THE PROPINQUITY OF DIVISORS

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Introduction

Let $\beta > \log 3 - 1$ be fixed. Erdös [1] stated without proof that the sequence of integers n having a divisors d, d' such that

$$d < d' < d\left(1 + (\log n)^{-\beta}\right)$$

has asymptotic density 0. It was also stated that if $\beta < \log 3 - 1$ then this density is 1, but this claim has had to be withdrawn.

In this note we prove a result which is more precise than the former one quoted above, particularly for the small d's (essentially those for which $\log d = o(\log n)$).

THEOREM. *Let $\varepsilon > 0$ be fixed and set*

$$\eta(x) = 3^{-(1+\varepsilon)\sqrt{(2 \log\log x . \log\log\log\log x)}},$$

and

$$\theta(x, d) = \eta(x)(\log d)^{1-\log 3}.$$

Then the number of integers $n < x$ having divisors d, d' such that

$$d < d' < d\left(1 + \theta(x, d)\right)$$

is $o(x)$.

Alternatively, the sequence of integers n having divisors d, d' such that

$$d < d' < d\left(1 + \theta(n, d)\right)$$

has asymptotic density 0.

Remarks. (i) The two forms are equivalent because of the slow decrease of $\eta(n)$. (ii) If we restrict our attention to divisors $d > x^{\delta}$ (or $> n^{\delta}$) for any fixed $\delta > 0$, the factor $\sqrt{\log\log\log\log x}$ may be replaced by any function of x tending to infinity in the definition of $\eta(x)$. This will be made clear in the proof of the theorem. (iii) Since the multiples of $d(d+1)$ have positive density, the theorem becomes false if $\theta(x, d)$ is any function of d alone, unless trivially $\theta(x, d) \leqslant 1/d$. But it is not clear that our $\eta(x)$ is the most slowly decreasing function of x which will do.

The idea of the proof is to consider weighted sums over the integers n, the weight being constructed so that only integers n with divisors d, d' close together have positive weight, but in such a way that the sum is not dominated by n's and d's in which we are not interested; these are integers with either an abnormal number, or abnormal distribution, of prime factors.

Proof of the theorem. Let n have divisors d and d' such that $d < d' < d\left(1 + \theta(x, d)\right)$, and suppose $(d, d') = t$. Since θ is a decreasing function of d, we have that $(d/t) < (d'/t) < (d/t)\left(1 + \theta(x, d/t)\right)$, so that n has a pair of such divisors which are relatively prime. We therefore assume in what follows that $(d, d') = 1$: actually all we need is that $dd'|n$.

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We get a better result if we restrict to $d > x^\delta$ and we deal with this case first. Let

$$S = \sum_{n<x} y^{\Omega(n)} \sum_{\substack{dd'|n \\ d>x^\delta}}^{\theta} 1$$

where $\sum^\theta$ denotes summation restricted by the condition $d < d' < d(1+\theta)$, $\Omega(n)$ denotes the total number of prime factors of n (i.e. counted according to multiplicity) and $0 < y \leqslant 1$. We have

$$S = \sum_{x^\delta < d < x^{1/2}} y^{\Omega(d)} \sum_{d<d'<d(1+\theta)} y^{\Omega(d')} \sum_{m<x/dd'} y^{\Omega(m)}$$

and to estimate the inner sum we use the formula, (valid for fixed $y \in (0,2)$),

$$\sum_{m<x} y^{\Omega(m)} = C(y)x(\log x)^{y-1}\left(1+O(1/\log x)\right),$$

where

$$C(y) = \frac{1}{\Gamma(y)} \prod_p \left(1-\frac{1}{p}\right)^y \left(1-\frac{y}{p}\right)^{-1}.$$

This formula is well-known and is proved by contour integration. It holds for all $x > 0$ if we interpret $\log x$ as 1 say for $x \leqslant e$: and we make this convention throughout the paper. Hence

$$S \ll x \sum_d \frac{y^{\Omega(d)}}{d^2}\left(\log \frac{x}{d^2}\right)^{y-1} \sum_{d<d'<d(1+\theta)} y^{\Omega(d')},$$

and we use the formula above again. Since $\theta(x,d) > 1/\log x$ there is no difficulty with the error term and

$$S \ll x \sum_d \theta(x,d) \frac{y^{\Omega(d)}}{d}\left(\log \frac{x}{d^2}\right)^{y-1} (\log d)^{y-1}$$

$$\ll_\delta x\eta(x)(\log x)^{y-\log 3} \sum_{x^\delta<d<x^{1/2}} y^{\Omega(d)}\left(\log \frac{x}{d^2}\right)^{y-1}$$

since for $d > x^\delta$ we have $\theta(x,d) \ll_\delta \theta(x,x)$. Now

$$\frac{\sqrt{x}}{d}\left(\log \frac{x}{d^2}\right)^{y-1} \ll \sum_{m<x^{1/2}/d} (\log m)^{y-1}$$

so that

$$S \ll_\delta \sqrt{x}\eta(x)(\log x)^{y-\log 3} \sum_{m<x^{1/2-\delta}} (\log m)^{y-1} \sum_{d<x^{1/2}/m} y^{\Omega(d)}$$

$$\ll_\delta x\eta(x)(\log x)^{2y-1-\log 3} \sum_{m<x^{1/2-\delta}} m^{-1}(\log m)^{y-1}$$

$$\ll_\delta x\eta(x)(\log x)^{3y-1-\log 3}.$$

We choose $y = 1/3$, and deduce that

$$\sum_{n<x} 3^{\log\log x - \Omega(n)} \sum_{\substack{dd'|n \\ d<x^\delta}}^{\theta} 1 \ll x\eta(x).$$

