Problems and Results on Polynomials and Interpolation

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This is not a survey paper. I am somewhat out of touch with this subject and therefore would not dare to attempt such a paper. I shall just discuss some of the problems my collaborators and I have worked on for more than 40 years. In particular, I shall concentrate on problems where there has been some progress recently – apart from this I shall discuss a few of my favourite problems.

Most of the problems discussed are mentioned in [5], [6] or [7]. These papers all contain extensive references and many solved and unsolved problems. Many of the problems in [7] were settled by Pommerenke and Elbert (for references see [6]). First of all, I shall discuss problems on polynomials and then problems on interpolation.

1 Problems on Polynomials

Here are two of my favourite problems mentioned in [7] which are still open.

Let $f_n(z) = z^n + \ldots + a_n$ be a polynomial of degree n. Consider the lemniscate $|f_n(z)| = 1$. Is it true that the length of this curve is maximal if $f_n(z) = z^n - 1$? I offer 100 dollars for the first proof or disproof. Perhaps a cleverly used variational technique will give a proof. Pommerenke has some inqualities for the length of the lemniscate, but they fall far short of the conjecture.

Let

$$g(z) = \prod_{i=1}^{n} (z - z_i), \quad |z_i| \leqslant 1, i = 1, 2, \ldots, n.$$

Denote by E_g the set $|g(z) \leqslant 1$. $A(E_g)$ denotes the area of E_g. Put $\epsilon_n = \min A(E_g)$, where the minimum is extended over all polynomials of degree $\leqslant n$ of the above kind. In [7] it was shown that $\epsilon_n \to 0$ $(n \to \infty)$. We have no satisfactory upper or lower bounds for ϵ_n. $n^\eta \epsilon_n$ should tend to ∞ for every $\eta > 0$ and perhaps the order of magnitude of ϵ_n is logarithmic, but we have no real evidence.

We conjectured also that a disk of radius λ/n, where $\lambda > 0$ is absolute, can always be placed in E_g. A much weaker result has been proved by Pommerenke. Our conjecture if true is best possible as $g(z) = z^n - 1$ shows. For further related results see my paper with E. Netanyahu (see [6]).

Reference [6] contains several further problems on the geometry of polynomials. Here is one of them. Assume that $g(z)$ has the above form and that E_g has n components. Is it true that $A(E_g)$ is maximal when $g(z) = z^n - 1$? Incidentally, as far as I know the area of $|z^n - 1| \leqslant 1$ has not been determined, but I do not think that it should be very difficult to do so.

An old conjecture of mine stated: Let $|z_n| = 1, n = 1, 2, \ldots$. Put

$$A_n = \max_{|z|=1} \left| \prod_{i=1}^{n} (z - z_i) \right|.$$

Then

$$\limsup_{n \to \infty} A_n = \infty.$$

This conjecture has recently been proved by G. Wagner [14].

Hayman observed that there is a sequence with $|z_n| = 1$ for which $A_n \leqslant n$ for all n and Linden [12] improved this to $A_n < n^{1-\alpha}$ for a positive α. It seems quite probable that there is a constant $c > 0$

so that for infinitely many n, $A_n > n^c$ holds for every sequence with $|z_n| = 1$. Perhaps it is always the case that

$$\lim_{n \to \infty} \left(\prod_{i=1}^{n} A_i \right)^{1/n} = \infty.$$

Is it true that to every B there corresponds a function $\phi(B)$ so that

$$\max_{n < m < n + \phi(B)} A_m > B?$$

If not, then there is the problem of estimating the smallest $f_n(B)$ for which

$$\max_{n < m < n + f_n(B)} A_m > B.$$

D. Newman and I considered long ago the following problem Let $|a_k| = 1$, $k = 0, 1, \ldots$. Is it true that

$$\max_{|z|=1} \left| \sum_{k=0}^{n-1} a_k z^k \right| > (1 + \lambda) n^{1/2},$$

for some absolute, positive constant λ? This conjecture has recently been disproved by T. Körner [**11**]. The conjecture for the special case where $a_k = \pm 1$, $k = 0, 1, 2, \ldots$, which we also put forward, is still open.

