A PROBLEM ON COMPLEMENTS AND DISJOINT EDGES IN A HYPERGRAPH

Richard A. Duke
Georgia Institute of Technology
(visiting at Emory University)

Paul Erdős
Mathematical Institute of the
Hungarian Academy of Sciences

The problem of our title was suggested by a question concerning the Ramsey numbers of pairs of k-uniform hypergraphs. We will describe this question first.

An early result in the generalized Ramsey theory for graphs is given by the following theorem [2].

<u>Theorem</u>. (Chvátal, Harary) Let $2K_2$ denote the graph with four vertices and two edges, these two edges being disjoint, and let G_N denote a graph with N vertices, none of which is isolated. Then the Ramsey number of this pair is given by

(1) $r(2K_2, G_N) = N + 2$ if G_N is complete, and

(1') $r(2K_2, G_N) = N + 1$ if G_N is not complete.

Let K_N^k denote the <u>complete</u> k-graph on N vertices, that is, the k-uniform hypergraph whose edges are the $\binom{N}{k}$ k-subsets of a set of N vertices. The Ramsey number of a pair of k-graphs is defined as for graphs in terms of the 2-colorings of the k-sets which are the edges of a complete k-graph. Let sK_k^k denote a k-graph which is a matching with s edges. That is, sK_k^k has sk vertices and s edges, no two of which intersect. The question of determining the Ramsey number of a pair of k-graphs, one of which is a matching, arose in connection with work done by the first author and S. Burr [1]. (The Ramsey number of a pair of <u>graphs</u>, one of which being a matching, has recently been considered by Faudree, Schelp, and Sheehan [6].)

Equation (1) can easily be generalized to k-graphs as follows:

(2) $r(sK_k^k, K_N^k) = N + k(s - 1)$ for $N \geq k$.

To show that the Ramsey number is at least $N + k(s - 1)$, we 2-color the edges of the complete k-graph on $N + k(s - 1) - 1$ vertices in such a way that all of the edges of some complete sub-k-graph with $ks-1$ vertices are given color one and all other edges are given color

Congressus Numeratium, Vol. 28 (1980), pp. 369 - 375

two. Then there is no matching of s edges all having color one, while each set of N vertices includes at least k vertices, and hence an edge, in the complete subgraph. An easy argument by induction on s establishes the inequality in the opposite direction.

If we replace the complete k-graph in (2) by a k-graph on N vertices which is not complete, the value of the Ramsey number may not decrease when $k \geq 3$. For example if $K_N^3 - K_3^3$ denotes the 3-graph obtained by deleting a single edge from K_N^3, we have

(3) $r(sK_3^3, K_N^3 - K_3^3) = r(sK_3^3, K_N^3) = N + 3(s-1)$, for $N \geq 3s-2$.

This follows from the existence of a system of triples on 6 points in which each set of four points spans two triples and no two of the triples are disjoint. The collection (1,2,3), (1,2,6), (1,3,4), (1,4,5), (1,5,6), (2,3,5), (2,4,5), (2,4,6), (3,4,6), and (3,5,6) forms such a system. Assigning one color to the triples of $s-1$ disjoint copies of this system, thought of as edges of the complete 3-graph on $N + 3(s-1) - 1$ vertices, yields the required 2-coloring, since then each set of N vertices must include at least four vertices in some one of these copies.

If in (2) the complete k-graph is replaced by a k-graph in which more than one of the possible edges is missing, the value of the Ramsey number may decrease. Let $r(s,N,t;k)$, $N > k$, denote the least integer m such that every 2-coloring of the edges of a complete k-graph on m vertices produces either a matching with s edges all in the first color, or <u>some</u> k-graph obtainable from K_N^k by the deletion of t edges which in this 2-coloring has all of its edges in the second color. Equivalently, each 2-coloring produces either a matching with s edges in the first color or a complete k-graph on N vertices with <u>at most</u> t edges in the first color. By fixing a set C of $ks-1$ vertices in the complete k-graph on $N+k(s-1)-2$ vertices and assigning color one to each edge has all of its vertices in C and a second color to all other edges, we obtain a 2-coloring in which there can be no matching of s edges all of the first color and such that each complete sub-k-graph on N vertices contains at least $k+1$ edges in that color. Similarly, by fixing a set C of $(k-1)s-1$ vertices in the complete k-graph on

$N + (k - 1)(s - 1) - 1$ vertices and assigning color one to those edges having at least $k - 1$ vertices in C we produce no matching of s edges in color one and no subgraph on N vertices having fewer than $n - k + 1$ edges in that color. Hence we have

(4) $r(s, N, t ; k) \geq N + k(s - 1) - 1$, for $t < k + 1$, and

(4') $r(s, N, t ; k) \geq N + (k - 1)(s - 1)$, for $t < N - k + 1$.

