Correction of some typografical errors of the paper P.ERDŐS and P.VÉRTESI, On the almost everywhere divergence of Lagrange interpolatory polynomials for arbitrary system of nodes, Acta Math. Acad.Sci.Hungar., 36/1-2/,/1980/, 71-89.

Page	Line	Read	Instead of
74	7	$1/\ell n\ \ell n\ m$; further	$1/\ell n\ \ell n\ m$. Further
75	11	$2\sqrt{\ell nn}\,\delta_n$	$\sqrt{\ell nn}\ \delta_n$
76	13	Dropping J_t	Dropping J_ℓ
79	3	$(n\geq n_1(q))$	$(n\geq n_o(q))$
80	7	$\sum_{i=1}^{s} f(x_{k_i})\ell_{k_i}(x)$	$\sum_{i=1}^{n} f(x_{k_s})\ell_{k_s}(x)$
81	/4.29/	$\sum_{t=1}^{\infty} \ldots$	$\sum_{t=k}^{\infty} \ldots$
83	24	that is	and that
86	/4.57/	$W=\bigcap_{k=1}^{\infty}\ \bigcup_{t=k}^{\infty} \ldots$	$W=\bigcup_{k=1}^{\infty}\ \bigcup_{t=k}^{\infty} \ldots$
87	5	by $W_{ti}=R_{ti}^{[o]}, W=G^{[o]}$	by $W=G^{[o]}$
87	6	$G_\rho \cup W$	$G_\rho \cap W$
88	3	ε and $M(M\geq 1$, integer)	ε and M
88	10	$\mu(\bigcap_{t=o}^{\infty} H_t)\geq\varepsilon$, which	$\mu(\bigcup_{t=o}^{\infty} H_t)\geq\varepsilon$ wich
88	15	Read:	

" $|L_{u_1(x)}(f_1, x)|\geq A_1 > 1^3\lambda_{N_o}^2$ whenever $x\in S_1$.

Here $m_1\leq u_1(x)\leq n_1$. Now we take the polynomial $\varphi_1(f_1,x)$ of degree $\leq N_1$, $||\varphi_1||\leq 32$, for which

$$|L_{u_1(x)}(\varphi_1,x)|\geq A_1>1^3\lambda^2_{N_o} \quad \text{whenever} \quad x\in S_1$$

/see 4.4.4/".

instead of :

$$"|L_{u_1(x)}(f_1,x)|\geq A_1>1^3\lambda^2_{N_o} \quad \text{whenever} \quad x\in S_1$$

/see 4.4.4/."

88	16	$\rho_k=2^{-k}, A_k>k^3\lambda^2_{N_{k-1}}$	$\delta_k=2^{-k}, A_k>k_\varepsilon\ \lambda^2_{N_{k-1}}$
88	18	$\geq 2-2\rho_k$	$\geq 2-2\delta_k$
88	23	$S=\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} S_i$	$S=\bigcap_{k=1}^{\infty}\bigcap_{i=k}^{\infty} S_i$

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 36 (1—2), (1980), pp. 71—89.

ON THE ALMOST EVERYWHERE DIVERGENCE OF LAGRANGE INTERPOLATORY POLYNOMIALS FOR ARBITRARY SYSTEM OF NODES

By

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Dedicated to the memory of John Curtiss

1. Introduction

In a previous paper P. ERDŐS [1] stated without proof that if $X=\{x_{in}\}$, $n=1, 2, \dots;\ 1\leqq i\leqq n$,

$$(1.1)\qquad -1\leqq x_{nn}<x_{n-1,n}<\dots<x_{1n}\leqq 1\quad (n=1, 2, \dots)$$

is a triangular matrix then there is a continuous function $F(x)$, $-1\leqq x\leqq 1$, so that the sequence of Lagrange interpolation polynomials

$$L_n(F, X, x)=L_n(F, x)=\sum_{k=1}^{n}F(x_{kn})\,l_{kn}(x)$$

diverges almost everywhere in $[-1, 1]$, and in fact

$$\overline{\lim_{n\to\infty}}\,|L_n(F, X, x)|=\infty$$

for almost all x. (Here, as usual,

$$(1.2)\qquad l_{kn}(x)=\frac{\omega_n(x)}{\omega_n'(x_{kn})(x-x_{kn})}\quad \left(k=1, 2, \dots, n;\ \omega_n(x)=\prod_{k=1}^{n}(x-x_{kn})\right)$$

are the corresponding fundamental polynomials,

$$(1.3)\qquad \lambda_n(x)=\sum_{k=1}^{n}|l_{kn}(x)|,\quad \lambda_n=\max_{-1\leqq x\leqq 1}\lambda_n(x)\quad (n=1, 2, \dots)$$

are the Lebesgue functions and Lebesgue constants of the interpolation, respectively.

We now prove this statement in full detail. The detailed proof turned out to be quite complicated and several unsuspected difficulties had to be overcome.

In the same paper P. Erdős also stated, that there is a pointgroup $\{x_{kn}\}$ so that for every continuous $f(x)$ $(-1\leqq x\leqq 1)$ $L_n(f, x_0)\to f(x_0)$ holds for at least one x_0 for which $\overline{\lim_{n\to\infty}}\sum_{k=1}^{n}|l_{kn}(x_0)|=\infty$. This is perhaps true, but at this moment we cannot prove it (the original "proof" was probably incomplete). We hope to settle it on another occasion.

2. Preliminary results

In his classical paper [2] G. FABER proved that for any matrix X

$$\overline{\lim_{n\to\infty}}\, \lambda_n = \infty$$

from where we immediately obtain that for every point group there exists a continuous function $f_1(x)$, $-1 \leqq x \leqq 1$ (shortly $f_1 \in C$) so that

$$\overline{\lim_{n\to\infty}}\, \|L_n(f_1, x)\| = \infty.$$

(Henceforward $\|g(x)\| = \|g\| = \max_{-1\leqq x\leqq 1} |g(x)|$ for $g \in C$.) Almost twenty years later, in 1931, S. BERNSTEIN [3] showed that for every X with (1.1) there is an $f_2 \in C$ and an x_0, $-1 \leqq x_0 \leqq 1$, such that

$$\overline{\lim_{n\to\infty}}\, |L_n(f_2, x_0)| = \infty.$$

Another problem is to prove divergence theorem on a set *of positive measure*.

In his paper [14] S. BERNSTEIN proved, that for the "bad" matrix $E = \{-1+2(k-1)/(n-1)\}$ and the function $|x|$

$$\overline{\lim_{n\to\infty}}\, |L_n(|t|, E, x)| = \infty \quad \text{if} \quad x \in (-1, 1), \quad x \neq 0.$$

Then, using the "good" Chebyshev matrix

$$T = \left\{x_{kn} = \cos\frac{2k-1}{2n}\pi;\ k = 1, 2, \ldots, n;\ n = 1, 2, \ldots\right\} \tag{2.1}$$

G. GRÜNWALD [4] got that there exists a function $f_3 \in C$, for which

$$\overline{\lim_{n\to\infty}}\, |L_n(f_3, T, x)| = \infty \tag{2.2}$$

holds for almost all x in $[-1, 1]$. Later he and (independently) J. MARCINKIEWICZ proved that for a suitable $f_4 \in C$, (2.2) is true for every x from $[-1, 1]$ (see [5] and [6]).

Very recently A. A. PRIVALOV [7] settled the case of Jacobi matrices

$$X^{(\alpha,\beta)} = \{x_{kn}^{(\alpha,\beta)},\ k = 1, 2, \ldots, n;\ n = 1, 2, \ldots\}, \quad \alpha, \beta > -1$$

(see e.g. [8], Part 2), showing that with a certain $f_4 \in C$

$$\overline{\lim_{n\to\infty}}\, |L_n(f_4, X^{(\alpha,\beta)}, x)| = \infty \quad \text{a.e. on } [-1, 1], \tag{2.3}$$

where "a.e." stands for almost everywhere. (He considered some further point groups, too.) His proof strongly depends on the properties of the Jacobi roots $x_{kn}^{(\alpha,\beta)}$. Finally, he proved (2.3) for the whole $(-1, 1)$ (see [13]).

3. Result

As indicated above we are going to prove (2.2) for *any* fixed point group X, i.e. we state

THEOREM. *For any matrix X with* (1.1) *one can find a function $F(x)\in C$ such that*

(3.1) $$\overline{\lim_{n\to\infty}}\, |L_n(F, X, x)| = \infty \quad \textit{for almost all } x \textit{ in } [-1, 1].$$

On the other hand, considering the special matrix

$$\begin{matrix} x_1 \\ x_1, x_2 \\ x_1, x_2, x_3 \\ \cdots\cdots \end{matrix}$$

we can say that (3.1) generally is not true for *all* $x\in[-1, 1]$ (see P. TURÁN [9], Problem III).

Finally, let us remark that the "$\overline{\lim}$" cannot be replaced by "lim" or "$\underline{\lim}$". Indeed, as P. ERDŐS showed, one can construct a point group so that for every $f\in C$ and every $x_0\in[-1, 1]$ there exists a sequence n_k (depending on f and x_0) so that

$$\lim_{k\to\infty} L_{n_k}(f, x_0) = f(x_0)$$

(see [1], p. 384).

