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TRANSVERSALS AND MULTITRANSVERSALS

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1. *Introduction*

A *transversal* of a family $\mathscr{F}$ of sets is a family of pairwise distinct elements, one from each member of $\mathscr{F}$, and a *multitransversal* of $\mathscr{F}$ is a family of pairwise disjoint subsets, one of each member of $\mathscr{F}$. The main result of this note, Theorem 4, gives necessary and sufficient conditions on families $\mathscr{A}$ and $\mathscr{B}$ of cardinals in order that every family $\mathscr{F}$ whose members have cardinals given by $\mathscr{A}$ should have (i) a transversal, (ii) a multitransversal whose members have cardinals given by $\mathscr{B}$. Our conditions turn out to involve the notion of a *weakly inaccessible* cardinal and that of a *stationary* set of ordinals. Our result (announced in [1]) amounts to saying that the test families $\mathscr{F}$, whose "good behaviour" implies that of every other family with the same cardinalities, are those whose members are sets of the form $\{x : x < \lambda\}$, where λ is an ordinal.

2. *Terminology and notation*

Capital letters denote sets. The relation $A \subset B$ denotes inclusion in the wide sense. If nothing is said to the contrary, small letters denote ordinals. For each α we put $\bar{\alpha} = \{x : x < \alpha\}$. For cardinals c put

$$\omega(c) = \min\{\alpha : |\alpha| = c\};$$

$$\bar{c} = \overline{\omega(c)}; \quad \bar{\bar{c}} = \{t = \text{cardinal} : t < c\}.$$

For every set S of cardinals put

$$\omega(S) = \{\omega(c) : c \in S\}.$$

For cardinals γ put

$$[A]^\gamma = \{X \subset A : |X| = \gamma\}.$$

The symbol $(a_0, \ldots, \hat{a}_n)$, where the a_ν are any objects, denotes the sequence $(a_\nu : \nu < n)$. Given a family $(a_i : i \in I)$ of cardinals and a family $(A_i : i \in I)$ of sets we put, for $J \subset I$,

$$a_J = \sum (j \in J) a_j ; A_J = \bigcup (j \in J) A_j .$$

Symbols such as $(a_0, \ldots, \hat{a}_n)_<$ or $(x_i : i \in I)_{\neq}$ are self-explanatory. For infinite cardinals x the symbol cf x, the *cofinality* of x, denotes the least cardinal t such that, for some

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cardinals $x_\tau < x$, we have $x = x_\tau$. The cardinal x is *regular*, if $\mathrm{cf}\, x = x$, and *singular*, if $\mathrm{cf}\, x < x$. For every cardinal x put

$$x^+ = \min\{y = \text{cardinal} : y > x\},$$

$$x^- = \min\{y = \text{cardinal} : y^+ \geqslant x\},$$

and similarly for ordinals. The infinite cardinal x is *weakly inaccessible* if $x = x^- = \mathrm{cf}\, x$.

Let λ be a regular cardinal and let $A \subset \bar{\lambda}$. A *regressive* function on A is a function $f: A - \{0\} \to \bar{\lambda}$ such that $f(x) < x$ for $0 < x \in A$. The set A is *stationary* on $\bar{\lambda}$ if $A \subset \bar{\lambda}$ and for every regressive function f on A there is $y \in \bar{\lambda}$ with $|f^{-1}(y)| = \lambda$. Let stat λ denote the set of all sets which are stationary on $\bar{\lambda}$.

The disjoint subset relation

$$(a_i : i \in I) \to (b_i : i \in I)_{ds} \tag{1}$$

means that the a_i and b_i are cardinals with the property that whenever $|A_i| = a_i$ for $i \in I$, there always exist pairwise disjoint sets $X_i \in [A_i]^{b_i}$ for $i \in I$. Thus if all $b_i = 1$ then (1) means that every family $(A_i : i \in I)$ with $|A_i| = a_i$ for $i \in I$ has a transversal. Families $(X_i : i \in I)$ as described above are called *multitransversals* of $(A_i : i \in I)$ of *size* $(b_i : i \in I)$.

If $\mathscr{F}_0, \ldots, \mathscr{F}_n$ are sequences, then $[\mathscr{F}_0, \ldots, \mathscr{F}_n]$ denotes the sequence obtained by *concatenation*, i.e., by arranging the terms of the $\mathscr{F}_\nu$ as a single sequence, maintaining in each $\mathscr{F}_\nu$ the given order and placing $\mathscr{F}_\mu$ in front of $\mathscr{F}_\nu$ if $\mu < \nu < n$. If x is an object and c a cardinal then $(x)_c$ denotes the sequence $(x_\nu : \nu \in \bar{c})$ in which $x_\nu = x$ for $\nu \in \bar{c}$.

