Maximum Degree in Graphs of Diameter 2

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It is well known that there are at most four Moore graphs of diameter 2, i.e., graphs of diameter 2, maximum degree d, and $d^2 + 1$ vertices. The purpose of this paper is to prove that with the exception of C_4, there are no graphs of diameter 2, of maximum degree d, and with d^2 vertices.

INTRODUCTION

The purpose of this paper is to prove that, with the exception of C_4, there are no graphs of diameter 2 and maximum degree d with d^2 vertices.

On one hand our paper is an extension of [4] where it was proved that there are at most four Moore graphs of diameter 2 (i.e. graphs of diameter 2, maximum degree d, and $d^2 + 1$ vertices). We also use the eigenvalue method developed in that paper.

On the other hand, our problem originated in [2]. The domination number of a graph G is the smallest integer k such that G has a k-element subset, S, for which every vertex of G either belongs to S or is adjacent to a vertex of S.

Authors of [2] constructed a number of graphs of diameter 2 which contained no three of four-cycles and for which the domination number was arbitrarily large. As a rule, the only lower bounds for the domination numbers were obtained from upper bounds on the maximum degree.

This suggested the following question: How small may the maximum degree be compared to the number of vertices in graphs of diameter 2?

Since a graph of a diameter 2 and maximum degree d may have at most $d^2 + 1$ vertices, the question can be formulated as follows: given non-negative numbers d and δ, is there a graph of diameter 2 and maximum degree d with $d^2 + 1 - \delta$ vertices? It was proved in [4] that if $\delta = 0$ then there are graphs corresponding to $d = 1, 3$, and 7 and that, moreover, only one more case, namely of $d = 57$, is possible. The case $\delta = 1$ is solved in the next section, and the last section contains some comments concerning the case $\delta > 1$.

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THE RESULT

Theorem. If G is a graph of diameter 2 with $n = d^2$ ($d \geqslant 2$) vertices and maximum degree d, then G is isomorphic to a four-element cycle.

Proof: First of all, let us notice that if G had a vertex of degree $k < d$, then G would have at most $1 + k + k \cdot (d - 1) = 1 + k \cdot d < d^2$ vertices. Thus G must be regular of degree d and in particular d must be even.

Since G has diameter two, the neighbors of any vertex dominate G. Thus, if G had a triangle it would have at most $1 + (d - 2)(d - 1) + 2(d - 2) < d^2$ vertices. Consequently, G is triangle-free and a similar argument shows that every vertex of G is contained in at most one C_4.

On the other hand if G had a vertex contained in no C_4, then G would have $1 + d + d(d - 1) > d^2$ vertices.

Thus, for each vertex r of G there is exactly one vertex r' of G at a distance 2 from r, such that there are exactly two paths of length 2 joining r and r'. Let K be the direct sum of 2×2 matrices of the form

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

i.e., K is the adjacency matrix of a 1-factor.

Then if A is the adjacency matrix of G, we have

$$A^2 + A - (d - 1)I = J + K \tag{1}$$

where J is a matrix all of whose entries are 1.

Since J is regular, A commutes with J, therefore with K, hence all matrices in (1) can be simultaneously diagonalized. Since K has eigenvalues 1 and -1 each of multiplicity $d^2/2$, and the eigenvalue $+1$ of K is paired with eigenvalues d of A and d^2 of J, we see from the fact that all other eigenvalues of J are 0, that the eigenvalues of A other than d are roots α satisfying:

$$\alpha^2 + \alpha - (d - 1) = +1, \qquad \text{occurring } d^2/2 - 1 \text{ times} \tag{2}$$

or

$$\alpha^2 + \alpha - (d - 1) = -1, \qquad \text{occurring } d^2/2 \text{ times.} \tag{3}$$

Thus the remaining eigenvalues of A are

$$\beta_1 = -1/2 + \sqrt{4d - 7}/2 \text{ and } \beta_2 = -1/2 - \sqrt{4d - 7}/2$$

with multiplicities m_1 and m_2 satisfying

$$m_1 + m_2 = d^2/2$$

and

$$\gamma_1 = -1/2 + \sqrt{1+4d}/2 \text{ and } \gamma_2 = -1/2 - \sqrt{1+4d}/2$$

with multiplicities n_1, n_2 satisfying

$$n_1 + n_2 = d^2/2 - 1.$$

Let $p = \sqrt{4d-7}$ and $q = \sqrt{4d+1}$.

From (2) $x^2 + x - d$ is a factor of the characteristic polynomial of A. Therefore if G is not an integer, $n_1 = n_2$, which would imply that $d^2/2 - 1$ is even. This contradiction shows that q is an integer.

If p is also an integer, then since $q^2 - p^2 = 8$, we must have that $p = 1$ and $q = 3$. Thus, $d = 2$ and in this case G is a cycle of length 4.

If p is not an integer, then $m_1 = m_2$.

Since $m_1 = m_2$, $m_1 + m_2 = d^2/2$, $n_1 + n_2 = d^2/2 - 1$, and the trace of $A = 0 =$ sum of eigenvalues of A, we have

$$\begin{aligned} 0 &= d + m_1 \frac{-1+p}{2} + m_2 \frac{-1-p}{2} + n_1 \frac{-1+q}{2} + n_2 \frac{-1-q}{2} \\ &= d - \frac{1}{2}(m_1 + m_2) - \frac{1}{2}(n_1 + n_2) + (n_1 - n_2)\frac{q}{2} \\ &= d - \frac{d^2}{2} + \frac{1}{2} + (n_1 - n_2)\frac{q}{2}. \end{aligned}$$

Since $d = (q^2 - 1)/4$, we can conclude that

$$q^4 - 10q^2 - 16q(n_1 - n_2) - 7 = 0.$$

Since q is an integer root of this equation, then $q = 1$ or $q = 7$, i.e., $d = 0$ or $d = 12$. If $d = 12$ then G has eigenvalues

12	with multiplicity 1
$\dfrac{-1+\sqrt{41}}{2}$	with multiplicity 36
$\dfrac{-1-\sqrt{41}}{2}$	with multiplicity 36
3	with multiplicity 44
-4	with multiplicity 27

Since G is triangle-free, the trace of A^3 is 0. But

$$\operatorname{tr} A^3 = \sum_i \lambda_i^3$$

where λ_i are eigenvalues of A. The sum of cubes of the above eigenvalues, however, is 72 and thus there is no graph with $d = 12$.

Thus the theorem is proved.

Let F be a finite field and P a projective plane over F, i.e., elements of P are proportional triples of nonzero elements of F^3. Brown [1] defines two elements of P to be adjacent iff their scalar product is zero. It is easy to see that if F has p elements then the resulting graph has maximum degree $p + 1$, diameter 2, and $p^2 + p + 1$ vertices (these graphs were introduced in a different form in [3]).

For our purposes, Brown's construction can be slightly improved. A vertex (x, y, z) of Brown's graph has degree p iff the norm $x^2 + y^2 + z^2 = 0$. Thus if F has characteristic 2 and $a \neq 0$ then the vector (a, b, c) is adjacent to the vector $(b + c, a + c, a + b)$, which has norm 0. If F has characteristic 2 then the function $f(x) = x^2$ is one-to-one and hence onto. Thus, up to proportionality, Brown's graph has $p + 1$ vertices of degree p. Adding a new vertex and joining it to all vertices of degree p we obtain a $(p + 1)$ regular graph with $p^2 + p + 2$ vertices.

We know only three examples of graphs in which $\delta = 2$, namely triangle, the 3-regular $R(3,3)$—critical Ramsey graph, and the graph corresponding to $p = 2$ in the above construction.

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