4. Proof

4.1. In what follows, sometimes omitting the superfluous notations, let $x_{0n}=1$, $x_{n+1,n}=-1$ and

(4.1) $$\Delta x_{kn} = x_{kn} - x_{k+1,n} \quad (k = 0, 1, \ldots, n;\ n = 1, 2, \ldots).$$

Let us define the index-sets K_{1n} and K_{2n} and sets D_{1n} and D_{2n} by

(4.2) $$\begin{cases} \Delta x_{kn} \begin{cases} \leqq \dfrac{1}{\ln n} \overset{\text{def}}{=} \delta_n & \text{iff} \quad k\in K_{1n}, \\ > \delta_n & \text{iff} \quad k\in K_{2n}, \end{cases} \\ D_{1n} = \bigcup\limits_{k\in K_{1n}} [x_{k+1}, x_k], \quad D_{2n} = [-1, 1]\setminus D_{1n}. \end{cases}$$

If $\Delta x_k \leqq \delta_n$ (which means $k\in K_{1n}$, $[x_{k+1}, x_k]\subset D_{1n}$) we say that the interval $[x_{k+1}, x_k]$ is *short*; the other ones are *long*.

The fact that for any given positive numbers A and ε the measure of those x $(-\infty<x<\infty)$ for which

$$\lambda_n(x) \leqq A$$

holds if $n \geqq n_0(\varepsilon, A)$, is less than ε, was shown by the first of us in [1]. But here we need a stronger statement. Namely, if

$$I_{lm} = \left[-1+\frac{2(l-1)}{m}, -1+\frac{2l}{m}\right) \quad (l = 1, 2, \ldots, m),$$

then for the short intervals we prove

LEMMA 4.1. *Let $A > 0$ be an arbitrary fixed number. Then with arbitrary $m \geqq \max[\exp(8A^3), \exp(\exp 100)] \overset{\text{def}}{=} m_0(A)$, for any $n \geqq n_0(m)$ there exists a set $H_{1n} \subset D_{1n}$ for which $\mu(H_{1n}) \leqq 1/\ln \ln m$. Further, whenever $x \in D_{1n} \setminus H_{1n}$,*

$$\sum_{\substack{1 \leqq k \leqq n \\ x_{kn} \in D_{1n} \\ x_{kn} \notin I_{j(x),m} \\ k \notin K_{3n}}} |l_{kn}(x)| \geqq (\ln m)^{1/3} \geqq 2A \quad \text{if} \quad n \geqq n_0(m). \tag{4.3}$$

Here $x \in I_{j(x),m}$ $(1 \leqq j \leqq m)$, K_{3n} is a certain index-set having $\sqrt{\ln n}$ elements at most, $\mu(\ldots)$ stands for the Lebesgue measure.

4.1.1. The proof of this lemma, which is one of the most important parts of our theorem, consists of severals steps.

First we settle Lemma 4.2 regarding both short and long intervals.

Let us introduce the following notation.

$$J_k(q) = J_{kn}(q) = [x_{k+1}+q\Delta x_k, x_k - q\Delta x_k], \quad J_k = J_k(0) = [x_{k+1}, x_k],$$

for $0 \leqq q \leqq 1/2$, $0 \leqq k \leqq n$. If $z_k = z_{kn}(q)$ is defined by

$$|\omega_n(z_k)| = \min_{x \in J_k(q)} |\omega_n(x)|, \quad k = 0, 1, \ldots, n \tag{4.4}$$

(obviously, z_k is one of the endpoints of $J_k(q)$), we state

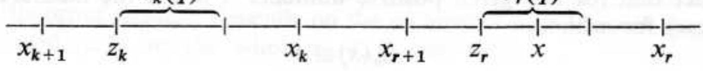
LEMMA 4.2. *If $x_k \leqq x_{r+1}$ $(1 \leqq r < k < n)$ then for arbitrary $0 < q \leqq 1/2$*

$$|l_k(x)| + |l_{k+1}(x)| \geqq q^2 \frac{|\omega_n(z_r)|}{|\omega_n(z_k)|} \frac{\Delta x_k}{x_r - x_{k+1}} \quad \text{if} \quad x \in J_r(q). \tag{4.5}$$

To prove (4.5), first we use

$$|l_s(x)| = \left|\frac{\omega(x)}{\omega'(x_s)(x-x_s)}\right| = \frac{|\omega(x)|}{|\omega(z_r)|} \frac{|z_r - x_s|}{|x-x_s|} |l_s(z_r)| \geqq q|l_s(z_r)| \tag{4.6}$$

if $s = k, k+1$ and $x \in J_r(q)$



(because $z_r - x_s \geqq q\Delta x_r + (x_{r+1} - x_s) \geqq q(\Delta x_r + x_{r+1} - x_s) \geqq q(x - x_s)]$, from where

$$|l_k(x)| + |l_{k+1}(x)| \geqq q[|l_k(z_r)| + |l_{k+1}(z_r)|] = \tag{4.7}$$

$$= q\frac{|\omega(z_r)|}{|\omega(z_k)|}\left[|l_k(z_k)|\frac{x_k - z_k}{z_r - x_k} + |l_{k+1}(z_k)|\frac{z_k - x_{k+1}}{z_r - x_{k+1}}\right] \geqq$$

$$\geqq q\frac{|\omega(z_r)|}{|\omega(z_k)|}\frac{\Delta x_k}{x_r - x_{k+1}}\frac{\min(z_k - x_{k+1}, x_k - z_k)}{\Delta x_k}[|l_k(z_k)| + |l_{k+1}(z_k)|] \geqq$$

$$\geqq q^2\frac{|\omega(z_r)|}{|\omega(z_k)|}\frac{\Delta x_k}{x_r - x_{k+1}} \quad (x \in J_r(q)),$$

using that $l_k(u) + l_{k+1}(u) \geqq 1$ if $u \in J_k$ (see [11], Lemma IV).

Similar estimation holds when $x_r \leqq x_{k+1}$.

4.1.2. We construct the set H_{1n} for $n \geqq n_0(m)$.

a) Any of J_{0n}, J_{nn} contained in D_{1n} should belong to H_{1n}. Further, if $J_{kn} \subset D_{1n}$ intersecting two I_{lm} $(1 \leqq l \leqq m)$ or whenever either k or $k+1 \in K_{3n}$, it should also belong to H_{1n}. The measure of these intervals J_{kn} is $\leqq 2\delta_n + (m-1)\delta_n + \sqrt{\ln n}\,\delta_n \leqq$ $\leqq (\ln\ln m)^{-2} \overset{\text{def}}{=} \varepsilon_m^2$, if, e.g. $n \geqq \exp(m^2) = n_0(m)$.

b) Let $q = q_m = \varepsilon_m/8$. The intervals $J_{kn}(q)$ or J_{kn} from D_{1n} not considered at a) will be called *exceptional* if there exists an $x = x(k, n) \in J_{kn}(q)$ for which the estimation (4.3) does not hold. The exceptional J_{kn}'s should also belong to H_{1n}. If $\sum_k' \mu(J_{kn}(q)) = 2c$ (where the dash indicates that the summation is extended only over the exceptional J_{kn}'s), we state that $c = c(n, m) \leqq \varepsilon_m^2$ if $n \geqq n_0(m)$ (whence the aggregate measure of the exceptional intervals J_{kn} is $< 3\varepsilon_m^2$).

Indeed, supposing $c > \varepsilon_m^2$ we shall obtain a contradiction.

Let us order the ψ_n exceptional $\bar{J}_1(q), \bar{J}_2(q), \dots, \bar{J}_{\psi_n}(q)$ such that

$$|\omega(\bar{z}_i)| \geqq \omega(\bar{z}_k) \quad (1 \leqq i \leqq k \leqq \psi_n),$$

where $\bar{z}_k$ stands for the corresponding minimum in $\bar{J}_k(q)$ (see (4.4)). Then for a certain φ_n, $1 \leqq \varphi_n \leqq \psi_n$,

$$\begin{cases} |\omega(\bar{z}_1)| \geqq |\omega(\bar{z}_k)| \geqq (\ln m)^{-1/2}|\omega(\bar{z}_1)| & \text{if} \quad 1 \leqq k \leqq \varphi_n, \\ |\omega(\bar{z}_1)| > (\ln m)^{1/2}|\omega(\bar{z}_k)| & \text{if} \quad \varphi_n < k \leqq \psi_n. \end{cases} \tag{4.8}$$

By a simple computation

$$\sum_{i=\varphi_n+1}^{\psi_n} \mu(\bar{J}_i(q)) \leqq c \quad \text{if} \quad n \geqq n_0(m) \tag{4.9}$$

(if, of course, $\varphi_n < \psi_n$).

