SYSTEMS OF DISTINCT REPRESENTATIVES AND MINIMAL BASES IN ADDITIVE NUMBER THEORY

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1. Introduction

The set A of nonnegative integers is an asymptotic basis of order h if every sufficiently large integer is the sum of h elements of A. For example, the squares form an asymptotic basis of order 4 and the square-free numbers form an asymptotic basis of order 2. If A is an asymptotic basis of order h, but no proper subset of A is an asymptotic basis of order h, then A is a minimal asymptotic basis of order h. Minimal asymptotic bases have been studied by Erdös, Härtter, Nathanson, and Stöhr [4, 5, 7, 8, 10-12, 18, 19, 21, 23, 24]. It usually is difficult to determine if a given asymptotic basis contains a minimal asymptotic basis. For example, it is not known if there is a minimal asymptotic basis consisting only of squares. In a previous paper [11], we proved that the set of square-free numbers does contain a minimal asymptotic basis of order 2. In this paper we prove the following more general result. Let A be an asymptotic basis of order 2. Let r(n) denote the number of representations of the integer n in the form $n = a_j + a_k$, where $a_j, a_k \in A$ and $a_j \leq a_k$. If $r(n) > c \log n$ for some constant $c > \log^{-1}(4/3)$ and all $n \geq N$, then A contains a minimal asymptotic basis of order 2. Perhaps this

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theorem is best possible in the sense that there may exist an absolute constant $C > 0$ such that, if $c < C$, then there exists a set A of integers with $r(n) > c \log n$ for $n \geq N$ such that A does not contain a minimal asymptotic basis of order 2. We are far from being able to prove this. But we do prove that with Lebesgue measure on the probability space of all sequences of nonnegative integers, almost every sequence contains a minimal asymptotic basis of order 2.

We obtain these and other results about bases in the following more general situation. Let U be an infinite set of positive integers. The set A is an asymptotic basis of order h for U if all but finitely many numbers $u \in U$ can be written as the sum of h elements of A. If A is an asymptotic basis of order h for U, but no proper subset of A has this property, then A is a minimal asymptotic basis of order h for U.

Dual to the concept of minimal basis is that of maximal nonbasis. The set A of nonnegative integers is an asymptotic nonbasis of order h if there are infinitely many positive integers that cannot be written as the sum of h elements of A. If A is an asymptotic nonbasis of order h, but no proper superset of A is an asymptotic nonbasis of order h, then A is a maximal asymptotic nonbasis of order h. In other words, A is a maximal asymptotic nonbasis of order h, but, for every nonnegative integer $b \notin A$, the set $A \cup \{b\}$ is an asymptotic basis of order h. Maximal nonbases were introduced by Nathanson [21], and they have been studied by Erdös, Deshouillers and Grekos, Hennefeld, Nathanson, and Turjányi [3, 6-9, 11, 20-23, 26].

It is usually difficult to decide if a given asymptotic nonbasis is contained in a maximal asymptotic nonbasis, or if a given sequence of integers contains a maximal asymptotic nonbasis. In this paper we prove that if A is an asymptotic basis of order 2 such that A contains arbitrarily long intervals and also $r(n) > c \log n$ for some constant

$c > \log^{-1}(4/3)$ and all $n \geq N$, then A contains a maximal asymptotic nonbasis of order 2. Moreover, with Lebesgue measure on the space of all sets of nonnegative integers, almost every set contains a maximal asymptotic nonbasis of order 2.

Many natural sets of integers do not contain maximal asymptotic nonbases. Indeed, any maximal asymptotic nonbasis of order 2 must contain arbitrarily long finite arithmetic progressions. We showed in [11] that there is no maximal asymptotic nonbasis of order 2 consisting only of square-free numbers, but that there does exist a set A^* of square-free numbers with the property that A^* is an asymptotic nonbasis of order 2, but, for any square-free number $b \notin A^*$, the set $A^* \cup \{b\}$ is an asymptotic basis of order 2.

Let A and A^* be sets of nonnegative integers. Then A^* is an <u>asymptotic nonbasis of order h maximal with respect to A</u> if A^* is an asymptotic nonbasis of order h, but, for every $b \in A \backslash A^*$, the set $A^* \cup \{b\}$ is an asymptotic basis of order h. We shall prove that if A is an asymptotic basis of order 2 such that (i) $r(n) > c \log n$ for $c > \log^{-1}(4/3)$ and $n \geq N$, and (ii) for every finite set $F \subseteq A$ there exist infinitely many n such that $n - a \in A$ for all $a \in F$, then A contains a subset A^* such that A^* is an asymptotic nonbasis of order 2 maximal with respect to A.