Let S be a set of infinite cardinals. An *S-sequence* is a sequence $(a_\nu : \nu < n)$ such that $\{a_\nu : \nu < n\} = S$ and, if $\nu_0 < n$, then $a_{\nu_0} > |\nu_0|$ and $|\{\nu < n : a_\nu = a_{\nu_0}\}| = a_{\nu_0}$.

3. *Results*

THEOREM 1. *Let S be a set of infinite cardinals. Then the conditions* (2), (3), (4), (5) *are equivalent, where*

(2) *for every weakly inaccessible cardinal λ, $\omega(S) \cap \bar{\lambda} \notin$ stat λ,*

(3) *there exists an S-sequence,*

(4) *every family of sets consisting, for each $\kappa \in S$, of κ members of cardinal κ, has a transversal,*

(5) *the family $(\bar{\kappa} : \kappa \in S)$ has a transversal.*

THEOREM 2. *Let I be a set; $a_i \geqslant \aleph_0$ for $i \in I$; $S = \{a_i : i \in I\}$. Then the conditions* (6), (7), (8), (9), (10) *are equivalent, where*

(6) $(a_i : i \in I) \to (a_i : i \in I)_{ds}$,

(7) $(a_i : i \in I) \to (1 : i \in I)_{ds}$,

(8) $(\bar{a}_i : i \in I)$ *has a transversal,*

(9) $(\bar{\kappa} : \kappa \in S)$ *has a transversal and* $|\{i \in I : a_i = \kappa\}| \leqslant \kappa$ *for every cardinal* κ,

(10) $|\{i \in I : a_i = \kappa\}| \leqslant \kappa$ *for every cardinal* κ, *and* $w(S) \cap \bar{\lambda} \notin \text{stat}\, \lambda$ *for every weakly inaccessible cardinal* λ.

Remark. The implication (7) ⇒ (6) seems to be interesting. Perhaps it can be proved directly.

COROLLARY. *Let I be a set and let a_i, b_i be cardinals for $i \in I$, where the a_i are infinite. Then* (11) *and* (12) *are equivalent, where*

(11) $(a_i : i \in I) \to (b_i : i \in I)_{ds}$,

(12) $(\bar{a}_i : i \in I)$ *has a multitransversal of size* $(b_i : i \in I)$.

THEOREM 3. *Let I be a set and let a_i, b_i be cardinals for $i \in I$ such that $a_i \leqslant \aleph_0$ for $i \in I$. Then* (13) ⇔ (14) ⇔ (15) ∧ (16), *where*

(13) $(a_i : i \in I)) \to (b_i : i \in I)_{ds}$,

(14) $(\bar{a}_i : i \in I)$ *has a multitransversal of size* $(b_i : i \in I)$,

(15) *if* $n \leqslant \aleph_0$, *then* $\sum (i \in I\,; a_i \leqslant n) b_i \leqslant n$,

(16) *if* $m < \omega$ *and* $m \leqslant \sum (i \in I\,; a_i = \aleph_0) b_i$, *then there is* $n_0 < \omega$ *such that, whenever* $n_0 \leqslant n < \omega$, *we have* $m + \sum (i \in I\,; a_i \leqslant n) b_i \leqslant n$.

Our main result is the following theorem.

THEOREM 4. *Let I be a set and a_i, b_i be arbitrary cardinals for $i \in I$. Put*

$$S = \{a_i : i \in I\,; b_i \geqslant 1\}.$$

Then (17) ⇔ (18) ⇔ (19) ∧ (20) ∧ (21), *where*

(17) $(a_i : i \in I) \to (b_i : i \in I)_{ds}$,

(18) $(\bar{a}_i : i \in I)$ *has a multitransversal of size* $(b_i : i \in I)$,

(19) $\sum (i \in I\,; a_i \leqslant \kappa) b_i \leqslant \kappa$ *for every cardinal* κ,

(20) $\omega(S) \cap \bar{\lambda} \notin \text{stat}\, \lambda$ *for every weakly inaccessible cardinal* λ,

(21) *if* $m < \omega$ *and* $m \leqslant \sum (i \in I\,; a_i = \aleph_0) b_i$, *then* $m + \sum (i \in I\,; a_i \leqslant n) b_i \leqslant n$ *for every sufficiently large finite* n.