Indeed, otherwise, using that

$$\sum_{i=\varphi_n+1}^{\psi_n} = \sum_{\substack{i=\varphi_n+1 \\ J_i \cap I_{jm} = \emptyset}}^{\psi_n} + \sum_{\substack{i=\varphi_n+1 \\ J_i \cap I_{jm} \neq \emptyset}}^{\psi_n} \overset{\text{def}}{=} \Sigma^{(1)} + \Sigma^{(2)}$$

where $\bar{z}_1 \in I_{jm} = I_{j(\bar{z}_1),m}$, we obtain

$$\sum^{(2)} \mu(\bar{J}_i(q)) \leqq 2m^{-1} < \varepsilon_m^2/2 < c/2,$$

from where $\sum^{(1)} \mu(\bar{J}_i(q)) > c/2$. Then by (4.5) and (4.8) for *any* $x \in \bar{J}_1(q)$ the sum (4.3) can be estimated as follows

$$\sum_{\dots} |l_k(x)| \geqq \frac{1}{2} \sum^{(1)} [|l_i(x)| + |l_{i+1}(x)|] \geqq \frac{q^2}{2} \sum^{(1)} \frac{|\omega(\bar{z}_1)|}{|\omega(\bar{z}_i)|} \frac{\Delta \bar{x}_i}{|\bar{x}_{i+1} - \bar{x}_1|} \geqq$$

$$\geqq \frac{q^2}{4} (\ln m)^{1/2} \sum^{(1)} \Delta \bar{x}_i \geqq \frac{q^2 c}{8} (\ln m)^{1/2} > \frac{\varepsilon_m^4}{8^3} (\ln m)^{1/2} > (\ln m)^{1/3}$$

which is a contradiction, i.e. (4.9) is true. (Here $\bar{x}_{i+1}$ and $\bar{x}_1$ are the "farthest" points of the corresponding intervals.)

4.1.3. Consequently, using the fact that the total measure γc $(1 \leqq \gamma \leqq 2)$ of the exceptional $\bar{J}_1(q), \dots, \bar{J}_{\varphi_n}(q)$ is bigger than ε_m^2, we should obtain a contradiction. Notice that for $\bar{J}_l$ we have (4.8), each $\bar{J}_l$ is in exactly one I_{km}; if $i = 0, n$, or when i or $i+1 \in K_{3n}$, then J_i cannot be exceptional. Obviously $\varphi_n \geqq c \ln n$.

Dropping $\bar{J}_l$ containing the middle point of $[-1, 1]$ and bisecting the same interval $[-1, 1]$, we have (say) in $[0, 1]$ a set of measure $\geqq [c - \mu(\bar{J}_t)]/2 \geqq (c - \delta_n)/2$, consisting of certain $\bar{J}_l(q)$'s $(1 \leqq l, t \leqq \varphi_n)$.

At the k-th bisection we obtain that interval of length 2^{1-k} which contains certain $\bar{J}_l(q)$'s $(1 \leqq l \leqq \varphi_n)$ of aggregate measure $\geqq 2^{-k} c - \delta_n \geqq 2^{-k-1} c$, if e.g. $1 \leqq k \leqq [\breve{\log}\, m] + 2 \overset{\text{def}}{=} p = p_m$.

Consider these intervals $L_1^*, L_2^*, \dots, L_p^*$.

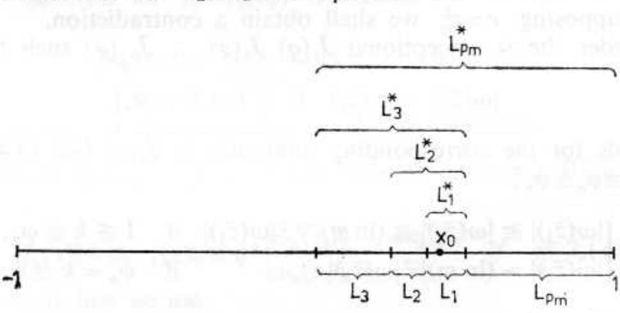


Fig. 1

Obviously $\mu(L_k^*) = 2^{k-p}$, each L_k^* contains at least k exceptional $\bar{J}_l(q)$'s, further

$$\sum_{\bar{J}_l(q) \subset L_k^*} \mu(\bar{J}_l(q)) \geqq 2^{k-p-2} c \quad (1 \leqq k \leqq p_m,\ 1 \leqq l \leqq \varphi_n).$$

Let $L_1 = L_1^*$, further $L_k = L_k^* \setminus L_{k-1}^*$ $(2 \leqq k \leqq p_m)$ (see Fig. 1). It is easy to see that $(2m)^{-1} \leqq \mu(L_1) \leqq m^{-1}$. Let us choose any fixed point x from any exceptional

$\bar{J}_{l_0,n}(q)$ contained in L_1 $(1\leq l_0\leq\varphi_n)$. Then, by (4.5) and (4.8) the sum (4.3) can be estimated as follows

$$(4.10)\quad \sum |l_t(x)| \geq \frac{1}{2}\sum [|l_t(x)|+|l_{t+1}(x)|] \geq \sum_{k=1}^{p}{}' \sum_{\substack{l\\ J_l(q)\subset L_k}} q^2 \frac{|\omega(\bar{z}_{l_0})|}{|\omega(\bar{z}_l)|}\cdot\frac{\Delta\bar{x}_l}{|\bar{x}_{l+1}-\bar{x}_0|} \geq$$

$$\geq q^2(\ln m)^{-1/2}\sum_{k=1}^{p}{}' \sum_{\substack{l\\ J_l(q)\subset L_k}} \frac{\Delta\bar{x}_l}{|\bar{x}_{l+1}-\bar{x}_0|} \overset{\text{def}}{=} q^2(\ln m)^{-1/2}B,$$

where $\bar{x}_{l+1}$ and $\bar{x}_0$ are the "farthest" points of the corresponding intervals, $1\leq l$, $l_0\leq\varphi_n$, the dash means that we exclude k whenever $I_{j(x),m}\cap L_k\neq\emptyset$. To estimate B, let

$$(4.11)\qquad \sum_{\substack{l\\ J_l(q)\subset L_k}} \Delta\bar{x}_l \overset{\text{def}}{=} c\alpha_k.$$

Using the construction, it is easy to see that

$$(4.12)\qquad c\sum_{k=1}^{i}\alpha_k \geq 2^{i-p-2}c \quad (1\leq i\leq p),$$

$$(4.13)\qquad |\bar{x}_0-\bar{x}_i| \leq 2^{k-p} \quad \text{if} \quad \bar{x}_0\in L_1 \quad \text{and} \quad \bar{x}_i\in L_k \quad (1\leq k\leq p).$$

By induction

$$(4.14)\qquad \alpha_k \leq 2^{k-2}\alpha_1 \quad (2\leq k\leq p).$$

(Indeed, by construction $\alpha_2\leq\alpha_1, \alpha_3\leq\alpha_1+\alpha_2\leq 2\alpha_1, \ldots,$ from where we get (4.14).)

Now, by (4.13), (4.11), (4.12), the Abel transformation and (4.14) we can write

$$B \geq c2^p \sum_{k=1}^{p}{}' 2^{-k}\alpha_k \geq c2^p\left[\sum_{k=1}^{p} 2^{-k}\alpha_k - 4\max_{1\leq k\leq p}\frac{\alpha_k}{2^k}\right] \geq$$

$$\geq c2^p\left[\sum_{k=1}^{p-1}\left(\sum_{i=1}^{k}\alpha_i\right)\frac{1}{2^{k+1}}+\left(\sum_{i=1}^{p}\alpha_i\right)\frac{1}{2^p}-4\alpha_1\right] \geq$$

$$\geq c2^p\left[\sum_{k=1}^{p-1}\frac{2^{k-p-2}}{2^{k+1}}+\frac{1}{2^{p+2}}-\frac{4}{mc}\right] \geq c\,\frac{\overset{2}{\log} m}{16}-16 \geq \frac{c\ln m}{20},$$

i.e., in virtue of (4.10),

$$\sum_{\ldots} |l_t(x)| \geq \frac{\varepsilon_m^4(\ln m)^{1/2}}{8^2\cdot 20} > (\ln m)^{1/3} \quad (n\geq n_0(m)),$$

i.e. for *any* $x\in\bar{J}_{l_0}(q)$ we have (4.3). But then $\bar{J}_{l_0}(q)$ is not exceptional which is a contradiction. So $c\leq\varepsilon_m^2$, as it was stated.

4.1.4. c) Clearly, *for any point* $x\in J_{kn}(q)(J_{kn}\subset D_{1n})$ *considered neither at* a) *nor at* b), *the estimation* (4.3) *will be true*. For these J_{kn} the sets $J_{kn}\setminus J_{kn}(q)$

of aggregate measure c_1 should belong to H_{1n}, too. Obviously, c_1 can be estimated as follows

$$c_1 \leqq \sum_{k\in K_{1n}} \left[\mu(J_{kn})-\mu(J_{kn}(q))\right] = 2q \sum_{k\in K_{1n}} \Delta x_k \leqq \frac{\varepsilon_m}{2}.$$

So by a), b) and c)

$$\mu(H_{1n}) \leqq \varepsilon_m^2+3\varepsilon_m^2+\varepsilon_m/2 \leqq \varepsilon_m$$

which proves Lemma 4.1.