As before, we obtain these and other results about maximal nonbases in a more general setting. Let U be an infinite set of positive integers. The set A is an <u>asymptotic nonbasis of order h for U</u> if there are infinitely many numbers belonging to U that cannot be written as the sum of h elements of A. If A is an asymptotic nonbasis of order h for U, but every proper superset of A is an asymptotic basis of order h for U, then A is a <u>maximal asymptotic nonbasis of order h for U</u>. If A and A^* are sets of nonnegative integers such that A^* is an asymptotic nonbasis of order h for U, but $A^* \cup \{b\}$ is an asymptotic basis of order h for U for every $b \in A \backslash A^*$, then A^* is an asymptotic nonbasis of order h for U maximal with respect to A.

In this paper we prove theorems only about asymptotic bases and nonbases of order 2. It is an unsolved problem to obtain results for bases and nonbases of orders $h \geq 3$.

Notation. Let $|S|$ denote the cardinality of the finite set S. Let $[a,b]$ denote the interval of integers $a \leq n \leq b$. Let L be a positive integer. The set A of nonnegative integers contains an interval of length L if $[b, b + L - 1] \subseteq A$ for some integer $b \geq 0$. The set A contains a gap of length L if $[b, b + L - 1] \cap A = \emptyset$ for some $b \geq 0$. The relative complement of B in A is denoted $A \backslash B$.

2. Systems of Distinct Representatives

The critical device used in this paper is the following estimate for simultaneous systems of distinct representatives.

LEMMA 1. Let $s \geq 1$ and $t \geq 0$. Let $S = \bigcup_{i=1}^{s} S_i$ and $T = \bigcup_{k=1}^{t} T_k$ be sets satisfying the following conditions:

(1a) $|S_i| = 1$ or 2 and $|T_k| = 1$ or 2 for $i = 1, \ldots, s$ and $k = 1, \ldots, t$;

(1b) $S_i \cap S_j = T_k \cap T_\ell = \emptyset$ for $1 \leq i < j \leq s$ and $1 \leq k < \ell \leq t$;

(1c) $S_i \neq T_k$ for all i and k.

Let $\Phi(S,T)$ denote the number of sets $X \subseteq S$ such that

(1d) $|X| = s$

(1e) $|X \cap S_i| = 1$ for $i = 1, \ldots, s$;

(1f) $X \cap T_k \neq \emptyset$ for $k = 1, \ldots, t$.

Then $\Phi(S,T) \leq 2^s(3/4)^t$. This estimate is best possible.

Proof. Let $\phi(s,t) = 2^s(3/4)^t$. Suppose that $S = \bigcup_{i=1}^{s} S_i$ and $T = \bigcup_{k=1}^{t} T_k$ satisfy (1a), (1b), (1c). We shall show that

$\Phi(S,T) \leq \phi(s,t)$. We can assume without loss of generality that $S_i \cap T \neq \emptyset$ for $i = 1, 2, \ldots, s$. The proof is by induction on t. If $t = 0$, then $T = \emptyset$ and

$$\Phi(S,T) \leq 2^s = \phi(s,0)$$

for all $s \geq 1$.

Now assume that $\Phi(S,T) \leq \phi(s,t')$ whenever $s \geq 1$ and $0 \leq t' < t$. We shall prove the Lemma for t. If $T_\ell \cap S = \emptyset$ for some ℓ, then $X \cap T_\ell = \emptyset$ for all $X \subseteq S$, and so

$$\Phi(S,T) = 0 < \phi(s,t).$$

If $|T_\ell \cap S| = 1$ for some ℓ, then $T_\ell \cap S_j = \{b\}$ for some unique j, and $b \in X$ for every set $X \subseteq S$ that satisfies (1d), (1e), (1f). Let $X' = X \backslash \{b\}$. Then $X = X' \cup \{b\}$. Let $S' = \cup_{i \neq j} S_i$ and $T' = \cup_{k \neq \ell} T_k$. Then S' and T' satisfy conditions (1a), (1b), (1c), and X' satisfies (1d), (1e), (1f) for S' and T'. Conversely, if X' satisfies (1d), (1e), (1f) for S' and T', then $X = X' \cup \{b\}$ satisfies (1d), (1e), (1f) for S and T. The induction hypothesis implies that

$$\Phi(S,T) = \Phi(S',T') \leq 2^{s-1}(3/4)^{t-1}$$

$$< 2^s(3/4)^t = \phi(s,t).$$

Finally, if $|T_k \cap S| > 1$ for all $k = 1, \ldots, t$, then condition (1a) implies that $|T_k| = 2$ and $T_k \subseteq S$, hence $T = \bigcup^*_{k=1} T_k \subseteq S$. There are two cases.