For $s = 2$ (2), (4), and (4') yield

(5) $r(s, N, t ; k) = N + k$ or $N + k - 1$ if either $t < k + 1$ or $t < N - k + 1$.

For N sufficiently large we have $r(s, N, t' ; k) \leq r(s, N, t ; k)$ whenever $t' \geq t$. It follows that there exists a value $t_0(k)$ such that

(6) $r(2, N, t ; k) = N + k$, for $0 \leq t \leq t_0(k)$, and
$r(2, N, t ; k) = N + k - 1$, for $t_0(k) \leq t < N - k + 1$.

The results (1) and (1') show that $t_0(2) = 1$, while (3) shows that $t_0(3) \geq 2$. By considering several cases it can be shown that any 2-coloring of the edges of the complete 3-graph on $N + 2$ vertices which is such that each pair of vertices miss at least three edges of color one must produce at least two disjoint edges in that color. It follows that $t_0(3) = 2$.

Examples which we will describe later establish that $t_0(4) \geq 3$. These facts suggest the possibility that $t_0(k) = k - 1$ in general. Before considering when this is so, it will be convenient to reformulate this question.

We have that $t_0(k) \leq k - 1$ if and only if whenever a 2-coloring of the edges of the complete k-graph on $n = N + k - 1 \geq 2k$ vertices is such that each complete sub-k-graph on N vertices contains at least k edges of color one, then there must be at least two disjoint edges in that color. Equivalently, $t_0(k) \leq k - 1$ if and only if whenever a 2-coloring of the edges of K_n^k, $n \geq 2k$, is such that each subset of $k - 1$ vertices misses at least k edges of color one, then there are at least two disjoint edges in color one. Considering those sets of edges given color one by the possible 2-

colorings of K_n^k we are led to the following conjecture.

Conjecture. Given k and n, $n \geq 2k$, if H is a k-graph on n vertices which is such that each subset of $k - 1$ vertices of H misses at least k edges of H, then H must possess at least two disjoint edges.

Note that if each subset of K vertices misses a single edge, then, trivially, there are two disjoint edges, while for each positive integer ℓ there is an integer $n(\ell)$ such that for any $n > n(\ell)$ there is a k-graph on n vertices with no two disjoint edges and in which each subset of $k - 2$ vertices misses at least ℓ edges.

The results concerning $t_0(k)$ show that the conjecture is true for $k = 2$ or 3 and any $n \geq 2k$. The conjecture is also correct for each $k \geq 2$ when $n = 2k$. This can be seen as follows. Consider a k-graph H whose vertices are the integers $1, 2, \ldots, 2k$, and suppose that each subset of $k - 1$ vertices misses at least k edges of H. Then there are at least k edges having their vertices in the set $C = \{1, 2, \ldots, k+1\}$. We may assume that each k-element subset of C, other than possibly $\{1, 2, \ldots, k\}$, is an edge of H. Similarly, the $(k+1)$-element set $\{k-1, k+2, k+3, \ldots, 2k\}$ must contain k edges, so at least one of $\{k-1, k+2, k+3, \ldots, 2k\}$ and $\{k, k+2, k+3, \ldots, 2k\}$ is an edge of H. But the complements in H of these two k-sets are edges in C. Thus H contains a pair of disjoint edges.

The conjecture is correct for arbitrary $k \geq 2$ and sufficiently large n if we assume that our k-graphs have no isolated vertices. In this case, for n large, our k-graph contains a large "Δ-system" (see [4] or [5]), that is, a large collection of edges which is such that the intersection of any two of them is equal to the intersection of all of them. If no two edges are disjoint, then the common intersection, say A, of the edges in our Δ-system satisfies $1 \leq |A| \leq k - 1$. Let B be a set of vertices with $A \subseteq B$, $|B| = k - 1$. An edge which misses B must meet each edge of the Δ-system, but can not meet any two of them in the same vertex, which is impossible. (When considering the sub-k-graph induced by one of the colors of an s-coloring of a complete k-graph we may have isolated vertices, so this result does not apply directly to the question of Ramsey numbers.)