4. *Proof of Theorem* 1

Proof of (3) ⇒ (4). Let $(\kappa_0, \ldots, \hat{\kappa}_n)$ be an S-sequence. Then every family $\mathscr{A}$ as described in (4) can be written in the form $(A_0, \ldots, \hat{A}_n)$, where $|A_\nu| = \kappa_\nu$ for $\nu < n$. Since $|A_\nu| = \kappa_\nu > |\nu|$, we can choose elements x_ν for $\nu < n$ so that $x_\nu \in A_\nu - \{x_0, \ldots, \hat{x}_\nu\}$ for $\nu < n$. Then $(x_0, \ldots, \hat{x}_n)$ is a transversal of $\mathscr{A}$.

Proof of (4) ⇒ (5). This is trivial.

Proof of (5) $\Rightarrow$ (2). Let $(x_\kappa : \kappa \in S)$ be a transversal of $(\bar{\kappa} : \kappa \in S)$. Then the function $\omega(\kappa) \mapsto x_\kappa$ is regressive on $\omega(S)$ and injective. Hence, clearly, (2) is satisfied. There only remains:

Proof of (2) $\Rightarrow$ (3). Let us call a set S *good* if S is a set of infinite cardinals satisfying (2). For $\lambda \geqslant \aleph_0$ let $P(\lambda)$ denote the statement: whenever S is good and $S \subset \bar{\lambda}$, then (3) holds. We have to show that $P(\lambda)$ holds for every $\lambda \geqslant \aleph_0$. We use induction over λ. We know that $P(\aleph_0)$ is true. Let $\lambda > \aleph_0$ and assume that $P(\lambda')$ holds for $\aleph_0 \leqslant \lambda' < \lambda$. We have to prove $P(\lambda)$. Let S be good and $S \subset \bar{\lambda}$. We have to construct an S-sequence.

Case 1: $\lambda > \lambda^-$. Put $\delta = \lambda^-$. We may assume that $S \not\subset \bar{\delta}$ so that $S = T \cup \{\delta\}$, where $T \subset \bar{\delta}$. By $P(\delta)$ there is a T-sequence $(\kappa_0, \ldots, \kappa_\tau)$. Then $|\tau| = \sum(\kappa \in T)\kappa \leqslant \delta$.

Case 1a: $|\tau| < \delta$. Put $\kappa'_\alpha = \kappa_\alpha$ for $\alpha < \tau$, and $\kappa'_\alpha = \delta$ for $\tau \leqslant \alpha \in \bar{\delta}$. Then $(\kappa'_\alpha : \alpha \in \bar{\delta})$ is an S-sequence.

Case 1b: $|\tau| = \delta$. For $\kappa \in T$ put

$$M_\kappa = \{\alpha < \tau : \kappa_\alpha = \kappa\}.$$

Then $\alpha \in M_\kappa$ implies that $\kappa = \kappa_\alpha > |\alpha|$, that is, $\alpha \in \bar{\kappa}$. Also, $|M_\kappa| = \kappa$ for $\kappa \in T$, and we can write $M_\kappa = P_\kappa \cup Q_\kappa$, where $P_\kappa \cap Q_\kappa = \varnothing$ and $|P_\kappa| = |Q_\kappa| = \kappa$. Put $\kappa'_\alpha = \kappa_\alpha$ for $\alpha \in P_T$ and $\kappa'_\alpha = \delta$ for $\alpha \in Q_T$. Then $(\kappa'_\alpha : \alpha < \tau)$ is an S-sequence.

Case 2: λ is weakly inaccessible. Then, since S is good, we have $\omega(S) \notin \operatorname{stat} \lambda$ and, by well known properties of inaccessible cardinals and stationary sets, there is a set $C = \{\delta_0, \ldots, \delta_{\omega(\lambda)}\}_<$ of infinite cardinals such that $\omega(C)$ is closed and cofinal in $\bar{\lambda}$ and $C \cap S = \varnothing$. (Here closure refers to the usual order topology.) For $\alpha \in \bar{\lambda}$ put $S_\alpha = S \cap \bar{\delta}_\alpha$ and $S'_\alpha = S_\alpha - S_{\bar{\alpha}}$. Then $S = S'_{\bar{\lambda}}; S'_\alpha \cap S'_\beta = \varnothing$ for $\alpha < \beta \in \bar{\lambda}; S'_\alpha \subset S_\alpha \subset \bar{\delta}_\alpha$, and $P(\delta_\alpha)$ holds for $\alpha \in \bar{\lambda}$. Hence there is an S'_α-sequence Λ_α. Put

$$\Lambda = [\Lambda_\alpha : \alpha \in \bar{\lambda}].$$

We claim that

(22) Λ is an S-sequence.