4.2. Here we introduce an important definition. The interval I_{km} and its index k will be called *good* for a certain $n\geqq n_0(m)$ if

$$\sum_{J_{in}\subset H_{1n}}{}' \mu(I_{km}\cap J_{in}) \leqq \frac{\varepsilon_m}{2m} \quad (n\geqq n_0(m)),$$

where the dash means that we take only such J_{in}'s which were considered in a) or b) $(1\leqq k\leqq m)$. (Observe that I_{km} is good whenever $I_{km}\cap D_{1n}=\emptyset$.) Using that

$$\sum_{J_{in}\subset H_{1n}}{}' \mu(J_{in}) \leqq 4\varepsilon_m^2,$$

for any $n\geqq n_0(m)$ *at most* $8m\varepsilon_m$ *intervals* I_{km} *are not good* (m is fixed).

If we can choose a subsequence $\{n_i\}_{i=1}^{\infty}$ such that I_{1m} is good whenever $n\in\{n_i\}$ we take it. Otherwise, let us define $\{n_i\}$ so that I_{1m} is *not* good if $n\in\{n_i\}$. Starting from $\{n_i\}$ let us make the analogous process for I_{2m}. So after the m-th step we essentially derive the following statement.

LEMMA 4.3. *For every fixed* $m\geqq m_0(A)$ *and sequence* $\{l_r\}_{r=1}^{\infty}$ $(l_r\geqq n_0(m)$ *are integers) one can select a subsequence* $\{n_i\}_{i=1}^{\infty}\subset\{l_r\}_{r=1}^{\infty}$ *such that for any* $n\in\{n_i\}_{i=1}^{\infty}$ *the intervals* $I_{1m}, I_{2m}, \ldots, I_{mm}$ *are good, apart from* $I_{k_1,m}, I_{k_2,m}, \ldots, I_{k_j,m}$. *Here* $1\leqq k_1<k_2<\ldots<k_j\leqq m$, $j=j(m)\leqq 8m\varepsilon_m$ *and, which is very important, the indices* k_s $(1\leqq s\leqq j)$ *depend only on* m. *(If* $j=0$, *every* I_{km} *is good.)*

4.3. Now we shall treat the long intervals, i.e. the case when $\Delta x_k>\delta_n$ or what is the same, $k\in K_{2n}$, $(x_{k+1}, x_k)\subset D_{2n}$.

The following estimation plays a similar role as Lemma 4.1.

LEMMA 4.4. *Let* $\Delta x_{kn}>\delta_n$ *(k is fixed,* $0\leqq k\leqq n$*). Then for any fixed* $0<q<1/2$ *we can define the index* $t=t(k,n)$ *and the set* $h_{kn}\subset J_{kn}$ *so that* $\mu(h_{kn})\leqq 4q\,\Delta x_{kn}$, *moreover*

$$|l_t(x)| \geqq 3^{n\delta_n^5} \stackrel{\text{def}}{=} \eta_n \quad \textit{if} \quad x\in J_{kn}\setminus h_{kn} \quad \textit{and} \quad n\geqq n_1(q). \tag{4.15}$$

In the proof we refine some ideas of the papers by ERDŐS and TURÁN [11] and ERDŐS and SZABADOS [12]. Take those roots $y_{in}=\cos\vartheta_{in}$ $(1\leqq i\leqq n)$ of the n-th Chebychev polynomials $T_n(x)=\cos n\vartheta=2^{n-1}x^n+\ldots$ $(x=\cos\vartheta)$ which are in $J_{kn}(q)$. Their number is not less than $(1-2q)n\delta_n/\pi$ because of $\vartheta_{i+1}-\vartheta_i=\pi/n$ $(1\leqq i\leqq n-1$; see (2.1)) and $\Delta x_k>\delta_n$. If

$$h_k = [J_k\setminus J_k(q)]\cup\left\{\bigcup_{y_i\in J_k(q)}\left[\cos\left(\vartheta_i+q\frac{\pi}{n}\right), \cos\left(\vartheta_i-q\frac{\pi}{n}\right)\right]\right\},$$

then $\mu(h_k)\leqq 4q\Delta x_k$ and for arbitrary $y\in J_k\setminus h_k=J_k(q)\setminus h_k$ we can write $|T_n(y)|\geqq \geqq|\sin n\vartheta_i \sin q\pi|\geqq 2q$. Consider now the interval $M=M(y)=\left[y-\frac{q}{4}\delta_n, y+\frac{q}{4}\delta_n\right]\subset \subset J_k$ which contains at least $\frac{q}{2\pi}n\delta_n>n\delta_n^2$ roots of $T_n(x)$ $(n\geqq n_0(q))$. Then the polynomial $p(y,n;\ x)=p(x)=\prod_{y_{in}\notin M(y)}(x-y_{in})$ of degree less than n, can be estimated at any $x\notin(x_{k+1},x_k)$ as follows

$$|p(x)|=\frac{|T_n(x)|}{2^{n-1}\prod_{y_i\in M}|x-y_i|}=\left|p(y)\frac{T_n(x)}{T_n(y)}\right|\prod_{y_i\in M}\frac{|y-y_i|}{|x-y_i|}\leqq$$

$$\leqq\frac{|p(y)|}{2q}\prod_{y_i\in M}\frac{1}{3}\leqq\frac{|p(y)|}{2q}\frac{1}{3^{n\delta_n^2}}<\frac{|p(y)|}{3^{n\delta_n^3}}.$$

Now, using the Lagrange interpolatory formula,

$$|p(y)|\leqq\sum_{i=1}^{n}|p(x_i)|\,|l_i(y)|\leqq|p(y)|\,3^{-n\delta_n^3}\sum_{i=1}^{n}|l_i(y)|$$

from where $\sum_{i=1}^{n}|l_i(y)|\geqq 3^{n\delta_n^3}$ if $n\geqq n_1(q)$, because $|p(y)|\neq 0$.

So for any fixed $y\in J_k(q)\setminus h_k$ there exists an index $t=t(y,k,n)$ such that

$$|l_t(y)|=\left|\frac{\omega(y)}{\omega'(x_t)(y-x_t)}\right|\geqq 3^{n\delta_n^4}\quad(n\geqq n_1(q)). \tag{4.16}$$

Let us choose the point $y=u_k$ such that

$$|\omega(u_k)|=\min_{x\in J_k(q)\setminus h_k}|\omega(x)|.$$

Then, for arbitrary $y\in J_k(q)\setminus h_k$

$$|l_t(y)|=|l_t(u_k)|\frac{|\omega(y)|}{|\omega(u_k)|}\frac{|u_k-x_t|}{|y-x_t|}.$$

If $t\neq k, k+1$ we obtain as in (4.6) that $|u_k-x_t|\geqq q\,|y-x_t|$. (This inequality is trivial if $t=k$ or $t=k+1$.)

I.e., in both cases for $n\geqq n_1(q)$

$$|l_t(y)|\geqq q\,|l_t(u_k)|>3^{n\delta_n^5}\quad\text{if}\quad y\in J_k(q)\setminus h_k,$$

which means that in (4.15) the index t *does not depend on* x.

4.4. In the following part we shall construct the function $F(x)$.

4.4.1. Let us consider the short intervals, the sequences $\{A_t\}_{t=1}^{\infty}$, $\{m_t\}_{t=1}^{\infty}$ satisfying $A_t\nearrow\infty$, $m_t=[m_0(A_t)]+1$ and the intervals I_{j,m_t} (I_j, for short) of length $2/m_t$ $(1\leqq j\leqq m_t)$.

Let $t=1$. Let us choose the subsequence Q fulfilling the requirements of Lemmas 4.1 and 4.3. If $n_{11}\in Q$, let us define $g_1(x)$ *only on the nodes* as follows.

$$g_1(x_{k,n_{11}})=\begin{cases}(-1)^{k+1} & \text{if } x_{k,n_{11}}\in D_{1,n_{11}}\setminus I_1,\\ 0 & \text{otherwise.}\end{cases} \tag{4.17}$$

Then, in virtue of Lemma 4.1

$$|L_{n_{11}}(g_1,x)|=\sum_{x_k\in D_1\setminus I_1}|l_k(x)|\geqq(\ln m_1)^{1/3}\geqq 2A_1 \tag{4.18}$$

if $x\in(I_1\cap D_{1,n_{11}})\setminus H_{1,n_{11}}\overset{\text{def}}{=}T_1$. (Generally, if $f(x)$ is defined only for certain $x_k=$ $=x_{kn}$, $k=k_1,k_2,\dots,k_s$, then let $L_n(f,x)\overset{\text{def}}{=}\sum_{i=1}^{s}f(x_{k_i})l_{k_i}(x)$. If $T_1=\emptyset$, (4.18) is meaningless.)