Case I. $S = T$. If $|S_i| = 1$ for all $i = 1, \ldots, s$, then $\Phi(S,T) = 1 < \phi(s,t)$. Suppose that $|S_i| = 2$ for some i, say, $i = 1$, and that $S_1 = \{a,b\}$. Then $a \in T_\ell$ and $b \in T_m$, and condition (1c) implies that $\ell \neq m$. Let $T_\ell = \{a,d\}$ and $T_m = \{b,c\}$. Then $c \in S_j$ for some $j \neq 1$. Let $X \subseteq S$ satisfy conditions (1d), (1e), (1f). Then $a \in X$ or $b \in X$. If $a \in X$, then $b \notin X$, hence $c \in X$ since $X \cap T_m \neq \emptyset$.

Let $X' = X\backslash\{a,c\}$. Let $S' = \bigcup_{i\neq 1,j} S_i$ and $T' = \bigcup_{k\neq \ell,m} T_k$. Then $X' \subseteq S'$ and X' satisfies (1d), (1e), (1f) for the sets S' and T'. Conversely, if $X' \subseteq S'$ satisfies (1d), (1e), (1f) for S' and T', then $X = X' \cup \{a,c\}$ satisfies (1d), (1e), (1f) for S and T. The induction hypothesis implies that there are at most $\phi(s - 2, t - 2)$ sets $X \subseteq S$ such that $a \in X$ and X satisfies (1d), (1e), (1f). Similarly, there are at most $\phi(s - 2, t - 2)$ sets $X \subseteq S$ such that $b \in X$ and X satisfies (1d), (1e), (1f). Therefore,

$$\Phi(S,T) \leq 2\phi(s - 2, t - 2) = 2\cdot 2^{s-2}(3/4)^{t-2}$$

$$< 2^s(3/4)^t = \phi(s,t).$$

Case II. $S \neq T$. Then $S_j \not\subseteq T$ for some j. But $S_j \cap T \neq \emptyset$, hence $|S_j| = 2$ and $|S_j \cap T| = 1$. Let $S_j = \{a,b\}$, where $a \in T_\ell \subseteq T$ and $b \notin T$. Let $T_\ell = \{a,c\}$. We divide the sets $X \subseteq S$ satisfying (1d), (1e), (1f) into two classes: Either $a \in X$ or $b \in X$. If $b \in X$, then $a \notin X$. Since $X \cap T_\ell \neq \emptyset$, it follows that $c \in X$. But $c \in S_m$ for some $m \neq j$. Let $S' = \bigcup_{i\neq j,m} S_i$ and $T' = \bigcup_{k\neq \ell} T_k$. Then $X' = X\backslash\{b,c\}$ satisfies (1d), (1e), (1f) for the sets S' and T'. It follows that there are at most $\phi(s - 2, t - 1)$ sets X such that $b \in X$ and X satisfies (1d), (1e), (1f) for S and T. Similarly, there are at most $\phi(s - 1, t - 1)$ sets X with $a \in X$. Therefore,

$$\Phi(S,T) \leq \phi(s - 2, t - 1) + \phi(s - 1, t - 1)$$

$$= 2^{s-2}(3/4)^{t-1} + 2^{s-1}(3/4)^{t-1}$$

$$= 2^s(3/4)^t = \phi(s,t).$$

This proves that $\Phi(S,T) \leq \phi(s,t)$ for all $s \geq 1$ and $t \geq 0$.

This estimate is best possible. If $s \geq 2t$, there exist sets S and T satisfying conditions (1a), (1b), (1c) such that

$\Phi(S,T) = \phi(s,t)$. Let $a_1, \ldots, a_s, b_1, \ldots, b_s$ be 2s distinct elements. Let $S_i = \{a_i, b_i\}$ for $i = 1, \ldots, s$ and let $T_k = \{a_k, a_{t+k}\}$ for $k = 1, \ldots, t$. Let $S = \bigcup_{i=1}^{s} S_i$ and let $T = \bigcup_{k=1}^{t} T_k$. It is easy to see that $\Phi(S,T) = 3^t 2^{s-2t} = \phi(s,t)$.

This completes the proof of Lemma 1.

Remark. It is an open combinatorial problem to find good estimates for $\Phi(S,T)$ in Lemma 1 when condition (1a) is replaced by the condition $1 \leq |S_i| \leq h$ and $1 \leq |T_k| \leq h$ for $h \geq 3$.

LEMMA 2. Let $n \geq N_0$ and let $R(n) = \bigcup_{i=1}^{r(n)} R_i(n)$ be a set of integers such that

(2a) $|R_i(n)| = 1$ or 2;

(2b) $|R_i(n)| = 1$ for at most one i;

(2c) $R_i(n) \cap R_j(n) = \emptyset$ for $1 \leq i < j \leq r(n)$;

(2d) $R_i(n) \neq R_k(m)$ for all $1 \leq i \leq r(n)$ and $1 \leq k \leq r(m)$, $m \neq n$;

(2e) $r(n) > c \log n$ for some constant $c > \log^{-1}(4/3)$ and all $n \geq N_1$.