Proof of (22). Let $\kappa \in S$. Then $\kappa \in S'_{\alpha_0}$ for some $\alpha_0 \in \bar{\lambda}$, and exactly κ terms of Λ are equal to κ. All these terms belong to Λ_{α_0}. We have to show that every occurrence of κ in Λ has an index in Λ which belongs to $\bar{\kappa}$. Now every occurrence of κ in Λ_{α_0} has an index in Λ_{α_0} which belongs to $\bar{\kappa}$. Hence it suffices to show that the sequence $[\Lambda_\alpha : \alpha < \alpha_0]$ has fewer than κ terms. This holds if $\alpha_0 = 0$. Now let $\alpha_0 \geqslant 1$. If $\alpha_0^- = \alpha_0$ then $\kappa \in S'_{\alpha_0} = \varnothing$ which is false. Hence $\alpha_0 = \alpha_1 + 1$ for some α_1. By definition of S'_{α_0} we have $\delta_{\alpha_1} \leqslant \kappa < \delta_{\alpha_1+1}$. Since $\delta_{\alpha_1} \in C, \kappa \in S, C \cap S = \varnothing$, we have $\delta_{\alpha_1} < \kappa$. Hence

$$\begin{aligned} &(\text{number of terms of } [\Lambda_\alpha : \alpha < \alpha_0]) \\ &= (\text{number of terms of } [\Lambda_\alpha : \alpha \leqslant \alpha_1]) \\ &= \sum(\kappa' \in S \cap \bar{\delta}_{\alpha_1})\kappa' \leqslant \delta_{\alpha_1} < \kappa . \end{aligned}$$

Case 3: $\lambda > \mathrm{cf}\lambda$. Put $\mathrm{cf}\,\lambda = \tau$. Then there is a set $D = \{\delta_0, \ldots, \hat{\delta}_{\omega(\tau)}\}_< \subset \bar{\lambda} - \bar{\tau}^+$ such that $\omega(D)$ is closed and cofinal in $\bar{\lambda}$. Put

$$A = \{\alpha \in \bar{\tau} : \delta_\alpha \in S\,;\, \sup \bar{\delta}_\alpha \cap S = \delta_\alpha\}\,;$$

$$\Delta = \{\delta_\alpha : \alpha \in A\}\,;\, S' = S - \Delta\,.$$

For $\alpha \in \bar{\tau}$ put $S_\alpha = S' \cap \bar{\delta}_\alpha$ and $S'_\alpha = S_\alpha - S_{\dot{\alpha}}$. Then $S' = S'_{\bar{\tau}}$ and $S'_\alpha \cap S'_\beta = \varnothing$ for $\alpha < \beta \in \bar{\tau}$. The set S is good and $S'_\alpha \subset S$. Hence S'_α is good. Since $S'_\alpha \subset \delta_\alpha$ and $P(\delta_\alpha)$ holds, it follows that there exists an S'_α-sequence Λ_α, for every $\alpha \in \bar{\tau}$. Put $\Lambda = [\Lambda_\alpha : \alpha \in \bar{\tau}]$. We claim that

(23) Λ is an S'-sequence.

Proof of (23). Let $\kappa \in S'$. Then there is exactly one $\alpha_0 \in \bar{\tau}$ with $\kappa \in S'_{\alpha_0}$, and exactly κ terms of Λ equal κ. All these terms are terms of Λ_{α_0}, and their indices in Λ_{α_0} lie in $\bar{\kappa}$. Hence it suffices to show that the sequence $[\Lambda_\alpha : \alpha < \alpha_0]$ has fewer than κ terms. This holds for $\alpha_0 = 0$. Now let $\alpha_0 \geqslant 1$. If $\alpha_0 = \alpha_0^-$, then $\kappa \in S'_{\alpha_0} = \varnothing$ which is false. Hence $\alpha_0 = \alpha_1 + 1$ for some α_1, and $\delta_{\alpha_1} \leqslant \kappa < \delta_{\alpha_1+1}$. If $\delta_{\alpha_1} < \kappa$, then

$$\text{(number of terms of } [\Lambda_\alpha : \alpha < \alpha_0])$$

$$= \text{(number of terms of } [\Lambda_\alpha : \alpha \leqslant \alpha_1])$$

$$= \sum (\kappa' \in S' \cap \bar{\delta}_{\alpha_1})\kappa' \leqslant \delta_{\alpha_1} < \kappa$$

as required. On the other hand, let $\delta_{\alpha_1} = \kappa$. Then

$$\delta_{\alpha_1} = \kappa \in S' = S - \Delta;\quad \delta_{\alpha_1} \notin \Delta;\quad \alpha_1 \notin A;\quad \delta_{\alpha_1} = \kappa \in S' \subset S\,.$$

Since $\alpha_1 \notin A$, we have

$$\sum (\kappa' \in S \cap \bar{\delta}_{\alpha_1})\kappa' < \delta_{\alpha_1}\,.$$

Hence

$$\text{(number of terms of } [\Lambda_\alpha : \alpha < \alpha_0])$$

$$= \text{(number of terms of } [\Lambda_\alpha : \alpha \leqslant \alpha_1])$$

$$= \sum (\kappa' \in S' \cap \bar{\delta}_{\alpha_1})\kappa' \leqslant \sum (\kappa' \in S \cap \bar{\delta}_{\alpha_1})\kappa' < \delta_{\alpha_1} = \kappa$$

as required. This proves (23).