4.4.2. Let $n_{12}>n_{11}$ $(n_{11},n_{12}\in Q)$ satisfy $\sqrt{\ln n_{12}}>n_{11}$. Let us define the set $\mathcal{T}_2$ by

$$2|L_{n_{12}}(g_1,x)|>(\ln m_1)^{1/3}\quad\text{if}\quad x\in\mathcal{T}_2\subset(I_2\cap D_{1,n_{12}})\setminus H_{1,n_{12}}\overset{\text{def}}{=}T_2. \tag{4.19}$$

Moreover, if $x\in T_2\setminus\mathcal{T}_2$, (4.19) should not hold.

a) *If* $2\mu(\mathcal{T}_2)\geqq\mu(T_2)$ or $T_2=\emptyset$ let $g_1(x_{k,n_{12}})=0$ *at* $x_{k,n_{12}}$ *not considered in* (4.17) (i.e. those for which does not exist l $(1\leqq l\leqq n_{11})$ such that $x_{k,n_{12}}=x_{l,n_{11}}$).

b) If $2\mu(\mathcal{T}_2)<\mu(T_2)$ then *for* $x_{k,n_{12}}$ *not considered in* (4.17) let, with $[a_j,a_{j+1})=I_j$,

$$g_1(x_{k,n_{12}})=\begin{cases}(-1)^k & \text{if } x_{k,n_{12}}\in D_{1,n_{12}}\setminus I_2 \text{ and } x_k<a_2,\\ (-1)^{k+1} & \text{if } x_{k,n_{12}}\in D_{1,n_{12}}\setminus I_2 \text{ but } x_k\geqq a_3,\\ 0 & \text{otherwise.}\end{cases} \tag{4.20}$$

By (4.19) and (4.3) if $x\in T_2\setminus\mathcal{T}_2$, then

$$|L_{n_{12}}(g_1,x)|\geqq\left|\pm\sum\nolimits^{(1)}|l_{k,n_{12}}(x)|+|\sum\nolimits^{(2)}g_1(x_{k,n_{12}})l_{k,n_{12}}(x)|\right|\geqq$$

$$\geqq(\ln m_1)^{1/3}-\frac{1}{2}(\ln m_1)^{1/3}=\frac{1}{2}(\ln m_1)^{1/3}\geqq A_1\quad(x\in T_2\setminus\mathcal{T}_2).$$

Here $\sum^{(1)}$ is extended over the x_k's considered in (4.20); for them Lemma 4.1 can be applied (because $\sqrt{\ln n_{12}}>n_{11}$); in $\sum^{(2)}$ we take those k's for which $x_{k,n_{12}}=x_{l,n_{11}}$ at certain $1\leqq l\leqq n_{11}$. So, by (4.19) $2|\sum^{(2)}|\leqq(\ln m_1)^{1/3}$, because $x\in T_2\setminus\mathcal{T}_2$.

Consequently, in both cases we can define the set $R_2\subset T_2$ and the function $g_1(x)$ such that $2\mu(R_2)\geqq\mu(T_2)$. Moreover

$$|L_{n_{12}}(g_1,x)|\geqq A_1\quad\text{whenever}\quad x\in R_2\subset T_2. \tag{4.21}$$

(At a) $R_2=\mathcal{T}_2$; at b) $R_2=T_2\setminus\mathcal{T}_2$; if $T_2=\emptyset$, the statement (4.21) is meaningless.)

4.4.3. By the above method we can obtain the sets $T_i=T_{1i}=(I_{i,m_1}\cap D_{1n_{1i}})\setminus H_{1n_{1i}}$, the subsets $R_i=R_{1i}\subset T_{1i}$ $(i=1, 2, \dots, m_1;\ R_1\equiv T_1)$ and the function $g_1(x)$ such that $2\mu(R_{1i})\geqq\mu(T_{1i})$ and

$$(4.22)\qquad |L_{n_{1i}}(g_1, x)|\geqq A_1 \quad\text{if}\quad x\in R_{1i}\subset T_{1i},\quad 1\leqq i\leqq m_1.$$

Let

$$(4.23)\qquad G_1 \overset{\text{def}}{=} \bigcup_{i=1}^{m_1} R_{1i}.$$

4.4.4. Now consider the *polynomial* $\varphi_1(x)=\varphi_1(g_1, x)$ satisfying $\varphi_1(x_{k,n_{1i}})=g_1(x_{k,n_{1i}})$ $(1\leqq k\leqq n_{1i};\ 1\leqq i\leqq m_1)$ and $\|\varphi_1\|\leqq 2$. Here $\deg\varphi_1\leqq N_1$, where N_1 depends only on the distribution of the nodes defining $g_1(x)$ (see [8], Part 3, II/§ 3).

4.4.5. Generally, starting from the subsequence obtained in the $(t-1)$-th step, let us make the above construction for (A_t, m_t) $(t=2, 3, \dots)$. We can suppose

$$(4.24)\qquad n_{t-1,m_{t-1}} < N_{t-1} < n_{t1} \quad (t=2, 3, \dots).$$

We successively get the sets T_{ti}, their parts R_{ti} with $2\mu(R_{ti})\geqq\mu(T_{ti})$ $(i=1, 2, \dots, m_t)$, the functions $g_t(x)$ for which

$$(4.25)\qquad |L_{n_{ti}}(g_t, x)|\geqq A_t \quad\text{if}\quad x\in R_{ti}\subset T_{ti},\quad 1\leqq i\leqq m_t,$$

further the sets

$$(4.26)\qquad G_t = \bigcup_{i=1}^{m_t} R_{ti}.$$

We can also construct the corresponding polynomials $\varphi_t(x)$, taking the values $g_t(x_{k,n_{ti}})$ $(1\leqq k\leqq n_{ti};\ 1\leqq i\leqq m_t)$ for which $\|\varphi_t\|\leqq 2$ and $\deg\varphi_t\leqq N_t$ $(t=2, 3, \dots)$.

Supposing

$$(4.27)\qquad A_t > t^3\lambda_{N_{t-1}}^2 \quad (\lambda_{N_0}\equiv 1,\ t=1, 2, \dots),$$

let us define the set

$$(4.28)\qquad G = \bigcap_{k=1}^{\infty}\left(\bigcup_{t=k}^{\infty} G_t\right)$$

and the function

$$(4.29)\qquad f(x) = \sum_{t=k}^{\infty}\frac{\varphi_t(x)}{t^2\lambda_{N_{t-1}}}.$$

We state that

$$(4.30)\qquad \varlimsup_{n\to\infty}|L_n(f, x)| = \infty \quad\text{whenever}\quad x\in G.$$

(Clearly $f\in C$, moreover $\|f\|\leqq 4$ can be attained.) If $G=\varnothing$, we have nothing to prove. Otherwise, if $x\in G$ there exists an index-set $\{r_k\}_{k=1}^{\infty}$ depending on x for which $x\in G_{r_k}$ $(k=1, 2, \dots)$. Then, by (4.26), for any fixed r_k we can find an s such that $x\in R_{r_k,s}$. By (4.29)

$$L_{n_{r_k,s}}(f, x) = \sum_{i=1}^{\infty}\frac{L_{n_{r_k,s}}(\varphi_i, x)}{i^2\lambda_{N_{i-1}}} = \sum_{i<r_k} + \sum_{i=r_k} + \sum_{i>r_k}.$$

Here by (4.24) $L_{n_{r_k},s}(\varphi_i,x)\equiv\varphi_i(x)$ if $i<r_k$, so

$$\Big|\sum_{i<r_k}\Big|\leqq 2\sum_{i=1}^{\infty} i^{-2}\lambda_{N_{i-1}}^{-1}\leqq c_1,$$

further, by (4.25) and (4.27)

$$\left|\frac{L_{n_{r_k},s}(\varphi_{r_k},x)}{r_k^2\lambda_{N_{r_k-1}}}\right|\geqq\frac{A_{r_k}}{r_k^2\lambda_{N_{r_k-1}}}>r_k\lambda_{N_{r_k-1}}.$$

Finally, supposing $\lambda_l>\lambda_j$ if $l>j$, $l, j\in\{n_{ti}\}\cup\{N_t\}$, we can write

$$\Big|\sum_{i>r_k}\Big|\leqq 2\lambda_{n_{r_k},s}\sum_{i=r_k+1}^{\infty} i^{-2}\lambda_{N_{i-1}}^{-1}\leqq 2\sum_{i=1}^{\infty} i^{-2}\leqq c_2,$$

because $\lambda_{n_{r_k},s}<\lambda_{N_{r_k}}$ (see (4.24)). Consequently,

$$|L_{n_{r_k},s}(f,x)|\geqq r_k\quad (k=2,3,\dots;\ x\in G)$$

which actually is more than (4.30).