Then there exists a number N_2 such that for all $n \geq N_0$ there is a set $X(n) \subseteq R(n)$ such that

(2f) $|X(n)| = r(n)$;

(2g) $|X(n) \cap R_i(n)| = 1$ for $i = 1, \ldots, r(n)$;

(2h) For every $m \geq N_2$, $m \neq n$, there is a $j \leq r(m)$ such that $X(n) \cap R_j(m) = \emptyset$.

Proof. Choose $\delta > 0$ so that $c \log(4/3) = 1 + \delta$. Choose $N_2 \geq N_1$ so that

$$\sum_{m=N_2}^{\infty} \frac{1}{m^{1+\delta}} < \frac{1}{2} .$$

We shall apply Lemma 1 with $S = R(n)$, $s = r(n)$, $T = R(m)$, and $t = r(m)$. Conditions (2a), (2c), (2d) indicate that the sets S and T satisfy (1a), (1b), (1c). It follows from Lemma 1 that there are at most $2^{r(n)}(3/4)^{r(m)}$ sets $X(n) \subseteq R(n)$ that satisfy (2f) and (2g) but violate (2h). Therefore, the number of sets $X(n) \subseteq R(n)$ that satisfy (2f) and (2g) but violate (2h) for <u>some</u> $m \geq N_2$ is at most

$$\begin{aligned} \sum_{m=N_2}^{\infty} 2^{r(n)}\left(\tfrac{3}{4}\right)^{r(m)} &\leq 2^{r(n)} \sum_{m=N_2}^{\infty} \left(\tfrac{3}{4}\right)^{c \log m} \\ &= 2^{r(n)} \sum_{m=N_2}^{\infty} \frac{1}{m^{c \log (4/3)}} \\ &= 2^{r(n)} \sum_{m=N_2}^{\infty} \frac{1}{m^{1+\delta}} \\ &< 2^{r(n)}\left(\tfrac{1}{2}\right) = 2^{r(n)-1} . \end{aligned}$$

By conditions (2a) and (2b), there are at least $2^{r(n)-1}$ sets $X(n) \subseteq R(n)$ satisfying (2f) and (2g). Therefore, there must exist a set $X(n) \subseteq R(n)$ satisfying (2f), (2g), and (2h). This completes the proof of Lemma 2.

Remark. Lemma 2 is the crucial tool used to obtain the results in this paper. The following is a typical application. Let $A = \{a_i\}$ be a strictly increasing sequence of positive integers. Let $r(n)$ denote the number of representations of n in the form $n = a_j + a_k$, where $a_j, a_k \in A$ and $a_j \leq a_k$. Let $R(n)$ be the union of the sets $R_i(n) = \{a_j, a_k\}$, where $a_j, a_k \in A$ and $a_j + a_k = n$. Notice that $|R_i(n)| = 1$ or 2, and that $|R_i(n)| = 1$ only if $R_i(n) = \{n/2\}$, where n is even and $n/2 \in A$. Clearly, $R(n) = \emptyset$ if and only if $n \notin 2A$. It is easy to see that (2a)-(2d) automatically hold for any sequence A, and that (2e) is true if A is an asymptotic basis of order 2 such that every sufficiently large integer n has at least $c \log n$ representations for $c > \log^{-1}(4/3)$. Let $X(n) \subseteq R(n)$ satisfy conditions (2g) and (2h). If we delete the numbers in $X(n)$ from A, then (2g) implies that we destroy every representation of n. Consequently, $n \notin 2\ (A\setminus X(n))$. But condition (2h) implies that no other number $m \geq N_2$ is destroyed, that is, $m \in 2\ (A\setminus X(n))$ for all $m \geq N_2$, $m \neq n$. This allows us to modify the sequence A in various ways.

3. Minimal Asymptotic Bases

THEOREM 1. Let $A = \{a_i\}$ be a strictly increasing sequence of nonnegative integers. Let $r(m)$ denote the number of representations of m in the form $m = a_j + a_k$, where a_j, $a_k \in A$ and $a_j \leq a_k$. Let $U = \{u_n\}$ be a strictly increasing sequence of positive integers. Let A be an asymptotic basis of order 2 for U; that is, $u_n \in 2A$ for $n \geq N_0$. Suppose that $r(u_n) > c \log n$ for some constant $c > \log^{-1}(4/3)$ and all $n \geq N_1$. Suppose also that for every $a_i \in A$ there are infinitely many $a_j \in A$ such that $a_i + a_j \in U$. Then A contains a minimal asymptotic basis of order 2 for U.