Let $\Lambda = (\kappa_0, \ldots, \hat{\kappa}_\sigma)$. For $\kappa \in S'$ put $M_\kappa = \{\mu < \sigma : \kappa_\mu = \kappa\}$. If $\mu \in M_\kappa$, then $\kappa = \kappa_\mu > |\mu|$. Hence $M_\kappa \subset \bar{\kappa}$. Also, $|M_\kappa| = \kappa$ for $\kappa \in S'$. If $\tau \leqslant \kappa \in S'$, then there is a representation $M_\kappa = \bigcup (\alpha \leqslant \omega(\tau)) M_\kappa^\alpha$ such that $|M_\kappa^\alpha| = \kappa$ for $\alpha \leqslant \omega(\tau)$ and $M_\kappa^\alpha \cap M_\kappa^\beta = \varnothing$ for $\alpha < \beta \leqslant \omega(\tau)$.

Let $\mu < \sigma$. We now define κ'_μ. If $\tau \leqslant \kappa_\mu$, then there is a unique $\alpha(\mu) \leqslant \omega(\tau)$ with $\mu \in M_{\kappa_\mu}^{\alpha(\mu)}$. If, in addition, $\alpha(\mu) < \omega(\tau)$ and $|\mu| < \delta_{\alpha(\mu)} \in S$, then we put $\kappa'_\mu = \delta_{\alpha(\mu)}$. For all other $\mu < \sigma$ we put $\kappa'_\mu = \kappa_\mu$. We claim that

(24) $$(\kappa'_\mu ; \mu < \sigma) \text{ is an } S\text{-sequence.}$$

Proof of (24). We have $\kappa'_\mu \in S$ for $\mu < \sigma$. Let $\mu < \sigma$. Then $\kappa'_\mu > |\mu|$. For if $\kappa'_\mu = \kappa_\mu$, then $\kappa'_\mu = \kappa_\mu > |\mu|$ since $(\kappa_0, \ldots, \hat{\kappa}_\sigma)$ is an S'-sequence, and if $\kappa'_\mu \neq \kappa_\mu$, then $\kappa'_\mu = \delta_{\alpha(\mu)} > |\mu|$. To complete the proof of Theorem 1 it suffices to show that, for $\kappa \in S$, we have

(25) $$|\{\mu < \sigma : \kappa'_\mu = \kappa\}| = \kappa .$$

Case 3a: $\tau \leqslant \kappa \in S'$. Then $\kappa'_\mu = \kappa$ for all $\mu \in M_\kappa^{\omega(\tau)}$, and (25) follows.

Case 3b: $\kappa < \tau$ and $\kappa \in S'$. Then $\mu \in M_\kappa$ implies that $\kappa_\mu = \kappa < \tau$ and hence $\kappa'_\mu = \kappa_\mu = \kappa$, so that $M_\kappa \subset \{\mu < \sigma : \kappa'_\mu = \kappa\}$. Since $|M_\kappa| = \kappa$, we conclude that (25) holds.

Case 3c: $\kappa \in S - S'$. Then $\kappa \in \Delta$. and $\kappa = \delta_\alpha$ for some $\alpha \in A$. Put $T = \{\kappa' \in S' : \tau \leqslant \kappa' < \delta_\alpha\}$. We claim that

(26) $$M_T^\alpha \subset \{\mu < \sigma : \kappa'_\mu = \kappa\} .$$

Proof of (26). Let $\kappa' \in S'; \tau \leqslant \kappa' < \delta_\alpha; \mu \in M_{\kappa'}^\alpha$. Then $\kappa_\mu = \kappa'$, so that $\tau \leqslant \kappa_\mu$ and $\alpha(\mu) = \alpha < \omega(\tau)$. Also, $|\mu| < \kappa_\mu = \kappa' < \delta_\alpha$, and we have $|\mu| < \delta_{\alpha(\mu)} \in S$. Hence $\kappa'_\mu = \delta_{\alpha(\mu)} = \delta_\alpha = \kappa$. This proves (26). Now, to complete the argument in Case 3c, it suffices to show that