4.4.6. Let us now take the sets $T_{ti}^{[2]}=T_{ti}^{[1]}\setminus R_{ti}^{[1]}$ $(i=1,2,\dots,m_t;\ t=1,2,\dots;$ $T_{ti}^{[1]}=T_{ti}, R_{ti}^{[1]}=R_{ti})$ given by the previous steps. If, e.g. $t=1$, let us begin the construction of $g_1^{[2]}(x)$ exactly as we did for $g_1(x)=g_1^{[1]}(x)$ in **4.4.1** (i.e., we use the *same* A_1, m_1, T_1 and nodes; compare (4.17)), but the distinctions a) and b) in **4.4.2** should be defined by the measure of $\mathscr{T}_{12}^{[2]}$ instead of $\mathscr{T}_2=\mathscr{T}_{12}^{[1]}$ where $\mathscr{T}_{12}^{[2]}$ collects those points of the set $T_{12}^{[2]}=T_{12}^{[1]}\setminus R_{12}^{[1]}$ for which $2|L_{n_{12}}(g_1^{[2]},x)|>$ $>(\ln m_1)^{1/3}$ (see (4.19)). Consequently, by the method analogous to **4.4.1—4.4.5** (using the same $\{n_{ti}\}$) we can construct the corresponding sets $R_{ti}^{[2]}, G_t^{[2]}$, the polynomials $\varphi_t^{[2]}(x)$ of degree $\leqq N_t$ and the continuous function

$$f^{[2]}(x)=\sum_{t=1}^{\infty}\frac{\varphi_t^{[2]}(x)}{t^2\lambda_{N_{t-1}}} \tag{4.31}$$

with $\|f^{[2]}\|\leqq 4$ such that on the set

$$G^{[2]}=\bigcap_{k=1}^{\infty}\left(\bigcup_{t=k}^{\infty}G_t^{[2]}\right) \tag{4.32}$$

we have

$$\overline{\lim_{n\to\infty}}\,|L_n(f^{[2]},x)|=\infty\quad (x\in G^{[2]}). \tag{4.33}$$

By the same considerations starting from the sets $T_{ti}^{[l]}=T_{ti}^{[l-1]}\setminus R_{ti}^{[l-1]}$ $(l=3,4,\dots,p$ where p will be defined later), we can successively define the functions $f^{[l]}\in C, \|f^{[l]}\|\leqq 4$ and the sets $G^{[l]}$ such that

$$\overline{\lim_{n\to\infty}}\,|L_n(f^{[l]},x)|=\infty\quad (x\in G^{[l]}) \tag{4.34}$$

$$(l=1,2,\dots,p;\ f^{[1]}=f,\ G^{[1]}=G).$$

Later we shall apply the fact that for *any* t and i

$$\mu(R_{ti}^{[l]}) \leqq \frac{1}{2^{l-2} m_t} \quad (l=1, 2, \dots, p) \tag{4.35}$$

and for any fixed t and i

$$R_{ti}^{[l_1]} \cap R_{ti}^{[l_2]} = \emptyset \quad (l_1 \neq l_2) \tag{4.36}$$

(see the definition of the sets $R_{ti}^{[l]}$).

Now let $\varrho > 0$ be arbitrarily small and $p = p_\varrho$ the smallest positive integer so that

$$\mu(R_{ti}^{[p_\varrho]}) \leqq \frac{\varrho}{m_t} \quad (i=1, 2, \dots, m_t;\ t=1, 2, \dots). \tag{4.37}$$

It is easy to see that $1 \leqq p_\varrho \leqq 3 + |\overset{2}{\log}\, \varrho|$.

4.4.7. To define the proper (linear) combination of the functions $f^{[1]}, f^{[2]}, \dots, f^{[p]}$ on $G^{[1]} \cup G^{[2]} \cup \dots \cup G^{[p]}$ we prove the following statement, which generalizes an idea of G. GRÜNWALD [4].

LEMMA 4.5. *If* $r_1(x), r_2(x) \in C$, *moreover*

$$\overline{\lim_{n\to\infty}}\, |L_n(r_1, x)| = \infty \quad \textit{if} \quad x \in B_1,\ \mu(B_1) < \infty, \tag{4.38}$$

$$\overline{\lim_{n\to\infty}}\, |L_n(r_2, x)| = \infty \quad \textit{if} \quad x \in B_2,\ \mu(B_2) < \infty, \tag{4.39}$$

then any fixed interval (β_1, β_2) $(\beta_1 < \beta_2)$ *contains an* α *such that*

$$\overline{\lim_{n\to\infty}}\, |L_n(\alpha r_1 + r_2, x)| = \infty \quad \textit{a.e. on } B_1 \cup B_2. \tag{4.40}$$

REMARK. An interesting special case can be obtained by $B_2 = \emptyset$. To prove the lemma let $\tilde{B}_1$ be the part of $B_1 \cup B_2$ fulfilling (4.38). Clearly $B_1 \subset \tilde{B}_1$. If

$$E_\lambda = \{x\colon x \in \tilde{B}_1,\ \overline{\lim_{n\to\infty}}\, |L_n(\lambda r_1 + r_2, x)| < \infty\} \quad (\beta_1 < \lambda < \beta_2)$$

then $E_\lambda \cap E_\mu = \emptyset$ $(\lambda \neq \mu)$. Indeed, otherwise we can write for $x \in E_\lambda \cap E_\mu$

$$\infty = \overline{\lim_{n\to\infty}}\, |(\lambda - \mu) L_n(r_1, x)| = \overline{\lim_{n\to\infty}}\, |L_n(\lambda r_1 + r_2, x) - L_n(\mu r_1 + r_2, x)| \leqq$$

$$\leqq \overline{\lim_{n\to\infty}}\, \big(|L_n(\lambda r_1 + r_2, x)| + |L_n(\mu r_1 + r_2, x)|\big) < \infty,$$

a contradiction. Using $\mu(\tilde{B}_1) < \infty$ and that only countable E_λ's have positive measure $(\beta_1 < \lambda < \beta_2)$, there exists $\alpha \in (\beta_1, \beta_2)$ such that $\mu(E_\alpha) = 0$ from where (4.40) is true a.e. on $\tilde{B}_1$. If $x \in (B_1 \cup B_2) \setminus \tilde{B}_1$ (when $x \in B_2$, too) both $|L_n(\alpha r_1, x)| \leqq K(x)$ $(0 \leqq K(x) < \infty)$, and $\overline{\lim_{n\to\infty}}\, |L_n(r_2, x)| = \infty$ hold which mean (4.40) for x. These prove the lemma.

4.4.8. Choosing $\beta_1 = 0$ and $\beta_2 = 0.5$, consider that $\alpha \in (0, 0.5)$ for which, with $e_2 = \alpha_1 f^{[1]} + f^{[2]}$,

$$\overline{\lim_{n\to\infty}}\, |L_n(e_2, x)| = \infty \quad \text{a.e. on} \quad G^{[1]} \cup G^{[2]}.$$

Obviously $\|e_2\| \leqq 2+4<8$. By this construction we succesively get the values $\alpha_{i-1} \in (0, 0.5)$ and the continuous functions $e_i = \alpha_{i-1} e_{i-1} + f^{[i]}$ satisfying

$$\overline{\lim_{n\to\infty}} |L_n(e_i, x)| = \infty \quad \text{a.e. on} \quad G^{[1]} \cup G^{[2]} \cup \ldots \cup G^{[i]}$$

and $\|e_i\| \leqq 0.5 \|e_{i-1}\| + \|f^{[i]}\| < 8$ $(i=3, 4, \ldots, p_\varrho)$.

I.e., if $i=p_\varrho$, we can say that *for every fixed* $\varrho>0$ *there exists a function* $f_\varrho \in C$, $\|f_\varrho\| \leqq 8$ *so that*

$$\overline{\lim_{n\to\infty}} |L_n(f_\varrho, x)| = \infty \quad \text{a.e. on } G_\varrho \tag{4.41}$$

where $G_\varrho = \bigcup_{i=1}^{p_\varrho} G^{[i]}$.

4.4.9. We go on with the construction of $F(x)$ for the long intervals ($\Delta x_{kn} > \delta_n$ i.e. $k \in K_{2n}$) employing the same A_t, m_t, n_{ti} and I_{im_t} $(i=1, 2, \ldots, m_t;\ t=1, 2, \ldots)$ as for the short intervals. First a simple note. If

$$H_{2n} \overset{\text{def}}{=} \bigcup_{k \in K_{2n}} h_{kn} \quad (n = 1, 2, \ldots) \tag{4.42}$$

and

$$q = q_t = \frac{\varepsilon_{m_t}}{8m_t}, \tag{4.43}$$

then by Lemma 4.4 for any t and i

$$\mu(H_{2,n_{ti}}) \leqq 2 \cdot 4q_t = \frac{\varepsilon_{m_t}}{m_t}, \quad \text{if} \quad n_{ti} \geqq n_1(m_t); \tag{4.44}$$

the latter should be supposed.

