top part of H_0



consists of a complete r-uniform hypergraph on $r/(2r-2) \cdot n$ points (i.e., all r-element subsets are edges). In addition, there are $n(r-2)/(2r-2)$ additional edges, each formed by adjoining a new point x_i to a fixed $(r-1)$-element subset X above. Thus,

$$\alpha_0(H_0) \sim \frac{r}{2r-2} n$$

$$\alpha_1(H_0) \sim \frac{n}{2}$$

and so,

$$\alpha_0(H_0)\alpha_1(H_0) \sim \frac{r}{4(r-1)} n^2$$

It would be interesting to characterize those hypergraphs H which achieve the maximum and minimum values of $\alpha_0(H)\alpha_1(H)$.

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