(27) $$|M_T^\alpha| = \kappa .$$

Proof of (27). Let $\kappa'' < \delta_\alpha$. Denote by κ''' the least cardinal in S satisfying $\max\{\kappa'', \tau\} < \kappa''' < \delta_\alpha$. This cardinal κ''' exists in view of

$$\sup \bar{\delta}_\alpha \cap S = \delta_\alpha .$$

Then $\sup \bar{\kappa}''' \cap S \leqslant \max\{\kappa'', \tau\} < \kappa'''$. If $\kappa''' \notin S'$ then $\kappa''' \in \Delta; \kappa''' = \delta_\beta$ for some $\beta \in A$; $\sup \bar{\delta}_\beta \cap S = \delta_\beta; \sup \bar{\kappa}''' \cap S = \kappa'''$ which is a contradiction. Hence $\kappa''' \in S'$. Now

$$|M_T^\alpha| = \sum (y \in S'; \tau \leqslant y < \delta_\alpha) y \geqslant \kappa''' > \kappa''.$$

Since κ'' is an arbitrary cardinal with $\kappa'' < \delta_\alpha$, we conclude that $|M_T^\alpha| \geqslant \delta_\alpha = \kappa$. This, together with the previously proved relation $M_\kappa \subset \bar{\kappa}$, establishes (27) and so completes the proof of Theorem 1.

5. *Proof of Theorem* 2

The implications (6) $\Rightarrow$ (7) $\Rightarrow$ (8) are trivial, and the implication (9) $\Rightarrow$ 10) follows from Theorem 1.

Proof of (8) ⇒ (9). Let $(x_i : i \in I)$ be a transversal of $(\bar{a}_i : i \in I)$. For each $\kappa \in S$ choose $i_\kappa \in I$ with $a_{i_\kappa} = \kappa$. Then $(x_{i_\kappa} : \kappa \in S)$ is a transversal of $(\bar{\kappa} : \kappa \in S)$. For every cardinal κ, we have $\{x_i : i \in I; a_i = \kappa\} \subset \bar{\kappa}$ and therefore

$$|\{i \in I : a_i = \kappa\}| = |\{x_i : i \in I; a_i = \kappa\}| \leqslant \kappa .$$

This proves (9).

Proof of (10) ⇒ (6). Let $|A_i| = a_i$ for $i \in I$. It suffices to show that the sequence $\mathscr{F} = [(A_i)_{a_i} : i \in I]$ has a transversal. Given any $\kappa \in S$, the family $\mathscr{F}$ contains at most $\kappa^2 (= \kappa)$ sets of cardinal κ. Since $\omega(S) \cap \bar{\lambda} \notin \operatorname{stat} \lambda$ for every weakly inaccessible cardinal λ, Theorem 1 shows that $\mathscr{F}$ has a transversal.

Proof of the Corollary. Clearly (11) ⇒ (12).

Proof of (12) ⇒ (11). We have $b_i \leqslant a_i$ for $i \in I$. Put $I^+ = \{i \in I : b_i \geqslant 1\}$. Then $(\bar{a}_i : i \in I^+)$ has a transversal. By Theorem 2, $(a_i : i \in I^+) \to (a_i : i \in I^+)_{ds}$. Since $b_i \leqslant a_i$, it follows that $(a_i : i \in I^+) \to (b_i : i \in I^+)_{ds}$. Since $b_i = 0$ for $i \in I - I^+$, (11) follows.

6. *Proof of Theorem* 3

The implications (13) ⇒ (14) ⇒ (15) are trivial.

Proof of (14) ⇒ (16). Let $(X_i : i \in I)$ be a multitransversal of $(\bar{a}_i : i \in I)$ of size $(b_i : i \in I)$. Let m satisfy $m < \omega$ and $m \leqslant \sum (i \in I; a_i = \aleph_0) b_i$. Then

$$|\bigcup (i \in I; a_i = \aleph_0) X_i| = \sum (i \in I; a_i = \aleph_0) b_i \geqslant m ,$$

and we can find a set M with

$$M \in [\bigcup (i \in I; a_i = \aleph_0) X_i]^m .$$

Then $M \subset \bar{\omega}$; $|M| = m < \omega$, and there is $n_0 < \omega$ with $M \subset \bar{n}_0$. Let $n_0 \leqslant n < \omega$. Then $M \subset \bar{n}_0 \subset \bar{n}$. Also, $\bigcup (i \in I; a_i \leqslant n) X_i \subset \bar{n}$. Therefore

$$m + \sum (i \in I; a_i \leqslant n) b_i = |M \cup \bigcup (i \in I; a_i \leqslant n) X_i| \leqslant n .$$