The first examples which we were able to find showing that the conjecture is not correct in all cases were a family of hypergraphs $H^{(k)}$, $k \geq 2$, constructed by Erdős and Lovász [3]. Here $H^{(k)}$ is a 3^k-graph on $n = 7^k$ vertices, with no two edges disjoint. These hypergraphs are defined recursively as follows: Let $H^{(1)}$ be the 7-point Fano plane, viewed as a 3-graph with vertex set $\{1, 2, \ldots, 7\}$ and the 7 lines of the plane as 3-edges. Let H_i, $i = 1, 2, \ldots 7$, be seven disjoint copies of $H^{(1)}$ and define $H^{(2)}$ to be the 9-graph having as edges all of the sets $E_{i_1} \cup E_{i_2} \cup E_{i_3}$, where E_i is an edge of H_i and $\{i_1, i_2, i_3\}$ is an edge of $H^{(1)}$. In general, $H^{(k)}$ is formed by taking one copy of $H^{(1)}$ for each vertex of $H^{(k-1)}$, and taking as an edge each 3^k-set which is the union of one edge from each of a set of 3^{k-1} copies of $H^{(1)}$ corresponding to the 3^{k-1} vertices of some edge of $H^{(k-1)}$.

Peter Frankl [private communication] was soon able to provide much simpler counterexamples for each even k, $k \geq 6$. For each such k Frankl constructed a k-graph H on $3k-3$ vertices by letting X_1, X_2, and X_3 be three disjoint $(k - 1)$-sets and taking as an edge of H each k-subset of $X_1 \cup X_2 \cup X_3$ having exactly $k/2$ vertices in each of two of the X_i.

By combining ideas from these two constructions we can now give counterexamples to the conjecture which are vertex 2-colorable as hypergraphs and for which the number of edges is given by a polynomial in k. To obtain such a k-graph we let X_1, X_2, and X_3 be three disjoint copies of a finite geometry on $p^2 + p + 1$ points. We then form a $(2p + 2)$-graph with vertex set $X_1 \cup X_2 \cup X_3$ by taking as an edge each set which is the union of one line from each of two of the X_i. The resulting hypergraph has $3(p^2 + p + 1)^2$ edges, no two of which are disjoint, and is such that each set of $2p + 1$ vertices misses at least $p(p^2 + p + 1)$ edges.

These examples show that the conjecture is not correct in general, but many questions remain. Some of these are the following:

Question 1. What is the smallest value of k for which there is a counterexample to the conjecture?

From the results above we know that this value is 4, 5 or 6.

Question 2. Given that there is a k-graph which is a counterexample to the conjecture, what is the smallest number, $m(k)$, of edges in such a counterexample?

The examples above show that for certain k we have $m(k) < ck^4$. Is $m(k) = o(k^4)$ possible?

Question 3. Given that the conjecture is not correct for some value of k, what is the smallest number, $n(k)$, of vertices in a counterexample for this k?

We know that $n(k) > 2k$ and that $n(k) < 3k$ for k even, $k \geq 6$.

Question 4. We have seen that if we do not allow "isolated vertices", then the conjecture is true for sufficiently large n. What is the smallest value, $n_1(k)$, such that the conjecture is true for all k-graphs with n vertices, $n > n_1(k)$, and no isolated vertices.

Many other questions may be asked. For example, suppose that each subset of $k - 1$ vertices misses more than k edges. When does this imply the existence of two disjoint edges? How many edges must each $(k - 1)$-set miss? What conditions imply the existence of s pairwise disjoint edges, $s > 2$?

Finally we mention a similar and apparently very difficult question which was posed several years ago in a paper by one of us with L. Lovász [3]. Suppose H is a k-graph in which each subset of $k - 1$ vertices misses at least one edge and H has no two disjoint edges. What is the minimum number, $m_1(k)$, of edges in such a k-graph? It is known that $m_1(k) \leq k^{3/2 + \epsilon}$, but perhaps $m_1(k) < ck$ for some constant c. In fact, it may be that $m_1(k) < 3k$.

REFERENCES

[1] S. A. Burr and R. A. Duke, Ramsey numbers for certain k-graphs, J. Austral. Math. Soc., to appear.

[2] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs, III. Small off-diagonal numbers. Pacific J. Math. 41(1972), 335-345.

[3] P. Erdös and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, Coll. Math. J. Bolyai, 10, Infinite and Finite Sets, 1973, 609-617.

[4] P. Erdös, E. C. Milner, and R. Rado, Intersection theorems for systems of sets, III, Austral. J. Math. 18(1974), 22-40.

[5] P. Erdös and R. Rado, Intersection theorems for systems of sets, J. London Math. Soc. 35(1960), 85-90.

[6] R. J. Faudree, R. H. Schelp, and J. Sheehan, Ramsey numbers for matchings, to appear.