Proof. Choose $\delta > 0$ so that $c \log(4/3) = 1 + 3\delta$. Let $u_n \in U$ and let $a_1 < a_2 < \ldots < a_{r(u_n)} \leq u_n/2$ be the numbers $a_i \in A$ such that $a_i \leq u_n/2$ and $u_n - a_i \in A$. We set $R_i(n) = \{a_i, u_n - a_i\}$ and $R(n) = \bigcup_{i=1}^{r(u_n)} R_i(n)$. The sets $R(n)$ satisfy conditions (2a)-(2e) of Lemma 2. We shall construct inductively a decreasing sequence of sets $A \supseteq A_1 \supseteq A_2 \supseteq \ldots$ such that $A^* = \bigcap_{n=1}^{\infty} A_n$ is a minimal asymptotic basis for U.

First we construct A_1. Let $a_1^* \in A$. Choose an integer $n(1) > N_0$ such that $u_{n(1)} > 2a_1^*$ and $u_{n(1)} - a_1^* \in A$. By Lemma 2, there exists a set $X(n(1)) \subseteq R(n(1))$ satisfying conditions (2f), (2g), and (2h). Since $X(n(1)) \cap R_i(n(1)) \neq \emptyset$ for $i = 1, 2, \ldots, r(u_{n(1)})$ it follows that $u_{n(1)} \notin 2\ (A\setminus X(n(1)))$. But for N_2 sufficiently large and all $m \geq N_2$, $m \neq n(1)$, there is a $j \leq r(u_m)$ such that $X(n(1)) \cap R_j(m) = \emptyset$, and so $u_m \in 2(A\backslash X(n(1)))$. We set

$$A_1 = (A\backslash X(n(1))) \cup \{a_1^*, u_{n(1)} - a_1^*\}.$$

Then $A_1 \subseteq A$ and $u_{n(1)} \in 2A_1$. Moreover, $u_{n(1)} = a_1^* + (u_{n(1)} - a_1^*)$ is the unique representation of $u_{n(1)}$ as a sum of two elements of A_1. Note that $A\backslash A_1 \subseteq [0, u_{n(1)}]$.

Suppose that we have determined integers $n(1) < n(2) < \ldots < n(k-1)$ and sets $A \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_{k-1}$ with the following properties:

(i) $A\backslash A_i \subseteq [0, u_{n(i)}]$ for $i = 1, 2, \ldots, k-1$;

(ii) $A_i \cap [0, u_{n(i)}] = A_{i+1} \cap [0, u_{n(i)}]$ for $i = 1, 2, \ldots, k-2$;

(iii) $u_m \in 2A_{k-1}$ for all $m \geq N_2$, $m \neq n(1), n(2), \ldots, n(k-1)$;

(iv) For each $i = 1, \ldots, k-1$ there is a number $a_i^* \in A_i$ such that $u_{n(i)} - a_i^* \in A_i$ and $u_{n(i)} = a_i^* + (u_{n(i)} - a_i^*)$ is the unique representation of $u_{n(i)}$ as a sum of two elements of A_i.

Now we construct the number $n(k)$ and the set A_k. Let $r_{k-1}(m)$ denote the number of representations of m as a sum of two elements of A_{k-1}. Property (i) implies that $r_{k-1}(m) \geq r(m) - u_{n(k-1)}$. Choose $w_1 > u_{n(k-1)}$ such that for $u_n \geq w_1$ we have

$$\begin{aligned} r_{k-1}(u_n) &\geq r(u_n) - u_{n(k-1)} \\ &> \frac{1 + 3\delta}{\log(4/3)} \log n - u_{n(k-1)} \\ &> \frac{1 + 2\delta}{\log(4/3)} \log n. \end{aligned}$$

Choose $w_2 > 2w_1$ such that for $u_n \geq w_2$ we have

$$\begin{aligned} r_{k-1}(u_n) - w_1 &> \frac{1 + 2\delta}{\log(4/3)} \log n - w_1 \\ &> \frac{1 + \delta}{\log(4/3)} \log n. \end{aligned}$$