Let $\psi(x) \to \infty$ arbitrarily slowly as $x \to \infty$. It is a well-known result of Hardy and

Ramanujan [5] that for all but $o(x)$ integers $n < x$, we have

$$|\Omega(n) - \log\log x| < \psi(x)\sqrt{\log\log x}.$$

We restrict the sum above to integers n for which this inequality is satisfied, and deduce that the number of integers n with such a divisor dd' is

$$\ll_\delta x\{o(1) + \eta(x)3^{\psi(x)\sqrt{\log\log x}}\}.$$

By definition of $\eta(x)$, this is $o(x)$ as required. We notice that for these "large" d's the factor $\sqrt{\log\log\log\log x}$ in $\eta(x)$ is not needed.

The proof is a little more difficult when we drop the condition $d > x^\delta$. Consider the sum

$$T = \sum_{n<x}\sum_{dd'|n}^{\theta} z^{\Omega(n,d)}(\log d)^{\log 3},$$

where $\Omega(n, d)$ denotes the total number of prime factors of n which do not exceed d, and $0 < z < \frac{1}{2}$. Notice that the condition $d < d' < d(1+\theta(x, d))$ implies that $\theta(x, d) > 1/d$, that is

$$d(\log d)^{\log 3 - 1} \geqslant 1/\eta(x), \quad \text{or} \quad d \geqslant d_0(x), \text{ say.}$$

Then

$$T = \sum_{d \leqslant x^{1/2}} \sum_{d<d'<d(1+\theta)} z^{\Omega(dd', d)}(\log d)^{\log 3} \sum_{m<x/dd'} z^{\Omega(m,d)}.$$

A result of Hall [4] states that if $f : Z^+ \to [0, 1]$ is multiplicative then

$$\sum_{n<x} f(n) \leqslant e^\gamma x \left(1 + O\left(\frac{\log\log x}{\log x}\right)\right) \prod_{p<x}\left(1 - \frac{1}{p}\right)\left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right),$$

and we deduce that $$\sum_{m<x/dd'} z^{\Omega(m,d)} \ll \frac{x}{d^2}(\log d)^{z-1}.$$

Substituting in the above, we obtain

$$T \ll \sum_{d\leqslant x^{1/2}} \frac{z^{\Omega(d)}}{d^2}(\log d)^{z-1+\log 3} \sum_{d<d'<d(1+\theta)} z^{\Omega(d',d)}.$$

Provided $\theta(x, d) < 1$, as we may assume, we have $\Omega(d', d) \geqslant \Omega(d') - 1$. Thus as before

$$\begin{aligned} T &\ll xz^{-1}\eta(x)\sum_{d<x}\frac{z^{\Omega(d)}}{d}(\log d)^{2z-1} \\ &\ll xz^{-1}\eta(x)\sum_{d<x}\frac{z^{\Omega(d)}}{d}\left\{(\log x)^{2z-1} + (1-2z)\int_d^x \frac{dt}{t(\log t)^{2-2z}}\right\} \\ &\ll xz^{-1}\eta(x)\left\{(\log x)^{3z-1} + (1-2z)\int_{d_0}^x \frac{dt}{t(\log t)^{2-3z}}\right\}. \end{aligned}$$

We choose $z = 1/3$, and deduce that

$$\sum_{n<x}\sum_{dd'|n}^{\theta} 3^{\log\log d - \Omega(n,d)} \ll x\eta(x)\log\log x.$$

We need the following lemma, which is an application of Theorem VI of Erdös [2]:

LEMMA. *Let $\lambda > 0$ be fixed, and let $N(x, \xi)$ denote the number of integers $n < x$ such that for some d, $\xi \leqslant d \leqslant n$, we have*

$$|\Omega(n, d) - \log\log d| > (1+\lambda)\sqrt{(2\log\log d . \log\log\log\log d)}.$$

Then

$$\lim_{\xi\to\infty} \limsup_{x\to\infty} x^{-1} N(x, \xi) = 0.$$

Since $N(x, \xi)$ is a decreasing function of ξ, if we substitute $\xi = \xi(x)$ where $\xi(x) \to \infty$ as $x \to \infty$, then

$$\lim_{x\to\infty} x^{-1} N(x, \xi(x)) = 0$$

and we apply this result with $\xi(x) = d_0(x)$. We restrict the sum above to integers n not counted by $N(x, d_0(x))$, with $\lambda = \varepsilon/2$, and deduce that the number of integers $n < x$ with a divisor dd' of the required type is

$$\ll N(x, d_0(x)) + x\eta(x)\log\log x . 3^{(1+\lambda)\sqrt{(2\log\log x . \log\log\log\log x)}}.$$

By the definition of $\eta(x)$ and the relation $\lambda = \varepsilon/2$, this is $o(x)$ as required.

References

1. P. Erdös, "On some applications of probability to analysis and number theory", *J. London Math. Soc.*, 39 (1964), 692–696.
2. P. Erdös, "On the distribution function of additive functions", *Ann. Math.*, 47 (1946), 1–20.
3. R. R. Hall, "The propinquity of divisors", *J. London Math. Soc.*, (2) 19 (1979), 35–40.
4. R. R. Hall, "Halving an estimate obtained from Selberg's upper bound method", *Acta Arithmetica*, 25 (1974), 347–351.
5. G. H. Hardy and S. Ramanujan, "The normal number of prime factors of a number n", *Quarterly J. of Math.*, 48 (1917), 76–92.

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