For random polynomials (i.e. the coefficients $a_k = \pm 1$ or $|a_k| = 1$ are chosen at random) much more is true. Salem and Zygmund [**13**] proved that for all but $o(2^n)$ choices of $a_k = \pm 1$,

$$c_1 (n \log n)^{1/2} < \max_{|z|=1} \left| \sum_{k=0}^{n} a_k z^k \right| < c_2 (n \log n)^{1/2}$$

for some absolute c_1, $c_2 > 0$. Halasz [**9**] strengthened this result by proving that one has

$$\max_{|z|=1} \left| \sum_{k=0}^{n} a_k z^k \right| = (1 + o(1))\, C (n \log n)^{1/2}$$

for some absolute $C > 0$.

Let $0 < t < 1$ and

$$t = \sum_{k=1}^{\infty} \frac{\epsilon_k(t)}{2^k}$$

be the binary expansion of t. Put

$$f_t(z) = \sum_{k=0}^{\infty} \{2\epsilon_k(t) - 1\} z^k .$$

Then Salem and Sygmund and Halasz show that in fact their respective results hold for the partial sums of $f_t(z)$ for almost all t.

Salem and Zygmund at the end of their paper pose the following problem. Estimate

$$M_n(t) = \max_{-1 \leqslant x \leqslant 1} \sum_{k=1}^{n} \{2\epsilon_k(t) - 1\} x^k$$

as well as possible for almost all t. I observed that a result of Chung [2] implies that for almost all t

$$M_n(t) < (1 + o(1)) \frac{\pi}{2\sqrt{2}} \left(\frac{n}{\log \log n}\right)^{1/2}$$

infinitely often, and I further showed that for almost all t and every $\epsilon > 0$,

$$\lim_{n \to \infty} M_n(t)/n^{\frac{1}{2} - \epsilon}$$

There is a big gap between the above results, which I can narrow somewhat, but a big gap still remains. The above results are referred to in the paper of Salem and Zygmund, but my proof of the latter result was never published.

Let $f_n(\theta)$ be a trigonometric polynomials of degree n satisfying $|f_n(\theta)| \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. I proved [4] that the length of the graph of $f_n(\theta)$ in $(0, 2\pi)$ is maximal for $\cos n\theta$. I stated that if $f_n(x)$ is a polynomial of degree n, $|f_n(x)| \leqslant 1$, $-1 \leqslant x \leqslant 1$, then the length of the graph of $f_n(x)$ in $(-1, +1)$ is maximal if $f_n(x) = T_n(x)$, the Chebyshev polynomial of degree n. This result is undoubtedly true, but I am unable to prove it.

The final problem: Let $f(z) = z^n + \ldots$. I noted that there is always a $z_0 \in E_f$ for which $|f'(z_0)| \geqslant n$, with equality for $f(z) = z^n$. Assume that E_f is connected. How large can $f'(z)$ be for $z \in E_f$?

I conjectured that the maximum is assumed if $f(z) = T_n(cz)$, where c is the unique real number chosen so that the interior of E_{T_n} consists of n components and the closures of two neighbouring ones have exactly one point in common. I mistakenly stated that the derivative in this case is less than $n^2/2$, but, of course, it is less than $(n^2/2)(1 + o(1))$, The somewhat weaker inequality

$$|f'(z)| < \frac{e\, n^2}{2}\ (z \in E_f)$$

was proved by Pommerenke.