Proof of (15) ∧ (16) ⇒ (13). Let $|A_i| = a_i$ for $i \in I$. For $n \leqslant \aleph_0$ put

$$I_n = \{i \in I ; a_i = n\}.$$

Let $p = b_{I_{\aleph_0}}$. Then $p \leqslant b_I \leqslant \aleph_0$ by (15). Put $P = \{r < \omega : 1 \leqslant r \leqslant p\}$. Then $|P| = p$. There is a mapping $f : P \to I_{\aleph_0}$ such that, for every $i \in I_{\aleph_0}$, $|\{r \in P : f(r) = i\}| = b_i$. This follows from the definition of p. For $n < \omega$ put $d_n = n - \sum (i \in I ; a_i \leqslant n) b_i$ and $e_n = \min \{d_n, d_{n+1}, \ldots, \hat{d}_\omega\}$. Then $0 \leqslant e_n \leqslant d_n$ and, since $d_{n+1} \leqslant d_n + 1$,

$$e_n \leqslant e_{n+1} \leqslant e_n + 1 .$$

By (16), given any $r \in P$, there is $n < \omega$ with $e_n = r$. (Here one uses that $e_0 = 0$.) For $n < \omega$ we shall define, by induction on n, a set F_n with $|F_n| \leqslant e_n + \sum (i \in I; a_i \leqslant n) b_i$, as well as sets $X_i \subset A_i \cap F_n$ for $i \in I_n$. Put $F_0 = \varnothing$ and $X_i = \varnothing$ for $i \in I_0$. Now let $0 < n < \omega$, and suppose that F_{n-1} and X_i have been defined for $i \in I_{\bar{n}}$. Then

$$|F_{n-1}|+b_{I_n} \leqslant e_{n-1}+\sum(i \in I; a_i \leqslant n-1)b_i+b_{I_n}$$

$$= e_{n-1}+\sum(i \in I; a_i \leqslant n)b_i \leqslant d_n+\sum(i \in I; a_i \leqslant n)b_i$$

$$= n.$$

(Here we have used the relation $e_{n-1} \leqslant d_n$ and the definition of d_n.) Thus $|F_{n-1}|+b_{I_n} \leqslant n$. Let $j \in I_n$. Then $|A_j| = a_j = n$; $|A_j-F_{n-1}| \geqslant n-|F_{n-1}| \geqslant b_{I_n}$, so that $b_{I_n} \leqslant |A_j-F_{n-1}|$ for $j \in I_n$. Therefore there are pairwise disjoint sets $X_i \in [A_i-F_{n-1}]^{b_i}$ for $i \in I_n$. Put $F'_n = F_{n-1} \cup X_{I_n}$. Then

$$|F'_n| = |F_{n-1}|+b_{I_n} \leqslant e_{n-1}+\sum(i \in I : a_i \leqslant n)b_i$$

$$\leqslant e_n+\sum(i \in I; a_i \leqslant n)b_i = e_n+(n-d_n) \leqslant n$$

by definition of e_n. Put $F_n = F'_n$ if either $e_{n-1} = e_n$ or $e_{n-1} < e_n \notin P$. In the remaining case, i.e. if $e_{n-1} < e_n \in P$, we choose, as is then possible, an element $x_n \in A_{f(e_n)}-F'_n$ and put $F_n = F'_n \cup \{x_n\}$. We have now defined X_i for every $i \in I_{\bar{\omega}}$. For $i \in I_{\aleph_0}$ put

$$X_i = \{x_n : 0 < n < \omega; e_{n-1} < e_n \in P; f(e_n) = i\}.$$

It follows that $(X_i : i \in I)$ is a multitransversal of $(A_i : i \in I)$ and that $|X_i| = b_i$ for $i \in I_{\bar{\omega}}$. It only remains to prove that $|X_i| = b_i$ for $i \in I_{\aleph_0}$. Let $r \in P$. Denote by $n(r)$ the least number $n < \omega$ with $e_n = r$, which clearly exists. Then, since $e_0 = d_0 = 0 \notin P$, we have $n(r) > 0$, so that $e_{n(r)-1} < e_{n(r)} \in P$. Hence the element $x_{n(r)}$ is defined and satisfies $x_{n(r)} \in X_{f(r)}$. Put $g(r) = x_{n(r)}$. Then the mapping

$$g : P \to \bigcup(i \in I_{\aleph_0})X_i$$

is bijective. For $i \in I_{\aleph_0}$ put

$$P_i = \{r \in P : f(r) = i\}.$$

Then $g(P_i) = X_i$ and hence $|X_i| = |P_i| = b_i$, and Theorem 3 is established.