Let $a_k^* \in A_{k-1} \cap [0, u_{n(k-1)}]$. A further constraint on the choice of a_k^* will appear later. Choose $n(k) > n(k - 1)$ such that $u_{n(k)} > w_1 + w_2$ and $u_{n(k)} - a_k^* \in A_{k-1}$. Let X' be the set consisting of all $a \in A_{k-1} \cap [u_{n(k)} - w_1 + 1, u_{n(k)}]$ such that $u_{n(k)} - a \in A_{k-1}$. Observe that $u_{n(k)} - a_k^* \in X'$ and that $|X'| \leq w_1$. Let $r'_{k-1}(m)$ denote the number of representations of m as the sum of two elements of $A_{k-1} \backslash X'$. Then for $u_n > u_{n(k)} - w_1$ we have

$$r'_{k-1}(u_n) \geq r_{k-1}(u_n) - w_1 > \frac{1 + \delta}{\log(4/3)} \log n.$$

For $w_1 \leq u_n \leq u_{n(k)} - w_1$ we have

$$r'_{k-1}(u_n) = r_{k-1}(u_n) > \frac{1 + 2\delta}{\log(4/3)} \log n.$$

Therefore,

$$r'_{k-1}(u_n) > \frac{1 + \delta}{\log(4/3)} \log n$$

for all $u_n \geq w_1$.

Let $u_n \geq w_1$ and let $a_1 < \ldots < a_{r'_{k-1}(u_n)} \leq u_n/2$ be the elements of $A_{k-1} \backslash X'$ such that $a_i \leq u_n/2$ and $u_n - a_i \in A_{k-1} \backslash X'$. Let $R_i(n) = \{a_i, u_n - a_i\}$ and let $R(n) = \bigcup_{i=1}^{r'_{k-1}(u_n)} R_i(n)$. Note that $R(n(k)) \subseteq [w_i, u_{n(k)} - w_1]$. The sets $R(n)$ satisfy conditions (2a)-(2e) of Lemma 2, where $N_0 = N_1$ is the least n such that $u_n \geq w_1$. Therefore, there exists a set $X(n(k)) \subseteq R(n(k))$ satisfying conditions (2f)-(2h). In particular, $u_{n(k)} \notin 2(A_{k-1} \backslash (X(n(k)) \cup X'))$, but $u_n \in 2(A_{k-1} \backslash (X(n(k)) \cup X'))$ for all $n \geq N_2$, $n \neq n(i)$ for $i = 1, 2, \ldots, k$. We set

$$A_k = (A_{k-1} \backslash (X(n(k)) \cup X')) \cup \{u_{n(k)} - a_k^*\}.$$

Then $u_{n(k)} \in 2A_k$, and $u_{n(k)} = a_k^* + (u_{n(k)} - a_k^*)$ is the unique representation of $u_{n(k)}$ as a sum of two elements of A_k.

Thus, we have determined inductively an increasing sequence of integers $n(1) < n(2) < \ldots$ and a decreasing sequence of sets $A_1 \supseteq A_2 \supset \ldots$ and a sequence of integers $\{a_k^*\}_{k=1}^{\infty}$ such that $a_k^* \in A^* = \bigcap_{j=1}^{\infty} A_j$ for all k, and the unique representation of $u_{n(k)}$ as a sum of two elements of A^* is $u_{n(k)} = a_k^* + (u_{n(k)} - a_k^*)$. Moreover, $u_n \in 2A^*$ for all $n \geq N_2$. Thus, A^* is an asymptotic basis of order 2 for U.

Recall that at the k-th step of the construction it was necessary to choose an integer $a_k^* \in A_{k-1} \cap [0, u_{n(k-1)}] = A^* \cap [0, u_{n(k-1)}]$. Now we impose the crucial constraint. Choose each integer $a^* \in A^*$ infinitely often as a number a_k^*. That is, if $a^* \in A^*$, then $a^* = a_k^*$ for infinitely many k. Then there will be infinitely many numbers k such that $a^* + (u_{n(k)} - a^*)$ is the unique representation of $u_{n(k)}$ as a sum of two elements of A^*. This implies that, for any $a^* \in A^*$, there are infinitely many numbers $u_{n(k)}$ such that $u_{n(k)} \notin 2(A^* \setminus \{a^*\})$. Therefore, A^* is a minimal asymptotic basis of order 2 for U. This completes the proof of Theorem 1.

THEOREM 2. Let A be an asymptotic basis of order 2 such that $r(n) > c \log n$ for some constant $c > \log^{-1}(4/3)$ and all $n \geq N_1$. Then A contains a minimal asymptotic basis of order 2.

Proof. Let U be the set of all positive integers. Then the assertion follows immediately from Theorem 1. Note that if $a_i \in A$, then $a_i + a_j \in U$ for all $a_j \in A$.

THEOREM 3. With Lebesgue measure on the probability space of all sequences of positive integers, a random sequence contains a minimal asymptotic basis of order 2 with probability 1.