2 Problems on Interpolation

Let $-1 \leqslant x_1 < \ldots < x_n \leqslant 1$ and denote by $l_k(x)$ the fundamental polynomial of Lagrange interpolation, i.e.

$$l_k(x_k) = 1, l_k(x_i) = 0 \text{ for } 1 \leqslant i \leqslant n, i \neq k.$$

Nearly 50 years ago S. Bernstein conjectured that

$$\min_{-1 \leqslant x_1 < \ldots < x_n \leqslant 1}\ \max_{-1 \leqslant x \leqslant 1}\ \sum_{k=1}^{n} |l_k(x)|$$

is assumed if all the $n + 1$ maxima in $(-1, 1)$ of

$$\sum_{k=1}^{n} |l_k(x)|$$

are the same and I conjectured that the smallest of these $n + 1$ maxima is largest when they are all equal.

These conjectures were recently proved in a series of remarkable papers by Kilgore [**10**], De Boor and Pincus [3] and Bratman [**1**].

I stated in previous papers the following theorem. Let

$$\begin{array}{cc} & x_1^{(1)} \\ x_1^{(2)} & x_2^{(2)} \\ \ldots & \ldots \end{array}$$

be a point group; all the $x_i^{(n)}, n = 1, 2, \ldots, 1 \leqslant i \leqslant n$, are in $(-1, +1)$ and the $x_i^{(n)}$, $1 \leqslant i \leqslant n$, are distinct. Then there is a continuous function $f(x)$ so that the sequence of Lagrange interpolation polynomials

$$(L_n f)(x) = \sum_{k=1}^{n} f(x_i^{(n)}) l_i^{(n)}(x)$$

diverges for almost all x. I now feel that my statement was a little "optimistic" and that there were gaps in my proof. In any case, Vértesi and I now have a complete proof which will appear soon in *Acta Hungarica*.

I also stated that there is a point group $\{x_i^{(n)}\}$ so that for every continuous function $f(x)$ there is a point $x_0, -1 < x < 1$, so that

$$(L_n f)(x_0) \underset{n}{\to} f(x_0), \limsup_{n \to \infty} \sum_{i=1}^{n} |l_i^{(n)}(x_0)| = \infty.$$

In other words, $(L_n f)(x)$ cannot diverge simultaneously at all points where divergence is possible. Vértesi and I tried to work out a proof of this, but unfortunately we failed. Thus at present it is safer to treat this "result" only as a conjecture.

Is it true that there is a point group $\{x_i^{(n)}\}$ so that for every x_0,

$$\limsup_{n \to \infty} \sum_{i=1}^{n} l_i^{(n)}(x_0) = \infty,$$

but for every continuous function $f(x)$ there is a y_0 so that

$$(L_n f)(y_0) \underset{n}{\to} f(y_0)?$$

This would be a most interesting result, if true. Unfortunately, I cannot prove it.

Szabados and I [8] proved that there is an absolute constant $c > 0$ so that, for $-1 \leqslant x_1 < \ldots < x_n \leqslant 1$,

$$\sum_{i=1}^{n} \int_{-1}^{1} |l_i(x)| \, \mathrm{d}x > c \log n.$$

The best value of c is not known. No doubt the roots of $T_n(x) = 0$, where $T_n(x)$ is the nth Chebyshev polynomial, give asymptotically the best value of c, but this has not been proved.

I stated that, for every point group and for almost all x and infinitely many n,

$$\sum_{i=1}^{n} |l_i(x)| > c \log n.$$

This is certainly true, but the proof I had in mind was incomplete. Vértesi and I hope to have a completely satisfactory proof soon. It is perhaps true that one can take any $c < 2/\pi$ and if so this would be best possible.

There are several other statements in some of my older papers which I should try to clear up before I "leave". The most important one is the following: G. Grunwald and I "proved" in a paper of ours that if the point group $\{x_i^{(n)}\}$ has the $x_i^{(n)}$ at the roots of $T_n(x)$ then there is a continuous function $f(x)$ so that

$$\frac{1}{n} \sum_{k=1}^{n} (L_k f)(x)$$

diverges everywhere. In fact our proof only gives the weaker result where the summands are replaced by their moduli. I have often tried to prove our earlier "result", but so far without success. Perhaps a proof will be difficult since I have shown that the arithmetic means of the $(L_k f)(x)$ certainly behave much more regularly than the $(L_n f)(x)$ themselves. G. Grunwald and Marcinkiewicz proved that for any $h(n) \underset{n}{\to} \infty$ there is a continuous function $f(x)$ so that for every x,