For simplicity's sake let $(D_{2,n_{11}} \setminus H_{2,n_{11}}) \cap I_{1,m_1} \neq \emptyset$, say, for the indices $j_1, j_2, \ldots, j_s \in K_{2,n_{11}}$

$$(J_{i,n_{11}} \setminus h_{i,n_{11}}) \cap I_{1,m_1} \neq \emptyset \quad (i = j_1, j_2, \ldots, j_s;\ s \geqq 1). \tag{4.45}$$

We take the indices $t(i, n_{11})$ $(i=j_1, j_2, \ldots, j_s)$ guaranteed by Lemma 4.4 and define the function $u_i(x)$ as follows.

$$u_i(x_{k,n_{11}}) = \begin{cases} 1 & \text{when} \quad k = t(i, n_{11}), \\ 0 & \text{otherwise}, \end{cases}$$

$|u_i(x)| \leqq 1$, $u_i \in C$. Then clearly

$$|L_{n_{11}}(u_i, x)| \geqq \eta_{n_{11}} \quad \text{if} \quad x \in J_{i,n_{11}} \setminus h_{i,n_{11}} \quad (i = j_1, j_2, \ldots, j_s). \tag{4.46}$$

4.4.10. To combine the at most $2m_1^{-1} \ln n_{11}$ functions $u_i(x)$ we need the following

LEMMA 4.6. *Let* $r_1, r_2 \in C$, *moreover*

$$|L_n(r_1, x)| \geqq M_1 \quad \text{if} \quad x \in B_1, \quad \mu(B_1) < \infty, \tag{4.47}$$

$$|L_n(r_2, x)| \geqq M_2 \quad \text{if} \quad x \in B_2, \quad \mu(B_2) < \infty. \tag{4.48}$$

Consider the fixed real numbers $\beta_1 < \beta_2$ *and the positive integer* k. *Further take*

$$\alpha_i = (\beta_2 - \beta_1)\frac{i}{k} + \beta_1 \quad (i = 0, 1, \ldots, k). \tag{4.49}$$

Then, if $M_2 \geqq M_1$ *and* $0 \leqq \beta_1 < \beta_2 \leqq 0.5$, *there exists an* α_j $(0 \leqq j \leqq k)$ *and E of measure at least* $\left(1-\frac{1}{k+1}\right)\mu(B_1 \cup B_2)$, $E \subset B_1 \cup B_2$, *so that*

$$(4.50) \qquad |L_n(\alpha_j r_1 + r_2, x)| \geqq \frac{\beta_2-\beta_1}{2k} M_1 \quad \text{if} \quad x \in E.$$

To prove this, we verify at first a statement which is slightly more than the special case corresponding to $B_2 = 0$.

Namely, *if we have only* (4.47), *then there exist* P_1 *of measure* $\geqq \left(1-\frac{1}{k+1}\right)\mu(B_1)$, $P_1 \subset B_1$ *and* α_j $(0 \leqq j \leqq k)$ *such that* (4.50) *is true for* $x \in P_1$.

Indeed, let

$$C_i = \left\{x\colon x \in B_1 \text{ and } |L_n(\alpha_i r_1 + r_2, x)| \geqq \frac{\beta_2-\beta_1}{2k} M_1\right\} \quad (i = 0, 1, \dots, k).$$

It is easy to see that any $x \in B_1$ can be contained in at most one $B_1 \setminus C_i$ (see (4.47), (4.50) and the similar part of **4.4.7**), from where $(B_1 \setminus C_i) \cap (B_1 \setminus C_l) = \varnothing$ $(i \neq l)$. By $B_1 \setminus C_i \subset B_1$, for certain $0 \leqq j \leqq k$ $\mu(B_1 \setminus C_j) \leqq \mu(B_1)(k+1)^{-1}$, which gives the special case with $P_1 = C_j$.

Now let $\tilde{B}_1$ be that part of $B_1 \cup B_2$ where (4.47) is satisfied. Take that α_j, for which (4.50) is true on certain $\tilde{P}_1 \subset \tilde{B}_1$. If $x \in (B_1 \cup B_2) \setminus \tilde{B}_1$ then by (4.48),

$$|L_n(\alpha_j r_1 + r_2, x)| \geqq |M_2 - 0.5 M_1| \geqq 0.5 M_2 > (\beta_2 - \beta_1) M_1$$

from where we obtain the lemma by $E = \tilde{P}_1 \cup ((B_1 \cup B_2) \setminus \tilde{B}_1)$.

4.4.11. Using this lemma with the cast

$$r_i(x) = u_{j_i}(x), \quad B_i = (J_{j_i, n_{11}} \setminus h_{j_i, n_{11}}) \cap I_{1, m_1}, \quad M_i = \eta_{n_{11}} \quad (i = 1, 2),$$
$$\beta_1 = 0, \quad \beta_2 = 0.5 \quad \text{and} \quad k = [\ln^2 n_{11}]$$

(see **4.4.9** and **4.4.10**), we obtain a $v_2(x) \in C$ for which

$$|L_{n_{11}}(v_2, x)| \geqq \frac{\eta_{n_{11}}}{4k} \quad \text{if} \quad x \in E_2$$

where (with the above cast)

$$0 \leqq \mu(B_1 \cup B_2) - \mu(E_2) \leqq \frac{\mu(B_1 \cup B_2)}{k+1} \leqq \frac{2\delta_{n_{11}}^2}{m_1},$$

$\|v_2\| \leqq \beta_2 \|r_1\| + \|r_2\| < 2$. At the next step, by $r_1 = v_2$, $B_1 = E_2$, $r_2 = u_{j_3}$ and $B_2 = (J_{j_3} \setminus h_{j_3}) \cap I_1$ we get the function $v_3(x) \in C$ and the set E_3. Finally, the $(s-1)$-th step gives the function $v_s(x) \overset{\text{def}}{=} w_1(x) \in C$, the set $E_s \overset{\text{def}}{=} W_1 \subset I_1$ so that

$$(4.51) \qquad |L_{n_{11}}(w_1, x)| \geqq \frac{\eta_{n_{11}}}{(4k)^{s-1}} \geqq \frac{\eta_{n_{11}}}{(4 \ln^2 n_{11})^{\ln n_{11}}} \overset{\text{def}}{=} \gamma_{n_{11}} \quad \text{if} \quad x \in W_1$$

where

$$\sum_{i=1}^{s} \mu[(J_{j_i} \setminus h_{j_i}) \cap I_1] - \mu(W_1) \leqq \frac{2s\delta_{n_{11}}^2}{m_1} \leqq \frac{2\delta_{n_{11}}}{m_1}, \tag{4.52}$$

because $s < \ln n_{11}$. Further notice that $\|w_1\| \leqq 2$. By definition $\gamma_n \nearrow \infty$ (e.g. $\gamma_n \gg 3^{\sqrt{n}}$) and

$$\mu[(D_{2,n_{11}} \setminus H_{2,n_{11}}) \cap I_{1,m_1}] - \mu(W_1) \leqq \frac{\varepsilon_{m_1}}{m_1} \tag{4.53}$$

if $n_{11} > n_1(m_1)$. (It is easy to see that the left hand sides of (4.52) and (4.53) are the same.)

Now consider the polynomial $\psi_1(x)$ for which $\|\psi_1\| \leqq 4$ and $\psi_1(x_{kn}) = w_1(x_{kn})$ $(k=1, 2, \dots, n;\ n=n_{11})$. Clearly we can suppose $n_{12} > \deg \psi_1$, too (compare with **4.4.4**).

By this construction one successively obtains the polynomials $\psi_i(x) = \psi_{1i}(x)$ and the sets $W_i = W_{1i}$ $(i=1, 2, \dots, m_1)$, then generally the polynomials $\psi_{ti}(x)$ and the sets W_{ti} $(i=1, 2, \dots, m_t,\ t=1, 2, \dots)$ such that $\|\psi_{ti}\| \leqq 4$, $\deg \psi_{ti} < n_{t,i+1}$ (where $n_{t,m_t+1} \equiv n_{t+1,1}$) and

$$|L_{n_{ti}}(\psi_{ti}, x)| \geqq \gamma_{n_{ti}} \quad \text{if} \quad x \in W_{ti} \subset I_{i,m_t}, \tag{4.54}$$

$$\mu[(D_{2,n_{ti}} \setminus H_{2,n_{ti}}) \cap I_{i,m_t}] - \mu(W_{ti}) \leqq \frac{\varepsilon_{m_t}}{m_t}. \tag{4.55}$$

(If $(D_{2,n_{ti}} \setminus H_{2,n_{ti}}) \cap I_{i,m_t} = \varnothing$ then the corresponding $W_{ti} = \varnothing$, further $w_{ti}(x) = \psi_{ti}(x) = 0$.)

4.4.12. We can define the sequence $\{n_{ti}\}$ (satisfying *all* the requirements mentioned above) such that

$$\gamma_{n_{ti}} > m_t^2 t^3 \lambda_{n_{t,i-1}}.$$

Consider the function

$$h(x) = \sum_{t=1}^{\infty} \frac{1}{t^2 m_t^2} \sum_{i=1}^{m_t} \frac{\psi_{ti}(x)}{\lambda_{n_{t,i-1}}} \tag{4.56}$$

(where $\lambda_{n_{10}} = 1$ and $\lambda_{n_{t,0}} = \lambda_{n_{t-1,m_{t-1}}}$) on the set

$$W = \bigcup_{k=1}^{\infty} \bigcup_{t=k}^{\infty} \left(\bigcup_{i=1}^{m_t} W_{ti} \right). \tag{4.57}$$

By the method applied in **4.4.5** we get

$$\overline{\lim_{n \to \infty}} |L_n(h, x)| = \infty \quad \text{if} \quad x \in W. \tag{4.58}$$

Moreover it is easy to fulfil the condition $\|h\| \leqq 8$. Now, using Lemma 4.5 for $f_\varrho \in C$ and the set G_ϱ (see **4.4.8**), further for $h \in C$ and W, we obtain as follows.

