AN INFINITE PRODUCT FOR e

Nicholas Pippenger

Wallis's infinite product [1, p. 180]

$$\frac{\pi}{2}=\frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\,\frac{8}{7}\cdots$$

was used by Stirling to determine the constant factor in his asymptotic formula [2, p. 137] $n!\sim(2\pi n)^{1/2}e^{-n}n^n$. A striking companion to Wallis's product is

$$\frac{e}{2}=\left(\frac{2}{1}\right)^{1/2}\left(\frac{2}{3}\,\frac{4}{3}\right)^{1/4}\left(\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\,\frac{8}{7}\right)^{1/8}\cdots,$$

which is proved as follows. For $\nu\geqslant 2$, the νth factor is $[2^{\nu-1}\cdots 2^{\nu}/(2^{\nu-1}+1)\cdots(2^{\nu}-1)]^{1/2^{\nu}}=[(2^{\nu-1}-1)!!^2 2^{\nu}!!^2/2\cdot 2^{\nu-1}!!^2(2^{\nu}-1)!!^2]^{1/2^{\nu}}$, where $n!!=n(n-2)\cdots 4\cdot 2$ if n is even, $n(n-2)\cdots 3\cdot 1$ if n is odd. Since $2^{\nu}!!=2^{2^{\nu-1}}2^{\nu-1}!$ and $(2^{\nu}-1)!!=2^{\nu}!/2^{\nu}!!=2^{\nu}!/2^{2^{\nu-1}}2^{\nu-1}!$, this expression becomes $[2^{2^{\nu}}2^{\nu-1}!^6/2\cdot 2^{\nu-2}!^4 2^{\nu}!^2]^{1/2^{\nu}}$. By induction on ν, the product of the first ν factors is $[2\cdot 2^{\nu 2^{\nu}}2^{\nu-1}!^2/2^{\nu}!^2]^{1/2^{\nu}}$. Applying Stirling's formula and letting $\nu\to\infty$ completes the proof.

References

1. J. Wallis, Arithmetica Infinitorum, Oxford, 1656.
2. J. Stirling, Methodus Differentialis, London 1730.

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RESEARCH PROBLEMS

Edited by Richard Guy

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4

HOW MANY PAIRS OF PRODUCTS OF CONSECUTIVE INTEGERS HAVE THE SAME PRIME FACTORS?

P. Erdős

Denote by $M(n,k)$ the least common multiple of the k consecutive integers $n+1, n+2,\ldots, n+k$. I conjectured that if $0<n<n+k\leqslant m$, then

$$M(n,k)\neq M(m,k), \tag{1}$$

and I thought that the following stronger result also holds.

If $k>2$, then $M(n,k)$ and $M(m,k)$ have the same prime factors on at most finitely many occasions.

For $k=2$, of course, $2n(n+1)=m(m+1)$ has infinitely many solutions, so that $n(n+1)$ and

$m(m+1)$ have the same prime factors. Several colleagues found examples with $k \geqslant 3$, but I know of no example for $k \geqslant 6$.

Are there infinitely many n, m, k with $1 \leqslant n$, $n+3 \leqslant n+k \leqslant m$ so that $M(n,k)$, $M(m,k)$ have the same prime factors?

Is there a k_0 such that for $k > k_0$ this never happens?

Finally, estimate the number of pairs (n,m) with $1 \leqslant n < m < x$ for which $n(n+1)$ and $m(m+1)$ have the same prime factors.

A well-known theorem of Størmer and Pólya states that if $a_1 < a_2 < a_3 < \cdots$ are all composed of the primes $p_1, p_2, \ldots, p_r$ then $a_{k+1} - a_k \to \infty$. Wintner conjectured more than 40 years ago that there is an infinite sequence of primes $p_1 < p_2 < p_3 < \cdots$ such that, if $b_1 < b_2 < b_3 < \cdots$ are the integers composed of the p_i, then $b_{k+1} - b_k \to \infty$. I conjecture that, if the sequence $p_1 < p_2 < p_3 < \cdots$ is sufficiently dense, then this can't happen, e.g., if $\Sigma 1/p_i$ diverges, then $b_{k+1} - b_k = 1$ infinitely often.

It is easy to see that for any function $f(k)$ tending to infinity as fast as we wish there is a sequence $\{p_k\}$ with $p_k > f(k)$ so that $b_{k+1} - b_k = 1$ has infinitely many solutions; I need Brun's method to prove this.

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HOW MANY *i-j* REDUCED LATIN RECTANGLES ARE THERE?

JOHN R. HAMILTON AND GARY L. MULLEN

There is a large literature [2] on **latin squares**, $n \times n$ squares with each of the numbers $1, 2, \ldots, n$ in each row and column. A **latin rectangle** is an array of m rows and n columns with $m \leqslant n$ in which each row is a permutation of $1, 2, \ldots, n$ and each column has distinct elements. A latin rectangle is **reduced** if the first row is in the standard order $1, 2, \ldots, n$ and we say that it is i-j **reduced** if the first i rows are cyclic permutations of $1, 2, \ldots, n$ and the first j columns are in the form $k, k+1, \ldots, k+m-1$ for $k = 1, \ldots, j$. Thus a latin rectangle of order $m \times n$ is i-j reduced if it has the following form.

$$
\begin{array}{ccccccc}
1 & 2 & \cdots & j & \cdots & n-1 & n \\
2 & 3 & \cdots & j+1 & \cdots & n & 1 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
i & i+1 & \cdots & j+i-1 & \cdots & i-2 & i-1 \\
\vdots & \vdots & & \vdots & & & \\
m-1 & m & \cdots & m+j-2 & & & \\
m & m+1 & \cdots & m+j-1 & & &
\end{array}
$$

It is understood that if a number in the rectangle exceeds n then it is reduced mod n. We can allow $i = 0$ or $j = 0$ so that a 1-0 reduced rectangle is reduced and a general rectangle is 0-0 reduced. As an illustration, the following rectangle is a 2-1 reduced latin rectangle of order 4×5.

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 1 \\
3 & 1 & 5 & 2 & 4 \\
4 & 5 & 2 & 1 & 3
\end{array}
$$

In [5] the second author considered the case $m = n$ and studied some elementary properties of i-j reduced latin squares of order n. In particular, the number $L(i,j,n)$ of i-j reduced latin