$$(L_n f)(x) > \frac{\log n}{h(n)}$$

infinitely often. On the other hand, I proved that for every continuous function $f(x)$,

$$\frac{1}{n} \sum_{k=1}^{n} (L_k f)(x) = o(\log \log n).$$

Therefore, taking arithmetic means clearly has a smoothing effect. I discovered the error in our earlier "proof" only after proving the last result above.

Marcinkiewicz proved that if the point group comes from the zeros of the polynomials $U_n(x) = T'_{n+1}(x)$, then for every continuous function $f(x)$ and every x_0 there is a subsequence (n_i) so that $(L_{n_i} f)(x_0) \underset{i}{\to} f(x_0)$. For Fourier series the analogous result that there is a subsequence of the partial sums which converges to $f(x_0)$ is a classical result of Fejér. Turán and I proved a similar result when the zeros of $U_n(x)$ are replaced by those of $T_n(x)$, and $x_0 \neq \cos(p/q)\pi$ with $p, q \equiv 1 \pmod 2$. I proved that if x_0 is such an exceptional point, there is a continuous function $f(x)$ for which $|(L_n f)(x_0)| \underset{n}{\to} \infty$. This is perhaps surprising since it was thought that the Lagrange interpolation polynomials based on the zeros of the $T_n(x)$ behaved similarly to the partial sums of the Fourier series. In fact, I claimed in my paper that for every α, $-\infty \leqslant \alpha \leqslant \infty$, there is a continuous function $f(x)$ for which $f(x_0) \neq \alpha$ and $(L_n f)(x_0) \underset{n}{\to} \alpha$. My oversight was discovered by Schoenberg and in the correction I published I showed that my original proof gave the weaker result $|(L_n f)(x_0)| \underset{n}{\to} \infty$.

In an addendum to the correction I claimed the following much stronger theorem: Let $x_0 = \cos(p/q)\pi$ with $p, q \equiv 1 \pmod 2$ and let S be an arbitrary closed set. Then there is always a continuous function $f(x)$ so that the set of limit points of $(L_n f)(x_0)$ is S. I never published a proof. I feel I will do this if three conditions are fulfilled: (1) I have time, i.e. I do not "leave" too soon; (2) I have enough energy; (3) my proof was correct and I can reconstruct it. I am optimistic enough to believe that (1) and (2) will more or less be fulfilled, but if I cannot fulfil (3) soon I shall withdraw my claim.

Turan asked the following question. Is it true that for an arbitrary point group and a continuous function $f(x)$, the set of x where $(L_n f)(x)$ converges to a value different from $f(x)$ is "small" – presumably of measure 0? I hope I can prove this; in fact, though this set may be of measure 0 it can have the power of the continuum.

To conclude, I restate a conjecture published in [5]. Is it true that to every A there is an $\epsilon > 0$ so that if $n > n_0(\epsilon)$, then for every $-1 \leqslant x_1 < \ldots < x_n \leqslant 1$ there is a set $y_1 \ldots, y_n$, $|y_i| \leqslant 1$, so that every polynomial $p_m(x)$ of degree $m < (1+\epsilon)n$ for which $p_m(x_i) = y_i$ holds for at least $(1-\epsilon)n$ values of i satisfies

$$\max_{-1 \leqslant x \leqslant 1} |p_m(x)| > A.$$

This conjecture, if true, clearly strengthens the classical theorem of Faber; in his theorem $m = n - 1$, $\epsilon = 0$.

A final note: many problems are contained in the posthumous paper of P. Turán, "Some open problems in the theory of approximation", *Mat. Lapok* **25** (1974) 21–75. This paper is written in Hungarian, but will be translated soon.

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