For arbitrary fixed $\varrho > 0$ there exists a continuous function $F_\varrho(x)$, $\|F_\varrho\| \leqq 16$ (if, e.g. $[\beta_1, \beta_2] = [0, 1]$) such that

$$\overline{\lim_{n \to \infty}} |L_n(F_\varrho, x)| = \infty \quad \textit{a.e. on } P_\varrho, \quad \mu(P_\varrho) \geqq 2 - \varrho, \tag{4.59}$$

where $P_\varrho = G_\varrho \cup W \subset [-1, 1]$.

Here the only thing we have to prove is that $\mu(P_\varrho)\geqq 2-\varrho$. For this aim let us see the definitions made in **4.4.6** and **4.4.8.** We can write

$$G_\varrho\cup W=\left(\bigcup_{j=1}^{p_\varrho} G^{[j]}\right)\cup W=\left(\bigcup_{j=1}^{p_\varrho}\bigcap_{k=1}^{\infty}\bigcup_{t=k}^{\infty}\bigcup_{i=1}^{m_t} R_{ti}^{[j]}\right)\cup$$

$$\cup\left(\bigcap_{k=1}^{\infty}\bigcup_{t=k}^{\infty}\bigcup_{i=1}^{m_t} W_{ti}\right)=\bigcap_{k=1}^{\infty}\bigcup_{t=k}^{\infty}\bigcup_{i=1}^{m_t}\left[\left(\bigcup_{j=1}^{p_\varrho} R_{ti}^{[j]}\right)\cup W_{ti}\right]\stackrel{\text{def}}{=}\bigcap_{k=1}^{\infty}\left(\bigcup_{t=k}^{\infty} V_t\right)\stackrel{\text{def}}{=}\bigcap_{k=1}^{\infty} Q_k.$$

(Indeed, by $W=G^{[0]}$ and $\bigcup_{i=1}^{m_t} R_{ti}^{[j]}=A_{tj}$,

$$G_\varrho\cap W=\bigcup_{j=0}^{p} G^{[j]}=\left\{\bigcup_{j=0}^{p}\bigcap_{k=1}^{\infty}\bigcup_{t=k}^{\infty} A_{tj}\right\}_1=\left\{\bigcap_{k=1}^{\infty}\bigcup_{t=k}^{\infty}\bigcup_{j=0}^{p} A_{tj}\right\}_2$$

because $x\in\{\ldots\}_s$ if and only if for a certain j there exist infinitely many t such that $x\in A_{tj}$ $(s=1,2)$. Of course, $\{\ldots\}_2=\bigcap_{k=1}^{\infty} Q_k.$)

Let us see the measure of $[\ldots]$ for a good interval I_{i,m_t} if $n=n_{ti}$.

The sets $R_{ti}^{[j]}$ $(j=1,2,\ldots,p_\varrho)$ overlap $(D_{1,n_{ti}}\setminus H_{1,n_{ti}})\cap I_{i,m_t}$ apart from a part of measure not exceeding ϱm_t^{-1} (see (4.35)—(4.37)). Moreover, the sets of type a) and b) from $H_{1,n_{ti}}\cap I_{i,m_t}$ have the measure not exceeding $\varepsilon_{m_t}(2m_t)^{-1}$ altogether (since i is good); the same is true for the parts of type c) (see **4.1.4** and **4.2**).

Further, by (4.55) the set W_{ti} contains the set $(D_{2,n_{ti}}\setminus H_{2,n_{ti}})\cap I_{i,m_t}$ excluding a part of measure not exceeding $\varepsilon_{m_t}(m_t)^{-1}$.

Using that $D_1\cap D_2=\emptyset$, $H_1\subset D_1$, $H_2\subset D_2$ and $D_1\cup D_2=[-1,1]$, by the above considerations and (4.44) we can estimate as follows $(I_i=I_{i,m_t})$.

$$\mu([\ldots])\geqq\mu((D_1\setminus H_1)\cap I_i)-\frac{\varrho}{m_t}+\mu((D_2\setminus H_2)\cap I_i)-\frac{\varepsilon_{m_t}}{m_t}=$$

$$=\mu(I_i\cap(D_1\cup D_2)\setminus(I_i\cap H_1)\setminus(I_i\cap H_2))-\frac{\varrho}{m_t}-\frac{\varepsilon_{m_t}}{m_t}\geqq\frac{2}{m_t}-\frac{1}{m_t}(3\varepsilon_{m_t}+\varrho).$$

By the construction and Lemma 4.3, *the good intervals I_{i,m_t} are uniquely determined by m_t, i.e. by t whenever $n=n_{Tk}$ $(k=1,2,\ldots,m_T;\ T\geqq t)$, its number is* $\geqq m_t-8m_t\varepsilon_{m_t}$. So we can write

$$\mu(V_t)=\sum_{i=1}^{m_t}\mu([\ldots])\geqq\sum_i{}'\mu([\ldots])\geqq(m_t-8m_t\varepsilon_{m_t})\frac{1}{m_t}(2-3\varepsilon_{m_t}-\varrho)=$$

$$=(1-8\varepsilon_{m_t})(2-3\varepsilon_{m_t}-\varrho)>2-19\varepsilon_{m_t}-\varrho\quad(t=1,2,\ldots),$$

where $\sum'$ means that we consider only the good indices i (t is fixed).

By this we obtain

$$\mu(Q_k)=\mu\left(\bigcup_{t=k}^{\infty} V_t\right)\geqq\mu(V_k)>2-19\varepsilon_{m_k}-\varrho.$$

On the other hand, $Q_1\supset Q_2\supset\ldots$ from where, as it is well-known, $\mu(Q_k)\to\mu(P_\varrho)$, which gives $\mu(P_\varrho)\geqq 2-\varrho$.

4.4.13. Now we state the following

LEMMA 4.7. *If* $g_1, g_2, \ldots \in C$ *and* $\overline{\lim_{n\to\infty}}\, g_n(x)=\infty$ *on B, then for arbitrary fixed A, ε and M there exist the set* $H \subseteq B$ *and the index N such that* $\mu(H) \leqq \varepsilon$; *moreover if* $x \in B \setminus H$ *then for a certain* $u(x)$ *we have*

$$g_{u(x)}(x) \geqq A \quad \textit{where} \quad M \leqq u(x) \leqq N. \tag{4.60}$$

Indeed, let

$$H_t = \{x\colon x \in B,\ g_{M+i}(x) < A,\ i = 0, 1, \ldots, t\} \quad (t = 0, 1, \ldots).$$

If for a certain $t=s$, $\mu(H_s) \leqq \varepsilon$, then we can choose $N=M+s$, because if $x \in B \setminus H_s$ then with suitable $u(x)$, $M \leqq u(x) \leqq N$, we obtain (4.60). On the other hand, if $\mu(H_t) > \varepsilon$ $(t=0, 1, \ldots)$ then using $H_t \supseteq H_{t+1}$ we get $\mu\left(\bigcup_{t=0}^{\infty} H_t\right) \geqq \varepsilon$ wich means that for $x \in \bigcap_{t=0}^{\infty} H_t \subseteq B$, $\overline{\lim_{t\to\infty}}\, g_t(x) \leqq A$ holds, a contradiction.

4.4.14. Now we construct the function $F(x)$. For this aim let $m_1 = \lambda_{N_0} = 1$, $A_1 = 2$ and $\varrho_1 = 2^{-1}$. By (4.59) and the previous lemma we can find an $f_1 \in C$, $\|f_1\| \leqq 16$, the index n_1 and the set $S_1 \subset [-1,1]$, $\mu(S_1) \geqq 2 - 2\varrho_1$ so that

$$|L_{u_1(x)}(f_1, x)| \geqq A_1 > 1^3 \lambda_{N_0}^2 \quad \text{whenever} \quad x \in S_1 \quad (\text{see } \mathbf{4.4.4}).$$

Generally, let $\delta_k = 2^{-k}$, $A_k > k_8 \lambda_{N_{k-1}}^2$ and choose $m_k = N_{k-1} + 1$. As above, we obtain the polynomial $\varphi_k(x)$ of degree $\leqq N_k$, $\|\varphi_k\| \leqq 32$, the set $S_k \subset [-1,1]$, $\mu(S_k) \geqq 2 - 2\delta_k$, and the index n_k so that

$$|L_{u_k(x)}(\varphi_k, x)| \geqq A_k > k^3 \lambda_{N_{k-1}}^2 \quad \text{if} \quad x \in S_k$$

with $m_k \leqq u_k(x) \leqq n_k$ $(k=2, 3, \ldots)$. Choosing N_k large enough compared to n_k, we obtain, using the arguments of **4.4.4-4.4.5,** that for the continuous function

$$F(x) = \sum_{k=1}^{\infty} \frac{\varphi_k(x)}{k^2 \lambda_{N_{k-1}}}$$

and for the set $S = \bigcap_{k=1}^{\infty} \bigcap_{i=k}^{\infty} S_i$ of measure 2

$$\overline{\lim_{n\to\infty}}\, |L_n(F, x)| = \infty \quad \text{on } S,$$

which is the statement of the theorem.

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