7.

LEMMA. *Let I be a set and a_i, b_i be cardinals for $i \in I$. Let*

$$c \geqslant \aleph_0; \quad I_0 = \{i \in I : \; a_i \leqslant c\}; \quad I_1 = I-I_0.$$

Then (28) $\Rightarrow$ (29) $\wedge$ (30), *where*

(28) $$(a_i : i \in I) \to (b_i : i \in I)_{ds},$$

(29) $$(a_i : i \in I_0) \to (b_i : i \in I_0)_{ds},$$

(30) $$(a_i : i \in I_1) \to (b_i : i \in I_1)_{ds}.$$

Proof. Trivially (28) $\Rightarrow$ (29) $\wedge$ (30). Now assume, *vice versa*, that (29) and (30) hold. Let $|A_i| = a_i$ for $i \in I$. Then, applying (29) to the family $(\bar{a}_i : i \in I_0)$, we find that $b_{I_0} \leqslant c$. Also, the family $(A_i : i \in I_0)$ has a multitransversal $(X_i : i \in I_0)$ of size $(b_i : i \in I_0)$. Then $|X_{I_0}| = b_{I_0} \leqslant c$. Hence, since $c \geqslant \aleph_0$, we have $|A_i - X_{I_0}| = a_i$ for $i \in I_1$. By (30), the family $(A_i - X_{I_0} : i \in I_1)$ has a multitransversal $(X_i : i \in I_1)$ of size $b_i : i \in I_1)$. Then $(X_i : i \in I)$ is a multitransversal of $(A_i : i \in I)$ of size $(b_i : i \in I)$, which proves (28).

8.

Proof of Theorem 4. The implications (17) $\Rightarrow$ (18) $\Rightarrow$ (19) are trivial.

Proof of (18) $\Rightarrow$ (20). Let $I' = \{i \in I : a_i \geqslant \aleph_0; b_i \geqslant 1\}$. Then $(\bar{a}_i : i \in I')$ has a multitransversal of size $(b_i : i \in I')$. Let $S' = \{a_i : i \in I'\}$. Then, by Theorem 2, $\omega(S') \cap \bar{\lambda} \notin \text{stat}\, \lambda$ for every weakly inaccessible cardinal λ. Since $\omega(S) \subset \omega(S') \cup \bar{\omega}$, it follows that $\omega(S) \cap \bar{\lambda} \notin \text{stat}\, \lambda$.

Proof of (18) $\Rightarrow$ (21). Let $I_0 = \{i \in I : a_i \leqslant \aleph_0\}$. Then $(\bar{a}_i : i \in I_0)$ has a multitransversal of size $(b_i : i \in I_0)$. Then (21) follows from Theorem 3, in view of the relations

$$\sum (i \in I; a_i = \aleph_0) b_i = \sum (i \in I_0; a_i = \aleph_0) b_i, \sum (i \in I; a_i \leqslant n) b_i = \sum (i \in I_0; a_i \leqslant n) b_i$$

for $n < \omega$.

Proof of (19) $\wedge$ (20) $\wedge$ (21) $\Rightarrow$ (17). Let $I_0 = \{i \in I : a_i \leqslant \aleph_0\}$; $I_1 = I - I_0$. Then $(a_i : i \in I_0) \rightarrow (b_i : i \in I_0)_{ds}$ by Theorem 3. Let $I_1^+ = \{i \in I_1 : b_i \geqslant 1\}$.

Then, for every cardinal κ,

$$|\{i \in I_1^+ : a_i = \kappa\}| = \sum (i \in I_1^+; a_1 = \kappa) 1$$

$$\leqslant \sum (i \in I_1^+; a_1 = \kappa) b_i \leqslant \sum (i \in I; a_i = \kappa) b_i \leqslant \kappa.$$

Since $a_i \in S$ for $i \in I_1^+$, we deduce from Theorem 2 that

$$(a_i : i \in I_1^+) \rightarrow (a_i : i \in I_1^+)_{ds}.$$

By (19) we have $b_i \leqslant a_i$ for $i \in I_1^+$, which implies that $(a_i : i \in I_1^+) \rightarrow (b_i : i \in I_1^+)_{ds}$. But $b_i = 0$ for $i \in I_1 - I_1^+$. Hence $(a_i : i \in I_1) \rightarrow (b_i : i \in I_1)_{ds}$. Now the lemma, with $c = \aleph_0$, yields $(a_i : i \in I) \rightarrow (b_i : i \in I)_{ds}$, and this concludes the proof of Theorem 4.

Reference

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