Proof. By the method of Erdös-Rényi [13, 17], there is a probability measure μ on the space of all strictly increasing sequences of positive integers such that, if $B^{(n)}$ denotes the set of all sequences containing n, then $\mu(B^{(n)}) = 1/2$ for all n. The law of large numbers implies that $r(n) \sim n/8$ for almost all sequences. Since $n/8 > c \log n$ for any $c > \log^{-1}(4/3)$ and all sufficiently large n, the result follows from Theorem 2.

THEOREM 4. The sequence of square-free numbers contains a minimal asymptotic basis of order 2.

Proof. A simple sieve argument [2, 14, 15] shows that there are at least cn representations of n as a sum of two square-free numbers for some $c > 0$ and all $n \geq N_1$. The result follows from Theorem 2.

THEOREM 5. Let A consist of all numbers of the form p or pq, where p and q are odd primes. Then A contains a minimal asymptotic basis of order 2 for the set of positive even integers.

Proof. Chen [1, 16] proved that there are at least $cn/\log^2 n$ representations of 2n as a sum of two elements of A. The result follows from Theorem 1.

Remark. If every sufficiently large even integer is the sum of two primes in at least $cn/\log^2 n$ different ways, that is, if the strong form of Goldbach's conjecture is true, then there is a subset of the primes that is a minimal asymptotic basis of order 2 for the even numbers.

4. Maximal Asymptotic Nonbases

THEOREM 6. Let $A = \{a_i\}$ be an asymptotic basis of order 2 for $U = \{u_n\}$. Let $r(u_n)$ denote the number of representations of u_n in the form $u_n = a_j + a_k$, where $a_j, a_k \in A$ and $a_j \leq a_k$. Suppose that $r(u_n) > c \log n$ for some constant $c > \log^{-1}(4/3)$ and all $n \geq N_1$.

Suppose also that for every $L \geq 1$ there are infinitely many n such that $[u_n - L, u_n] \subseteq A$. Then A contains a maximal asymptotic nonbasis of order 2 for U.

Proof. Repeating the proof of Theorem 1, we can construct inductively an increasing sequence of numbers $\{u_{n(k)}\}_{k=1}^{\infty} \subseteq U$ and a decreasing sequence of sets $A \supseteq A_1 \supseteq A_2 \supseteq \ldots$ such that $A^* = \bigcap_{n=1}^{\infty} A_n$ will be a maximal asymptotic nonbasis of order 2 for U. The essential difference is the following. Suppose that the number $u_{n(k-1)} \in U$ and the set A_{k-1} have been determined. Let $L = u_{n(k-1)}$. Choose $u_{n(k)}$ sufficiently large that $u_{n(k)} > 2u_{n(k-1)}$ and A_{k-1} contains the interval $[u_{n(k)} - u_{n(k-1)}, u_{n(k)}]$. Apply Lemma 2 to destroy all representations of $u_{n(k)}$ as a sum of two elements of A_{k-1}. This produces the set A_k. This set has the property that if $b \notin A_k$ and $b \in [0, u_{n(k-1)}]$, then $u_{n(k)} - b \in A_k$. Let $A^* = \bigcap_{k=1}^{\infty} A_k$. If $u_n \in U$ and $n \geq N_2$, then $u_n \in 2A^*$ if and only if u_n is not one of the numbers $u_{n(k)}$. Thus, A^* is an asymptotic nonbasis of order 2 for U. But if $b \notin A^*$, then $u_{n(k)} - b \in A^*$ for all sufficiently large k. This means that A^* is a maximal asymptotic nonbasis of order 2 for U. This concludes the proof of Theorem 6.

THEOREM 7. Let A be an asymptotic basis of order 2 that contains arbitrarily long intervals. Suppose that $r(n) > c \log n$ for some constant $c > \log^{-1}(4/3)$. Then A contains a maximal asymptotic nonbasis of order 2.

Proof. This follows immediately from Theorem 6 with U equal to the set of all positive integers.

THEOREM 8. Let U be an infinite set of positive integers. With Lebesgue measure on the probability space of all sequences of positive integers, a random sequence contains a maximal asymptotic nonbasis of order 2 for U with probability 1.

Proof. Using the method of Erdös-Rényi [13, 17], we let the number n belong to a random sequence with probability 1/2. The law of large numbers implies that $r(n) \sim n/8$ for almost all sequences, and the Borel-Cantelli lemma implies that for any $L \geq 1$ almost all sequences contain infinitely many intervals of the form $[u_n - L, u_n]$ with $u_n \in U$. The result follows from Theorem 6.

THEOREM 9. Let $A = \{a_i\}$ be an asymptotic basis of order 2 for $U = \{u_n\}$. Suppose that $r(u_n) > c \log n$ for some constant $c > \log^{-1}(4/3)$ and all $n \geq N_1$. Suppose also that for any finite subset $F \subseteq A$ there are infinitely many $u_n \in U$ such that $u_n - a \in A$ for all $a \in F$. Then A contains a subset A* that is an asymptotic nonbasis of order 2 for U maximal with respect to A.

Proof. The proof is similar to that of Theorem 6. We construct inductively an increasing sequence of numbers $\{u_{n(k)}\}_{k=1}^{\infty} \subseteq U$ and a decreasing sequence of sets $A \supseteq A_1 \supseteq A_2 \supseteq \ldots$ such that $A^* = \bigcap_{n=1}^{\infty} A_n$ has the desired properties. Suppose that $u_{n(k-1)} \in U$ and $A_{k-1} \subseteq A$ have been determined. Let $F = A \setminus A_{k-1}$. Then F is a finite subset of A. Choose $u_{n(k)}$ sufficiently large such that $u_{n(k)} - a \in A_{k-1}$ for all $a \in F$. Now construct the set A_k as in the proof of Theorem 6. Then $u_{n(k)} \notin 2A_k$, but $u_{n(k)} - a \in A_k$ for all $a \in A$ such that $a \notin A_{k-1}$ and $a \leq u_{n(k-1)}$. The set $A^* = \bigcap_{n=1}^{\infty} A_n$ is an asymptotic nonbasis of order 2 for U that is maximal with respect to A. This completes the proof of Theorem 9.

THEOREM 10. Let $A = \{a_i\}$ be an asymptotic basis of order 2 such that, for any finite set $F \subseteq A$, there are infinitely many integers n such that $n - a \in A$ for all $a \in F$. Suppose that $r(n) > c \log n$ for some constant $c > \log^{-1}(4/3)$ and all $n \geq N_1$. Then A contains a subset A^* that is an asymptotic nonbasis of order 2 maximal with respect to A.

Proof. This follows at once from Theorem 9.

THEOREM 11. There exists a sequence of square-free integers that is an asymptotic nonbasis of order 2 maximal with respect to the set of all square-free numbers.

Proof. Simple sieve arguments [11] show that the sequence of square-free numbers satisfies both conditions of Theorem 10. The result follows immediately.

THEOREM 12. Let A be a sequence of integers containing a gap of length L. If A contains a maximal asymptotic nonbasis of order 2 for an infinite set U, then A contains infinitely many intervals of length L.

Proof. Let A^* be the subset of A that is a maximal asymptotic nonbasis of order 2 for U. Let $\{u_{n(k)}\}_{k=1}^{\infty}$ be the infinite subset of U such that $u_{n(k)} \notin 2A^*$. Since A has a gap of length L, there is an integer $b \geq 0$ such that $[b, b+L-1] \cap A = \emptyset$. Since A^* is maximal, it follows that for each $i = 0, 1, \ldots, L-1$ there is an integer K_i such that $u_{n(k)} - b - i \in A^*$ for all $k \geq K_i$. Let $K^* = \max\{K_i \mid i = 0, 1, \ldots, L-1\}$. Then $u_{n(k)} - b - i \in A^*$ for all $k \geq K^*$ and $i = 0, 1, \ldots, L-1$. Thus, A^* contains the interval $[u_{n(k)} - b - L + 1, u_{n(k)} - b]$ for all $k \geq K^*$. This concludes the proof of Theorem 12.

THEOREM 13. The sequence of square-free numbers does not contain a maximal asymptotic nonbasis of order 2.

Proof. Every interval of length 4 contains a multiple of 4, hence the sequence of square-free numbers contains no interval of length 4. But the sequence of square-free does contain gaps of length 4. The result follows from Theorem 12.

THEOREM 14. Let A be a sequence of integers containing arbitrarily long gaps. If A contains a maximal asymptotic nonbasis of order 2 for an infinite set U, then A contains arbitrarily long intervals.

Proof. This follows from Theorem 12.

THEOREM 15. Let A be a sequence of integers of lower asymptotic density zero. If A contains a maximal asymptotic nonbasis of order 2 for an infinite set U, then A contains arbitrarily long intervals.

Proof. If A has lower asymptotic density zero, then A contains arbitrarily long gaps. The result follows from Theorem 14.

Remark. It is not necessary that a maximal asymptotic nonbasis of order 2 contain arbitrarily long intervals. For example, the set of all even integers contains no interval of length 2. But it is true that a maximal asymptotic nonbasis of order 2 for an infinite set U must contain arbitrarily long finite arithmetic progressions.

THEOREM 16. Let A be a maximal asymptotic nonbasis of order 2 for the infinite set U. Then A contains arbitrarily long finite arithmetic progressions.

Proof. If the lower asymptotic density of A is zero, the result follows from Theorem 15. If the lower asymptotic density of A is positive, the result follows from Szemerêdi's theorem [25]. This concludes the proof.

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